Exercises 2.1

1. (a) [BB] $\{-\sqrt{5}, \sqrt{5}\}$
   (b) $\{1, 3, 5, 15, -1, -3, -5, -15\}$
   (c) [BB] $\{0, -\frac{3}{2}\}$ (Although $\pm\sqrt{2}$ are solutions to the equation, they are not rational.)
   (d) $\{-1, 0, 1, 2, 3\}$
   (e) This is the empty set. There are no numbers less than -4 **and** bigger than $+4$.

2. (a) [BB] For example, $1 + i$, $1 + 2i$, $1 + 3i$, $-8 - 5i$ and $17 - 43i$.
   (b) For example, $\{1 - 2\sqrt{2}, 1 - 5\sqrt{2}, 1 - 7\sqrt{2}, 316 - 2\sqrt{2}$ and $394 - 7\sqrt{2}\}$.
   (c) If $x = 0$, $y = \pm 5$ and $x/y = 0$. If $x = 1$, $y = \pm\sqrt{24}$, and $x/y = \pm 1/\sqrt{24} = \pm\sqrt{24}/24$. If $x = 2$, $y = \pm\sqrt{21}$ and $x/y = \pm 2\sqrt{21}/21$. Five elements of the given set are 0, $\sqrt{24}/24$, $-\sqrt{24}/24$, $2\sqrt{21}/21$ and $-2\sqrt{21}/21$.
   (d) $\{2, 3, 5, 6, 8\}$.

3. (a) [BB] $\{1,2\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,3,4\}$
   (b) $\emptyset$, $\{1\}$, $\{2\}$, $\{1,2\}$
   (c) $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{1,3,4\}$, $\{2,3,4\}$
   (d) $\{3\}$, $\{4\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,2,3,4\}$
   (e) $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,3,4\}$
   (f) $\emptyset$, $\{1\}$, $\{2\}$.

4. [BB] Only (c) is true. The set A contains one element, $\{a, b\}$.

5. (a) [BB] True. 3 belongs to the set $\{1, 3, 5\}$.
   (b) False. $\{3\}$ is a subset of $\{1, 3, 5\}$ but not a member of this set.
   (c) True. $\{3\}$ is a proper subset of $\{1, 3, 5\}$.
   (d) [BB] False. $\{3, 5\}$ is a subset of $\{1, 3, 5\}$.
   (e) False. Although $\{1, 3, 5\}$ is a subset of itself, it is not a **proper** subset.
   (f) False. If $a + 2b$ is in the given set, a is even, so $a + 2b$ is even and can't equal 1.
   (g) False. If $a + b\sqrt{2} = 0$ and $b \neq 0$, then $\sqrt{2} = -\frac{a}{b}$ is the quotient of rational numbers and, hence, rational. But this is not true.

6. (a) [BB] $\{\emptyset\}$; (b) $\{\emptyset, \{\emptyset\}\}$; (c) $\{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}, \{\{\emptyset\}\}\}$.

7. (a) [BB] True. The empty set is a subset of every set.
   (b) True. The empty set is a subset of every set.
   (c) False. The empty set does not contain any elements.
   (d) True. $\{\emptyset\}$ is a set containing one element, namely, $\emptyset$.
   (e) [BB] False. $\{1, 2\}$ is a subset of $\{1, 2, 3, \{1, 2, 3\}\}$.
   (f) False. $\{1, 2\}$ is not an element of $\{1, 2, 3, \{1, 2, 3\}\}$.
   (g) True. $\{1, 2\}$ is a proper subset of $\{1, 2, \{\{1, 2\}\}\}$.
   (h) [BB] False. $\{1, 2\}$ is not an element of $\{1, 2, \{\{1, 2\}\}\}$.

(i) True. $\{\{1,2\}\}$ contains just one element, $\{1,2\}$, and this is an element of $\{1,2,\{1,2\}\}$.

8. [BB] Yes it is; for example, let $x = \{1\}$ and $A = \{1, \{1\}\}$.

9. (a) i. $\{a,b,c,d\}$ ii. [BB] $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$
iii. $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$ iv. $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ v. $\emptyset$

(b) 16

10. (a) If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ is a set containing one element, so **its** power set contains two elements.

(b) $\mathcal{P}(A)$ contains two elements; $\mathcal{P}(\mathcal{P}(A))$ has four elements.

11. (a) [BB] 4; (b) [BB] 8.

(c) [BB] There are 2^n subsets of a set of n elements. (See Exercise 15 in Section 5.1 for a proof.)

12. (a) [BB] False. Let $A = \{2\}$, $B = \{\{2\}\}$, $C = \{\{\{2\}\}\}$. Then A is an element of B (that is, $A \in B$) and B is an element of C ($B \in C$), but A is not an element of C (since B is C's only element).

(b) True. If $x \in A$, then $x \in B$ since $A \subseteq B$. But since $x \in B$, then $x \in C$ since $B \subseteq C$.

(c) True. As in the previous part, we know that $A \subseteq C$. To prove $A \neq C$, we note that there is some $x \in C$ such that $x \notin B$ (since $B \subsetneqq C$). Then, since $x \notin B$, $x \notin A$. Therefore, x is an element of C which is not in A, proving $A \neq C$.

(d) [BB] True. $A \in B$ means that A belongs to the set B. Since B is a subset of C, any element of B also belongs to C. Hence, $A \in C$.

(e) False. For example, let $A = \{1\}$, $B = \{\{1\}, 2\}$ and $C = \{\{1\}, 2, 3\}$. Then $A \in B$, $B \subseteq C$, but $A \not\subseteq C$.

(f) False. Let $A = \{1\}$, $B = \{1,2\}$, $C = \{\{1,2\}, 3\}$. Then $A \subseteq B$ and $B \in C$, but $A \notin C$.

(g) False. Same example as 12(f) where $A \not\subseteq C$.

13. (a) This is false. As a counter-example, consider $A = \{1\}$, $B = \{2\}$. Then A is not a subset of B and B is not a proper subset of A.

(b) The converse of the implication in (a) is the implication $B \subsetneqq A \rightarrow A \not\subseteq B$. This is true. Since $B \subsetneqq A$, there exists some element $a \in A$ which is not in B. Thus A is not a subset of B.

14. (a) [BB] True. ($\longrightarrow$) If $C \in \mathcal{P}(A)$, then by definition of "power set," C is a subset of A; that is, $C \subseteq A$.

($\longleftarrow$) If $C \subseteq A$, then C is a subset of A and so, again by definition of "power set," $C \in \mathcal{P}(A)$.

(b) True. ($\longrightarrow$) Suppose $A \subseteq B$. We prove $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. For this, let $X \in \mathcal{P}(A)$. Therefore, X is a subset of A; that is, every element of X is an element of B. Since $A \subseteq B$, every element of X must be an element of B. So $X \subseteq B$; hence, $X \in \mathcal{P}(B)$.

($\longleftarrow$) Conversely, assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We must prove $A \subseteq B$. For any set A, we know that $A \subseteq A$ and, hence, $A \in \mathcal{P}(A)$. Here, with $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have, therefore, $A \in \mathcal{P}(B)$; that is, $A \subseteq B$, as desired.

(c) The double implication here is false because the implication $\longrightarrow$ is false. If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ and $\{\emptyset\} \neq \emptyset$.

Exercises 2.2

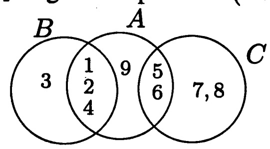
1. (a) [BB] $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{-1, 0, 1, 2, 3, 4, 5\}$, $C = \{0, 2, -2\}$.
   (b) $A \cup C = \{-2, 0, 1, 2, 3, 4, 5, 6\}$, $B \cap C = \{0, 2\}$,
   $B \setminus C = \{-1, 1, 3, 4, 5\}$, $A \oplus B = \{-1, 6, 0\}$,
   $C \times (B \cap C) = \{(0,0), (0,2), (2,0), (2,2), (-2,0), (-2,2)\}$,
   $(A \setminus B) \setminus C = \{6\}$, $A \setminus (B \setminus C) = \{2, 6\}$,
   $(B \cup \emptyset) \cap \{\emptyset\} = \emptyset$.
   (c) $S = \{(1,-1), (2,0), (3,1), (4,2), (5,3), (6,4)\}$; $T = \{(1,2), (2,2)\}$.
2. (a) [BB] $S \cap T = \{\sqrt{2}, 25\}$, $S \cup T = \{2, 5, \sqrt{2}, 25, \pi, \frac{5}{2}, 4, 6, \frac{3}{2}\}$,
   $T \times (S \cap T) = \{(4, \sqrt{2}), (4, 25), (25, \sqrt{2}), (25, 25), (\sqrt{2}, \sqrt{2}),$
   $(\sqrt{2}, 25), (6, \sqrt{2}), (6, 25), (\frac{3}{2}, \sqrt{2}), (\frac{3}{2}, 25)\}$.
   (b) [BB] $\mathsf{Z} \cup S = \{\sqrt{2}, \pi, \frac{5}{2}, 0, 1, -1, 2, -2, \ldots\}$; $\mathsf{Z} \cap S = \{2, 5, 25\}$;
   $\mathsf{Z} \cup T = \{\sqrt{2}, \frac{3}{2}, 0, 1, -1, 2, -2, \ldots\}$; $\mathsf{Z} \cap T = \{4, 25, 6\}$.
   (c) $\mathsf{Z} \cap (S \cup T) = \{2, 5, 25, 4, 6\} = (\mathsf{Z} \cap S) \cup (\mathsf{Z} \cap T)$. The two sets are equal.
   (d) $\mathsf{Z} \cup (S \cap T) = \{\sqrt{2}, 0, 1, -1, 2, -2, \ldots\} = \mathsf{Z} \cup \{\sqrt{2}\} = (\mathsf{Z} \cup S) \cap (\mathsf{Z} \cup T)$.
   The two sets are equal.
3. (a) [BB] $\{1, 9, 0, 6, 7\}$; (b) $\{4, 6, 5\}$; (c) $\{0, 1\}$.
4. $A = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$ and $B = \{\pm\frac{1}{2}, \pm 1, \pm 2\}$.
5. (a) [BB] $\{c, \{a, b\}\}$; (b) $\{\emptyset\}$; (c) A; (d) $\emptyset$;
   (e) [BB] $\emptyset$; (f) $\{A\}$.
6. (a) [BB] $A^c = (-2, 1]$; (b) $A^c = (-\infty, -3] \cup (4, \infty)$; (c) $A^c = \mathsf{R}$.
7. (a) $Y \cap Z = \{3, 4, 5\}$, so $X \oplus (Y \cap Z) = \{1, 2, 5\}$.
   (b) $(X^c \cup Y)^c = X \cap Y^c = X \setminus Y = \{1\}$.
8. (a) [BB] The subsets of A containing $\{1, 2\}$ are obtained by taking the union of $\{1, 2\}$ with a subset of $\{3, 4, 5, \ldots, n\}$. Their number is the number of subsets of $\{3, 4, 5, \ldots, n\}$ which is 2^{n-2}. (See Exercise 11 of Section 2.1.)
   (b) The subsets B which have the property that $B \cap \{1, 2\} = \emptyset$ are exactly the subsets of $\{3, 4, 5, \ldots, n\}$ and these number 2^{n-2}.
   (c) The subsets B which have the property that $B \cup \{1, 2\} = A$ are precisely those subsets which contain $\{3, 4, 5, \ldots, n\}$ and these correspond, as in (a), to the subsets of $\{1, 2\}$. There are four.
9. [BB] $(a, b)^c = (-\infty, a] \cup [b, \infty)$, $[a, b)^c = (-\infty, a) \cup [b, \infty)$, $(a, \infty)^c = (-\infty, a]$,
   $(-\infty, b]^c = (b, \infty)$.
10. (a) [BB] $CS \subseteq T$; (b) [BB] $M \cap P = \emptyset$; (c) $M \not\subseteq P$; (d) $CS \setminus T \subseteq P$;
    (e) $(M \cup CS) \cap P \subseteq T^c$.
11. (a) Negation: $CS \not\subseteq T$; Converse: $T \subseteq CS$.
    (b) Negation: $M \cap P \neq \emptyset$; Converse: $P \subseteq M^c$ or $P \cap M = \emptyset$,

(c) Negation: $M \subseteq P$; No converse since the statement is not an implication.
(d) Negation: $(M \cup CS) \cap P \not\subseteq T^c$; Converse: $T^c \subseteq (M \cup CS) \cap P$.
(e) Negation: $(M \cup CS) \cap P \cap T \neq \emptyset$; Converse $T^c \subseteq (M \cup CS) \cap P$.

12. (a) [BB] $\mathsf{E} \cap \mathsf{P} \neq \emptyset$ (b) $0 \in \mathsf{Z} \setminus \mathsf{N}$ (c) $\mathsf{N} \subseteq \mathsf{Z}$
(d) $\mathsf{Z} \not\subseteq \mathsf{N}$ (e) $(\mathsf{P} \setminus \{2\}) \subseteq \mathsf{E}^c$ (f) $2 \in \mathsf{E} \cap \mathsf{P}$
(g) $\mathsf{E} \cap \mathsf{P} = \{2\}$

13. (a) [BB] Since $A_{-3} \subseteq A_3$, $A_3 \cup A_{-3} = A_3$.
(b) Since $A_{-3} \subseteq A_3$, $A_3 \cap A_{-3} = A_{-3}$.
(c) $A_3 \cap (A_{-3})^c = \{a \in \mathsf{Z} \mid -3 < a \leq 3\} = \{-2, -1, 0, 1, 2, 3\}$.
(d) Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$, we have $\bigcap_{i=0}^4 A_i = A_0$.

14. [BB] Region 2 represents $(A \cap C) \setminus B$. Region 3 represents $A \cap B \cap C$; region 4 represents $(A \cap B) \setminus C$.

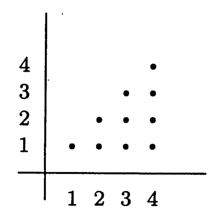
15. (a) 

(b) i. $(A \cup B) \cap C = \{5, 6\}$
ii. $A \setminus (B \setminus A) = A = \{1, 2, 4, 7, 8, 9\}$
iii. $(A \cup B) \setminus (A \cap C) = \{1, 2, 3, 4, 9\}$
iv. $A \oplus C = \{1, 2, 4, 7, 8, 9\}$
v. $(A \cap C) \times (A \cap B) = \{(5, 1), (5, 2), (5, 4), (6, 1), (6, 2), (6, 4)\}$

16. (a) [BB] $A \subseteq B$, by Problem 7; (b) $B \subseteq A$, by PAUSE 4 with A and B reversed.

17. [BB] Think of listing the elements of the given set. There are n pairs of the form $(1, b)$, $n - 1$ pairs of the form $(2, b)$, $n - 2$ pairs of the form $(3, b)$, and so on until finally we list the only pair of the form (n, b). The answer is $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.

18. Since $(1, 1) \in A$, $(2, 1)$ and $(2, 2)$ are in A. Since $(2, 1) \in A$, we get $(3, 1)$ and $(3, 2)$ in A and since $(2, 2) \in A$, $(3, 3) \in A$. Now $(3, 1) \in A \to (4, 1)$ and $(4, 2) \in A$; $(3, 2) \in A \to (4, 3) \in A$ and $(3, 3) \in A \to (4, 4) \in A$. The points shown so far which belong to A are plotted in the picture to the right and this makes it seem very plausible that A contains the set $\{(m, n) \in \mathsf{N} \times \mathsf{N} \mid m \geq n\}$.



19. (a) Let $x \in B$. Certainly x is also in A or in A^c. This suggests cases.
Case 1: If $x \in A$, then $x \in A \cap B$, so $x \in C$.
Case 2: If $x \notin A$, then $x \in A^c \cap B$, so $x \in C$.
In either case, $x \in C$, so $B \subseteq C$.

(b) [BB] Yes. Given $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, certainly we have $A \cap B \subseteq C$ and $A^c \cap B \subseteq C$ so, from (a), we have that $B \subseteq C$. Reversing the roles of B and C in (a), we can also conclude that $C \subseteq B$; hence, $B = C$.

20. (a) The Venn diagram shown in Fig. 2.1 suggests the following counterexample: Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{2, 3, 5, 7\}$. Then $A \cup (B \cap C) = A \cup \{3, 5\} = \{1, 2, 3, 4, 5\}$ whereas $(A \cup B) \cap C = \{1, 2, 3, 4, 5, 6\} \cap C = \{2, 3, 5\}$.

(b) First, we prove $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
So let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$ so $x \in A \cup B$ and $x \in A \cup C$; that is, $x \in (A \cup B) \cap (A \cup C)$. In either case, $x \in (A \cup B) \cap (A \cup C)$ giving the desired inclusion.
Second, we prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
So let $x \in (A \cup B) \cap (A \cup C)$. Thus, $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then we must have $x \in B$ and $x \in C$; that is, $x \in B \cap C$, so $x \in A \cup (B \cap C)$. In either case, $x \in A \cup (B \cap C)$ giving the desired inclusion and equality.

21. We use the fact that $(X^c)^c = X$ for any set X.
Let $X = A^c$ and $Y = B^c$. Then $A = X^c$ and $B = Y^c$, so $(A \cap B)^c = [X^c \cap Y^c]^c = [(X \cup Y)^c]^c$ (by the first law of De Morgan) $= X \cup Y = A^c \cup B^c$, as required.

22. [BB] Using the fact that $X \setminus Y = X \cap Y^c$, we have

$$(A \setminus B) \setminus C = (A \cap B^c) \cap C^c = A \cap (B^c \cap C^c) = A \cap (B \cup C)^c = A \setminus (B \cup C).$$

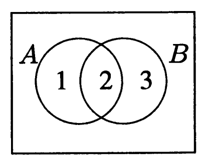
23. We use the laws of De Morgan and the facts that $(X^c)^c = X$ and $X \cap X^c = \emptyset$ for any set X. We have $[(A \cup B)^c \cap (A^c \cup C)^c]^c \setminus D^c = [(A^c \cap B^c) \cap (A \cap C^c)]^c \setminus D^c = \emptyset^c \setminus D^c = U \setminus D^c = U \cap (D^c)^c = U \cap D = D$.

24. $A \setminus (B \setminus C) = A \setminus (B \cap C^c) = A \cap (B \cap C^c)^c = A \cap (B^c \cup C) = (A \cap B^c) \cup (A \cap C) = (A \cap B^c) \cup (A \cap (C^c)^c) = (A \setminus B) \cup (A \setminus C^c)$.

25. (a) [BB] $(A \cup B \cup C)^c = [A \cup (B \cup C)]^c = A^c \cap (B \cup C)^c = A^c \cap (B^c \cap C^c) = A^c \cap B^c \cap C^c$.
$(A \cap B \cap C)^c = [A \cap (B \cap C)]^c = A^c \cup (B \cap C)^c = A^c \cup (B^c \cup C^c) = A^c \cup B^c \cup C^c$.

(b)
$$\begin{aligned}(A \cap (B \setminus C))^c \cap A &= (A \cap B \cap C^c)^c \cap A = (A^c \cup B^c \cup C) \cap A\\ &= (A \cap (A^c \cup B^c)) \cup (A \cap C) = (A \cap A^c) \cup (A \cap B^c) \cup (A \cap C)\\ &= \emptyset \cup (A \cap B^c) \cup (A \cap C) = (A \cap B^c) \cup (A \cap C)\\ &= (A \setminus B) \cup (A \cap C)\end{aligned}$$

26. (a) [BB] Looking at the Venn diagram at the right, $A \oplus B$ consists of the points in regions 1 and 3. To have $A \oplus B = A$, we must have both regions 2 and 3 empty; that is, $B = \emptyset$. On the other hand, since $A \oplus \emptyset = A$, this condition is necessary and sufficient.



(b) Looking at the Venn diagram, $A \cap B$ is the set of points in region 2 while $A \cup B$ is the set of points in regions 1, 2 and 3. Hence, $A \cap B = A \cup B$ if and only if regions 1 and 3 are both empty; that is, if and only if $A = B$.

27. (a) [BB] This does not imply $B = C$. For example, let $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$. Then $A \cup B = A \cup C$, but $B \neq C$.

(b) This does not imply $B = C$. For example, let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 3\}$. Then $A \cap B = A \cap C = A$, but $B \neq C$.

(c) This does imply $B = C$, and here is a proof. First let $b \in B$. Then, in addition, either $b \in A$ or $b \notin A$.

Case 1: $b \notin A$

In this case, $b \in A \oplus B$, so $b \in A \oplus C$ and since $b \notin A$ it follows that $b \in C$.

Case 2: $b \in A$. Here we have $b \in B \cap A$ and, hence, $b \notin A \oplus B$, so $b \notin A \oplus C$. Since $b \in A$, we must have $b \in C$ (otherwise, $b \in A \setminus C \subseteq A \oplus C$).

In either case, we obtain $b \in C$. It follows that $B \subseteq C$. A similar argument shows $C \subseteq B$ and, hence, $C = B$.

(d) This is false since for $A = \emptyset$, $A \times B = A \times C = \emptyset$ regardless of B and C.

28. (a) True. Let $(a, b) \in A \times B$. Since $a \in A$ and $A \subseteq C$, we have $a \in C$. Since $b \in B$ and $B \subseteq D$, $b \in D$. Thus, $(a, b) \in C \times D$ and $A \times B \subseteq C \times D$.

(b) False: Consider $A = \{1\}$, $B = \{2, 3\}$, $C = \{1, 2, 3\}$.

(c) False. Let $A = \{1\}$, $B = \emptyset$, $C = \{2\}$, $D = \{3\}$. Then $A \times B = \emptyset \subseteq \{(2, 3)\} = C \times D$, but $A \not\subseteq C$.

(d) False since, by (b), the implication $\leftarrow$ is false.

(e) [BB] True. Let $x \in A$. Then $x \in A \cup B$, so $x \in A \cap B$ and, in particular, $x \in B$. Thus, $A \subseteq B$. Similarly, we have $B \subseteq A$, so $A = B$.

29. Let $(x, y) \in (A \cap B) \times C$. This means $x \in A \cap B$ and $y \in C$. Hence, $x \in A$, $x \in B$, $y \in C$. Thus, $(x, y) \in A \times C$ and $(x, y) \in B \times C$; i.e., $(x, y) \in (A \times C) \cap (B \times C)$. Therefore, $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.

Now let $(x, y) \in (A \times C) \cap (B \times C)$. This means that $(x, y) \in A \times C$ and $(x, y) \in B \times C$); that is, $x \in A$, $x \in B$, $y \in C$, so $x \in A \cap B$ and $y \in C$. Hence, $(x, y) \in (A \cap B) \times C$. Therefore, $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$ and we have equality, as desired.

30. (a) False. For example, let $A = \{1, 2\}$, $B = \{1\}$ and $C = \{2\}$. Then $A \setminus (B \cup C) = \{1, 2\} \setminus \{1, 2\} = \emptyset$, but $(A \setminus B) \cup (A \setminus C) = \{2\} \cup \{1\} = \{1, 2\}$.

(b) True. Let $(x, y) \in (A \setminus B) \times C$. This means that $x \in A \setminus B$ and $y \in C$; that is, $x \in A$, $x \notin B$, $y \in C$. Hence, $(x, y) \in A \times C$, but $(x, y) \notin B \times C$, so $(x, y) \in (A \times C) \setminus (B \times C)$. Therefore, $(A \setminus B) \times C \subseteq (A \times C) \setminus (B \times C)$.

Now let $(x, y) \in (A \times C) \setminus (B \times C)$. This means that $(x, y) \in A \times C$, but $(x, y) \notin B \times C$. Since $(x, y) \in A \times C$, we have $x \in A$, $y \in C$. Since $y \in C$ and $(x, y) \notin B \times C$, we must have $x \notin B$; that is, $x \in A \setminus B$, $y \in C$, so $(x, y) \in (A \setminus B) \times C$. Therefore, $(A \times C) \setminus (B \times C) \subseteq (A \setminus B) \times C$ and we have equality as claimed.

(c) [BB] True. Let $(x, y) \in (A \oplus B) \times C$. This means that $x \in A \oplus B$ and $y \in C$; that is, $x \in A \cup B$, $x \notin A \cap B$, $y \in C$. If $x \in A$, then $x \notin B$, so $(x, y) \in (A \times C) \setminus (B \times C)$. If $x \in B$, then $x \notin A$, so $(x, y) \in (B \times C) \setminus (A \times C)$. In either case, $(x, y) \in (A \times C) \oplus (B \times C)$. So $(A \oplus B) \times C \subseteq (A \times C) \oplus (B \times C)$.

Now, let $(x, y) \in (A \times C) \oplus (B \times C)$. This means that $(x, y) \in (A \times C) \cup (B \times C)$, but $(x, y) \notin (A \times C) \cap (B \times C)$. If $(x, y) \in A \times C$, then $(x, y) \notin B \times C$, so $x \in A$, $y \in C$ and, therefore, $x \notin B$. If $(x, y) \in B \times C$, then $(x, y) \notin A \times C$, so $x \in B$, $y \in C$ and, therefore, $x \notin A$. In either case, $x \in A \oplus B$ and $y \in C$, so $(x, y) \in (A \oplus B) \times C$. Therefore, $(A \times C) \oplus (B \times C) \subseteq (A \oplus B) \times C$ and we have equality, as claimed.

(d) False. Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$. Then $1 \in A \cup B$, $4 \in C \cup D$, so $(1, 4) \in (A \cup B) \times (C \cup D)$. But $(1, 4) \notin A \times C$ and $(1, 4) \notin B \times D$, so $(1, 4) \notin (A \times C) \cup (B \times D)$.

(e) False. Let $A = \{1, 2\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$. Then, since $3 \notin D$, $(2, 3) \in (A \times C) \setminus (B \times D)$. However, because $2 \in B$, $2 \notin A \setminus B$, so $(2, 3) \notin (A \setminus B) \times (C \setminus D)$.

31. George Boole (1815-1864) was one of the greatest mathematicians of the nineteenth century. He was the first Professor of Mathematics at University College Cork (then called Queen's College) and is best known today as the inventor of a subject called *mathematical logic*. Indeed he introduced much of the symbolic language and notation we use today. Like Charles Babbage and Alan Turing, Boole also had a great impact in computer science, long before the computer was even a dream. He invented an *algebra of logic* known as Boolean Algebra, which is used widely today and forms the basis of much of the internal logic of computers. His books, "The Mathematical Analysis of Logic" and "An Investigation of the Laws of Thought" form the basis of present-day computer science.

Exercises 2.3

1. [BB] $S \times B$ is the set of ordered pairs (s, b), where s is a student and b is a book; thus, $S \times B$ represents all possible pairs of students and books. One sensible example of a binary relation is $\{(s, b) \mid s$ has used book $b\}$.

2. $A \times B$ is the set of all ordered pairs (a, b) where a is a street and b is a person. One binary relation would be $\{(a, b) \mid b$ lives on street $a\}$.

3. (a) [BB] not reflexive, not symmetric, not transitive.

   (b) (in most cases) reflexive, (in somewhat fewer cases) symmetric, certainly not transitive!

   (c) [BB] not reflexive, not symmetric, but it is transitive.

   (d) reflexive, symmetric, transitive.

   (e) not reflexive, not symmetric, not transitive

4. (a) [BB]

	a	b	c	d
a	×	×		
b	×	×		
c			×	
d				×

(b)

	a	b	c	d
a	×	×	×	
b		×		
c	×		×	
d				

(c)

	a	b	c	d
a	×	×	×	
b				
c			×	×
d				×

(d)

	a	b	c	d
a	×	×	×	×
b		×	×	×
c			×	
d			×	×

5. (a) [BB] $\{(1,1),(1,2),(2,3)\}$; (b) $\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$;
   (c) $\{(1,2),(2,3),(2,1),(3,2)\}$; (d) $\{(1,2),(1,3),(2,3)\}$;
   (e) $\{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$;
   (f) $\{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$; (g) [BB] $\{(1,2),(2,1),(1,1),(2,2)\}$;
   (h) $\{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3),(2,1),(3,2),(3,1)\}$.

6. The answer is yes and the only such binary relations are subsets of the equality binary relation. To see why, let $\mathcal{R}$ be a binary relation on a set A which is both symmetric and antisymmetric. Let $(a, b) \in \mathcal{R}$. Then $(b, a) \in \mathcal{R}$ by symmetry, so $a = b$ by antisymmetry.

7. [BB] The argument assumes that for $a \in \mathcal{R}$ there exists a b such that $(a,b) \in \mathcal{R}$. This need not be the case: See Exercise 5(g).

8. (a) [BB] **Reflexive**: Every word has at least one letter in common with itself.
   Symmetric: If a and b have at least one letter in common, then so do b and a.
   Not antisymmetric: (cat, dot) and (dot, cat) are both in the relation but dot $\neq$ cat!!
   Not transitive: (cat, dot) and (dot, mouse) are both in the relation but (cat, mouse) is not.

   (b) **Reflexive**: Let a be a person. If a is not enrolled at Miskatonic University, then $(a,a) \in \mathcal{R}$. On the other hand, if a is enrolled at MU, then a is taking at least one course with himself, so again $(a,a) \in \mathcal{R}$.
   Symmetric: If $(a,b) \in \mathcal{R}$, then either it is the case that neither a nor b is enrolled at MU (so neither is b or a, hence, $(b,a) \in \mathcal{R}$) or it is the case that a and b are both enrolled and are taking at least one course together (in which case b and a are enrolled and taking a common course, so $(b,a) \in \mathcal{R}$). In any case, if $(a,b) \in \mathcal{R}$, then $(b,a) \in \mathcal{R}$.
   Not antisymmetric: If a and b are two different students in the same class at Miskatonic University, then $(a,b) \in \mathcal{R}$ and $(b,a) \in \mathcal{R}$, but $a \neq b$.
   At most universities, this is not a transitive relation. Let a, b and c be three students enrolled at MU such that a and b are enrolled in some course together and b and c are enrolled in some (other) course together, but a and c are taking no courses together. Then (a,b) and (b,c) are in $\mathcal{R}$ but $(a,c) \notin \mathcal{R}$.

9. (a) **Not reflexive**: $(1,1) \notin \mathcal{R}$.
   Not symmetric: $(1,2) \in \mathcal{R}$ but $(2,1) \notin \mathcal{R}$.
   Antisymmetric: It is never the case that for two different elements a and b in A we have both (a,b) and (b,a) in $\mathcal{R}$.
   Transitive vacuously; that is, there exists no counterexample to disprove transitivity: The situation $(a,b) \in \mathcal{R}$ and $(b,c) \in \mathcal{R}$ never occurs.

   (b) [BB] **Not reflexive**: $(2,2) \notin \mathcal{R}$.
   Not symmetric: $(3,4) \in \mathcal{R}$ but $(4,3) \notin \mathcal{R}$.
   Not antisymmetric: $(1,2)$ and $(2,1)$ are both in $\mathcal{R}$ but $1 \neq 2$.
   Not transitive: $(2,1)$ and $(1,2)$ are in $\mathcal{R}$ but $(2,2)$ is not.

   (c) [BB] **Reflexive**: For any $a \in \mathsf{Z}$, it is true that $a^2 \geq 0$. Thus, $(a,a) \in \mathcal{R}$.
   Symmetric: If $(a,b) \in \mathcal{R}$, then $ab \geq 0$, so $ba \geq 0$ and hence, $(b,a) \in \mathcal{R}$.
   Not antisymmetric: $(5,2) \in \mathcal{R}$ because $5(2) = 10 \geq 0$ and similarly $(2,5) \in \mathcal{R}$, but $5 \neq 2$.
   Not transitive: $(5,0) \in \mathcal{R}$ because $5(0) = 0 \geq 0$ and similarly, $(0,-6) \in \mathcal{R}$; however, $(5,-6) \notin \mathcal{R}$ because $5(-6) \not\geq 0$.

   (d) **Reflexive**: For any $a \in \mathsf{R}$, $a^2 = a^2$, so $(a,a) \in \mathcal{R}$.
   Symmetric: If $(a,b) \in \mathcal{R}$ then $a^2 = b^2$, so $b^2 = a^2$ which says that $(b,a) \in \mathcal{R}$.
   Not antisymmetric: $(1,-1) \in \mathcal{R}$ and $(-1,1) \in \mathcal{R}$ but $1 \neq -1$.
   Transitive: If (a,b) and (b,c) are both in $\mathcal{R}$, then $a^2 = b^2$ and $b^2 = c^2$, so $a^2 = c^2$ which says $(a,c) \in \mathcal{R}$.

   (e) **Reflexive**: For any $a \in \mathsf{R}$, $a - a = 0 \leq 3$ and so $(a,a) \in \mathcal{R}$.
   Not symmetric: For example, $(0,7) \in \mathcal{R}$ because $0 - 7 = -7 \leq 3$, but $(7,0) \notin \mathcal{R}$ because $7 - 0 = 7 \not\leq 3$.

Not antisymmetric: $(2,1) \in \mathcal{R}$ because $2-1=1 \leq 3$ and $(1,2) \in \mathcal{R}$ because $1-2=-1 \leq 3$, but $1 \neq 2$.

Not transitive: $(5,3) \in \mathcal{R}$ because $5-3=2 \leq 3$ and $(3,1) \in \mathcal{R}$ because $3-1=2 \leq 3$, but $(5,1) \notin \mathcal{R}$ because $5-1=4 \not\leq 3$.

(f) **Reflexive**: For any $(a,b) \in A$, $a-a=b-b$; thus, $((a,b),(a,b)) \in \mathcal{R}$.

Symmetric: If $((a,b),(c,d)) \in \mathcal{R}$, then $a-c=b-d$, so $c-a=d-b$ and, hence, $((c,d),(a,b)) \in \mathcal{R}$.

Not antisymmetric: $((5,2),(15,12)) \in \mathcal{R}$ because $5-15=2-12$ and similarly, $((15,12),(5,2)) \in \mathcal{R}$; however, $(15,12) \neq (5,2)$.

If $((a,b),(c,d)) \in \mathcal{R}$ and $((c,d),(e,f)) \in \mathcal{R}$ then $a-c=b-d$ and $c-e=d-f$. Thus, $a-e=(a-c)+(c-e)=(b-d)+(d-f)=b-f$ and so $((a,b),(e,f)) \in \mathcal{R}$.

(g) **Not reflexive**: If $n \in \mathsf{N}$, then $n \neq n$ is not true.

Symmetric: If $n_1 \neq n_2$, then $n_2 \neq n_1$.

Not antisymmetric: $1 \neq 2$ and $2 \neq 1$ so both $(1,2)$ and $(2,1)$ are in $\mathcal{R}$, yet $1 \neq 2$.

Not transitive: $1 \neq 2$, $2 \neq 1$, but $1 = 1$.

(h) **Not reflexive**: $(2,2) \notin \mathcal{R}$ because $2+2 \neq 10$.

Symmetric: If $(x,y) \in \mathcal{R}$, then $x+y=10$, so $y+x=10$, and hence, $(y,x) \in \mathcal{R}$.

Not antisymmetric: $(6,4) \in \mathcal{R}$ because $6+4=10$ and similarly, $(4,6) \in \mathcal{R}$, but $6 \neq 4$.

Not transitive: $(6,4) \in \mathcal{R}$ because $6+4=10$ and similarly, $(4,6) \in \mathcal{R}$, but $(6,6) \notin \mathcal{R}$ because $6+6 \neq 10$.

(i) [BB] **Reflexive**: If $(x,y) \in \mathsf{R}^2$, then $x+y \leq x+y$, so $((x,y),(x,y)) \in \mathcal{R}$.

Not symmetric: $((1,2),(3,4)) \in \mathcal{R}$ since $1+2 \leq 3+4$, but

$((3,4),(1,2)) \notin \mathcal{R}$ since $3+4 \not\leq 1+2$.

Not antisymmetric: $((1,2),(0,3)) \in \mathcal{R}$ since $1+2 \leq 0+3$

and $((0,3),(1,2)) \in \mathcal{R}$ since $0+3 \leq 1+2$, but $(1,2) \neq (0,3)$.

Transitive: If $((a,b),(c,d))$ and $((c,d),(e,f))$ are both in $\mathcal{R}$, then $a+b \leq c+d$ and $c+d \leq e+f$, so $a+b \leq e+f$ (by transitivity of $\leq$) which says $((a,b),(e,f)) \in \mathcal{R}$.

(j) **Reflexive**: $\frac{a}{a} = 1 \in \mathsf{N}$ for any $a \in \mathsf{N}$.

Not symmetric: $(4,2) \in \mathcal{R}$ but $(2,4) \notin \mathcal{R}$.

Antisymmetric: If $\frac{a}{b} = n$ and $\frac{b}{a} = m$ are integers then $nm = 1$ so $n, m \in \{\pm 1\}$. Since a and b are positive, so are n and m. Therefore, $n = m = 1$ and $a = b$.

Transitive: The argument given in Example 24 for Z works the same way for N.

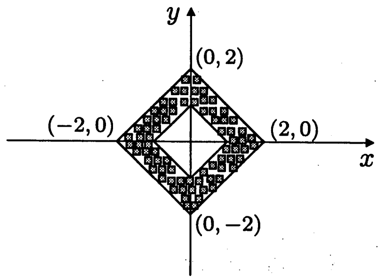
(k) **Not reflexive**: $\frac{0}{0}$ is not defined, let alone an integer!

Not symmetric: As before.

Not antisymmetric: $(4,-4)$ and $(-4,4)$ are both in $\mathcal{R}$.

Transitive: As shown in Example 24.

10. (a)



(b) The relation is not reflexive because, for example, $(2,2) \notin \mathcal{R}$. It is not transitive because, for example, $(2,0) \in \mathcal{R}$ and $(0,1) \in \mathcal{R}$ but $(2,1) \notin \mathcal{R}$.

(c) The relation is symmetric since if $(x,y) \in \mathcal{R}$, then $1 \le |x| + |y| \le 2$, so $1 \le |y| + |x| \le 2$, so $(y,x) \in \mathcal{R}$. It is not antisymmetric since, for example, $(0,1) \in \mathcal{R}$ and $(1,0) \in \mathcal{R}$, but $0 \ne 1$.

11. (a) [BB] **Reflexive**: For any set X, we have $X \subseteq X$.
Not symmetric: Let $a, b \in S$. Then $\{a\} \subseteq \{a,b\}$ but $\{a,b\} \not\subseteq \{a\}$.
Antisymmetric: If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.
Transitive: If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(b) **Not reflexive**: For no set X is it true that $X \subsetneq X$.
Not symmetric: As before.
Antisymmetric "vacuously": It is impossible for $X \subsetneq Y$ and $Y \subsetneq X$. (Recall that an implication is false only when the hypothesis is true and the conclusion is false.)
Transitive: As before.

(c) **Not reflexive**: Since $S \ne \emptyset$, there is some element $a \in S$, and so some set $X = \{a\} \ne \emptyset \in \mathcal{P}(S)$. For this X, however, $X \cap X = X \ne \emptyset$, so $(X,X) \notin \mathcal{R}$.
Symmetric: If $(X,Y) \in \mathcal{R}$, then $X \cap Y = \emptyset$, so $Y \cap X = \emptyset$, hence, $(Y,X) \in \mathcal{R}$.
Not antisymmetric: Let a, b be two elements in S and let $X = \{a\}$, $Y = \{b\}$. Then $(X,Y) \in \mathcal{R}$ and $(Y,X) \in \mathcal{R}$, but $X \ne Y$.
Not transitive: Let a, b be two elements in S and let $X = \{a\}$, $Y = \{b\}$, $Z = \{a\}$. Then $(X,Y) \in \mathcal{R}$, $(Y,Z) \in \mathcal{R}$, but $(X,Z) \notin \mathcal{R}$.

12. (a) [BB] **Reflexive**: Any book has price $\ge$ its own price and length $\ge$ its own length, so $(a,a) \in \mathcal{R}$ for any book a.
Not symmetric: $(Y,Z) \in \mathcal{R}$ because the price of Y is greater than the price of Z and the length of Y is greater than the length of Z, but for these same reasons, $(Z,Y) \notin \mathcal{R}$.
Antisymmetric: If (a,b) and (b,a) are both in $\mathcal{R}$, then a and b must have the same price and length. This is not the case here unless $a = b$.
Transitive: If (a,b) and (b,c) are in $\mathcal{R}$, then the price of a is $\ge$ the price of b and the price of b is $\ge$ the price of c, so the price of a is $\ge$ the price of c. Also the length of a is $\ge$ the length of b and the length of b is $\ge$ the length of c, so the length of a is $\ge$ the length of c. Hence, $(a,c) \in \mathcal{R}$.

(b) **Reflexive**: For any book a, the price of a is $\ge$ the price of a so $(a,a) \in \mathcal{R}$. (One could also use a similar argument concerning length.)
Not symmetric: As in part (a), $(Y,Z) \in \mathcal{R}$, but $(Z,Y) \notin \mathcal{R}$.
Not antisymmetric: $(W,X) \in \mathcal{R}$ because the price of W is greater than or equal to the price of X, and $(X,W) \in \mathcal{R}$ because the length of W is greater than or equal to the length of X, but $W \ne X$.
Not transitive: $(Z,U) \in \mathcal{R}$ because the length of Z is $\ge$ the length of U and $(U,Y) \in \mathcal{R}$ because the price of U is $\ge$ the price of Y, but $(Z,Y) \notin \mathcal{R}$ because neither is the price of $Z \ge$ the price of Y nor is the length of $Z \ge$ the length of Y.

13. Now the second binary relation would have an extra term, {Mike, 120}, and the third would have the extra term, {Pippy Park, 120}. But, in addition, the entry {Pippy Park, 74} would be deleted. So Mike is now clearly identified as the one who shot 120, and Pippy Park is where that occurred. Hence, Mike's round of 74 was at Clovelly. Since Edgar has only one entry in binary relation two, he must have shot 72 at both courses. Finally, Bruce's 74 must have been at Clovelly and hence his 72 was at Pippy Park. All information has been retrieved in this case.

Exercises 2.4

1. **Reflexive:** For any citizen a of New York City, either a does not own a cell phone (in which case $a \sim a$) or a has a cell phone and a's exchange is the same as a's exchange (in which case again $a \sim a$).

   Symmetric: If $a \sim b$ and a does not have a cell phone, then neither does b, so $b \sim a$; on the other hand, if a does have a cell phone, then so does b and their exchanges are the same, so again, $b \sim a$.

   Transitive: Suppose $a \sim b$ and $b \sim c$. If a does not have a cell phone, then neither does b and, since $b \sim c$, neither does c, so $a \sim c$. On the other hand, if a does have a cell phone then so does b and a's and b's exchanges are the same. Since $b \sim c$, c has a cell phone with the same exchange as b. It follows that a and c have the same exchange and so, in this case as well, $a \sim c$.

   There is one equivalence class consisting of all residents of New York who do not own a cell phone and one equivalence class for each New York City exchange consisting of all residents who have cell phones in that exchange.

2. (a) [BB] This is not reflexive: $(2,2) \notin \mathcal{R}$.

   (b) This is not symmetric: $(2,3) \in \mathcal{R}$ but $(3,2) \notin \mathcal{R}$.
   It would also be acceptable to not that $\mathcal{R}$ is not transitive. $(3,1) \in \mathcal{R}$ and $(1,2) \in \mathcal{R}$, but $(3,2) \notin \mathcal{R}$.

   (c) This is not symmetric: $(1,3)$ is in the relation but $(3,1)$ is not.
   It would also be acceptable to note that this relation is not transitive: $(2,1) \in \mathcal{R}$, $(1,3) \in \mathcal{R}$, but $(2,3) \notin \mathcal{R}$.

3. [BB] Equality! The equivalence classes specify that $x \sim y$ if and only if $x = y$.

4. (a) **Reflexive**: If $a \in S$, then a and a have the same number of elements, so $a \sim a$.
   Symmetric: If $a \sim b$, then a and b have the same number of elements, so b and a have the same number of elements. Thus $b \sim a$.
   Transitive: If $a \sim b$ and $b \sim c$, then a and b have the same number of elements, and b and c have the same number of elements, so a and c have the same number of elements. Thus $a \sim c$.

   (b) There are **seven** equivalence classes, represented by $\emptyset$, $\{1\}$, $\{1,2\}$, $\{1,2,3\}$, $\{1,2,3,4\}$, $\{1,2,3,4,5\}$, $\{1,2,3,4,5,6\}$.

5. (a) [BB] **Reflexive:** If $a \in \mathsf{R} \setminus \{0\}$, then $a \sim a$ because $\frac{a}{a} = 1 \in \mathsf{Q}$.
   Symmetric: If $a \sim b$, then $\frac{a}{b} \in \mathsf{Q}$ and this fraction is not zero (because $0 \notin A$). So it can be inverted and we see that $\frac{b}{a} = 1/\frac{a}{b} \in \mathsf{Q}$ too. Therefore, $b \sim a$.
   Transitive: If $a \sim b$ and $b \sim c$, then $\frac{a}{b} \in \mathsf{Q}$ and $\frac{b}{c} \in \mathsf{Q}$. Since the product of rational numbers is rational, $\frac{a}{c} = \frac{a}{b}\frac{b}{c}$ is in Q, so $a \sim c$.

(b) [BB] $\overline{1} = \{a \mid a \sim 1\} = \{a \mid \frac{a}{1} \in \mathsf{Q}\} = \{a \mid a \in \mathsf{Q}\} = \mathsf{Q} \setminus \{0\}$.

(c) [BB] $\frac{\sqrt{12}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}} = 2 \in \mathsf{Q}$, so $\sqrt{3} \sim \sqrt{12}$ and hence $\overline{\sqrt{3}} = \overline{\sqrt{12}}$.

6. **Reflexive:** For any $a \in \mathsf{N}$, $a \sim a$ since $a^2 + a = a(a+1)$ is even, as the product of consecutive natural numbers.

Symmetric: If $a \sim b$, then $a^2 + b$ is even. It follows that either a and b are both even or both are odd. If they are both even, $b^2 + a$ is the sum of even numbers, hence, even. If they are both odd, $b^2 + a$ is the sum of odd numbers and, hence, again, even. In both cases $b^2 + a$ is even, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $a^2 + b$ and $b^2 + c$ are even, so $(a^2 + b) + (b^2 + c)$ is even; in other words, $(a^2 + c) + (b^2 + b)$ is even. Since $b^2 + b$ is even, $a^2 + c$ is even too; therefore, $a \sim c$.

The quotient set is the set of equivalence classes. Now

$$\overline{a} = \{x \mid x^2 + a \text{ is even}\} = \begin{cases} \text{evens} & \text{if } a \text{ is even} \\ \text{odds} & \text{if } a \text{ is odd} \end{cases}$$

So $A/{\sim} = \{2\mathsf{Z}, 2\mathsf{Z}+1\}$.

7. (a) [BB] **Reflexive:** For any $a \in \mathsf{R}$, $a \sim a$ because $a - a = 0 \in \mathsf{Z}$.
Symmetric: If $a \sim b$, then $a - b \in \mathsf{Z}$, so $b - a \in \mathsf{Z}$ (because $b - a = -(a-b)$) and, hence, $b \sim a$.
Transitive: If $a \sim b$ and $b \sim c$, then both $a - b$ and $b - c$ are integers; hence, so is their sum, $(a-b) + (b-c) = a - c$. Thus, $a \sim c$.

(b) [BB] The equivalence class of 5 is $\overline{5} = \{x \in \mathsf{R} \mid x \sim 5\} = \{x \mid x - 5 \in \mathsf{Z}\} = \mathsf{Z}$, because $x - 5 \in \mathsf{Z}$ implies $x \in \mathsf{Z}$.

$$\begin{aligned} \overline{5\tfrac{1}{2}} &= \{x \in \mathsf{R} \mid x \sim 5\tfrac{1}{2}\} \\ &= \{x \mid x - 5\tfrac{1}{2} \in \mathsf{Z}\} \\ &= \{x \mid x = 5\tfrac{1}{2} + k, \text{ for some } k \in \mathsf{Z}\} \\ &= \{x \mid x = 5 + k + \tfrac{1}{2}, \text{ for some } k \in \mathsf{Z}\} \\ &= \{x \mid x = n + \tfrac{1}{2}, \text{ for some } n \in \mathsf{Z}\} \end{aligned}$$

(c) [BB] For each $a \in \mathsf{R}$, $0 \le a < 1$, there is one equivalence class,

$$\overline{a} = \{x \in \mathsf{R} \mid x = a + n \text{ for some integer } n\}.$$

The quotient set is $\{\overline{a} \mid 0 \le a < 1\}$.

8. [BB] **Reflexive:** For any $a \in \mathsf{Z}$, $a \sim a$ because $2a + 3a = 5a$.

Symmetric: If $a \sim b$, then $2a + 3b = 5n$ for some integer n. So $2b + 3a = (5a + 5b) - (2a + 3b) = 5(a+b) - 5n = 5(a + b - n)$. Since $a + b - n$ is an integer, $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $2a + 3b = 5n$ and $2b + 3c = 5m$ for integers n and m. Therefore, $(2a + 3b) + (2b + 3c) = 5(n + m)$ and $2a + 3c = 5(n+m) - 5b = 5(n + m - b)$. Since $n + m - b$ is an integer, $a \sim c$.

9. (a) **Reflexive:** For any $a \in \mathsf{Z}$, $3a + a = 4a$ is a multiple of 4, so $a \sim a$.
Symmetric: If $a \sim b$, then $3a + b = 4k$ for some integer k. Since $(3a + b) + (3b + a) = 4(a + b)$, we see that $3b + a = 4(a + b) - 4k$ is a multiple of 4, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $3a + b = 4k$ for some integer k and $3b + c = 4\ell$ for some integer ℓ. Since $4(k+\ell) = (3a+b)+(3b+c) = (3a+c)+4b$, we see that $3a+c = 4(k+\ell)-4b$ is a multiple of 4 and, hence, that $a \sim c$.

(b) $\overline{0} = \{x \in Z \mid x \sim 0\} = \{x \mid 3x = 4k \text{ for some integer } k\}$. Now if $3x = 4k$, k must be a multiple of 3. So $3x = 12\ell$ for some $\ell \in Z$ and $x = 4\ell$. $\overline{0} = 4Z$.

(c) $\overline{2} = \{x \in Z \mid x \sim 2\} = \{x \mid 3x + 2 = 4k \text{ for some integer } k\} = \{x \mid 3x = 4k - 2 \text{ for some integer } k\}$ Now if $3x = 4k - 2$, then $3x = 3k + k - 2$ and so $k - 2$ is a multiple of 3. Therefore, $k = 3\ell + 2$ for some integer ℓ, $3x = 4(3\ell + 2) - 2 = 12\ell + 6$ and $x = 4\ell + 2$. So $\overline{2} = 4Z + 2$.

(d) The quotient set is $\{4Z, 4Z + 1, 4Z + 2, 4Z + 3\}$.

10. (a) **Reflexive:** For any $a \in Z$, $a \sim a$ because $3a + 4a = 7a$ and a is an integer.

Symmetric: If $a, b \in Z$ and $a \sim b$, then $3a + 4b = 7n$ for some integer n. Then $3b + 4a = (7a + 7b) - (3a + 4b) = 7a + 7b - 7n = 7(a + b - n)$ and $a + b - n$ is an integer. Thus $b \sim a$.

Transitive: Suppose $a, b, c \in Z$ with $a \sim b$ and $b \sim c$. Then $3a + 4b = 7n$ and $3b + 4c = 7m$ for some integers n and m. Then $7n + 7m = (3a + 4b) + (3b + 4c) = (3a + 4c) + 7b$, so $3a + 4c = 7n + 7m - 7b = 7(n + m - b)$ and $n + m - b$ is an integer. Thus $a \sim c$.

(b) $\overline{0} = \{x \in Z \mid x \sim 0\} = \{x \in Z \mid 3x = 7n \text{ for some integer } n\}$. Now if $3x = 7n$, n must be a multiple of 3. So $3x = 21k$ for some $k \in Z$ and $x = 7k$. We conclude that $\overline{0} = 7Z$.

11. (a) [BB] **Reflexive:** If $a \in Z \setminus \{0\}$, then $aa = a^2 > 0$, so $a \sim a$.

Symmetric: If $a \sim b$, then $ab > 0$. So $ba > 0$ and $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $ab > 0$ and $bc > 0$. Also $b^2 > 0$ since $b \neq 0$. Hence,

$$ac = \frac{(ac)b^2}{b^2} = \frac{(ab)(bc)}{b^2} > 0$$

since $ab > 0$, $bc > 0$. Hence, $a \sim c$.

(b) [BB]
$$\begin{aligned}\overline{5} &= \{x \in Z \setminus \{0\} \mid x \sim 5\} = \{x \mid 5x > 0\} = \{x \mid x > 0\} \\ \overline{-5} &= \{x \in Z \setminus \{0\} \mid x \sim -5\} = \{x \mid -5x > 0\} = \{x \mid x < 0\}\end{aligned}$$

(c) [BB] This equivalence relation partitions $Z \setminus \{0\}$ into the positive and the negative integers.

12. (a) [BB] **Reflexive:** For any $a \in Z$, $a^2 - a^2 = 0$ is divisible by 3, so $a \sim a$.

Symmetric: If $a \sim b$, then $a^2 - b^2$ is divisible by 3, so $b^2 - a^2$ is divisible by 3, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $a^2 - b^2$ is divisible by 3 and $b^2 - c^2$ is divisible by 3, so $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2)$ is divisible by 3.

(b)
$$\begin{aligned}\overline{0} &= \{x \in Z \mid x \sim 0\} \\ &= \{x \in Z \mid x^2 \text{ is divisible by } 3\} \\ &= \{x \in Z \mid x \text{ is divisible by } 3\} = 3Z\end{aligned}$$

$$\begin{aligned}\overline{1} &= \{x \in Z \mid x \sim 1\} \\ &= \{x \mid x^2 - 1 \text{ is divisible by } 3\} \\ &= \{x \mid (x-1)(x+1) \text{ is divisible by } 3\} \\ &= \{x \mid x - 1 \text{ or } x + 1 \text{ is divisible by } 3\} \\ &= 3Z + 1 \cup 3Z + 2\end{aligned}$$

(c) This equivalence relation partitions the integers into the two disjoint sets $3Z$ and $(3Z+1) \cup (3Z+2)$.

13. (a) [BB] Yes, this is an equivalence relation.

    Reflexive: Note that if a is any triangle, $a \sim a$ because a is congruent to itself.

    Symmetric: Assume $a \sim b$. Then a and b are congruent. Therefore, b and a are congruent, so $b \sim a$.

    Transitive: If $a \sim b$ and $b \sim c$, then a and b are congruent and b and c are congruent, so a and c are congruent. Thus, $a \sim c$.

    (b) Yes, this is an equivalence relation.

    Reflexive: If a is a circle, then $a \sim a$ because a has the same center as itself.

    Symmetric: Assume $a \sim b$. Then a and b have the same center. Thus, b and a have the same center, so $b \sim a$.

    Transitive: Assume $a \sim b$ and $b \sim c$. Then a and b have the same center and b and c have the same center, so a and c have the same center. Thus, $a \sim c$.

    (c) Yes, this is an equivalence relation.

    Reflexive: If a is a line, then a is parallel to itself, so $a \sim a$.

    Symmetric: If $a \sim b$, then a is parallel to b. Thus, b is parallel to a. Hence, $b \sim a$.

    Transitive: If $a \sim b$ and $b \sim c$, then a is parallel to b and b is parallel to c, so a is parallel to c. Thus, $a \sim c$.

    (d) No, this is not an equivalence relation. The reflexive property does not hold because no line is perpendicular to itself. Neither is this relation transitive; if ℓ_1 is perpendicular to ℓ_2 and ℓ_2 is perpendicular to ℓ_3, then ℓ_1 and ℓ_3 are **parallel**, not perpendicular to one another.

14. (a) [BB] $\mathcal{R} = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,4),(4,5),(5,5),(4,3),(5,3),(5,4)\}$

    (b) $\mathcal{R} = \{((1,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5)\}$

    (c) $\mathcal{R} = \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$

15. (a) As suggested in the text, we list the partitions of $\{a\}$. There is only one; namely, $\{a\}$.

    (b) As suggested in the text, we list the partitions of $\{a,b\}$. There are two; namely, $\{a,b\}$ and $\{a\},\{b\}$.

    (c) [BB] As suggested in the text, a good way to list the equivalence relations on $\{a,b,c\}$ is to list the partitions of this set. Here they are:

    $\{\ \{a\},\{b\},\{c\}\ \}$;
    $\{\ \{a,b,c\}\ \}$;
    $\{\ \{a,b\},\{c\}\ \}$; $\{\ \{a,c\},\{b\}\ \}$; $\{\ \{b,c\},\{a\}\ \}$
    There are five in all.

    (d) As suggested in the text, we list the partitions of $\{a,b,c,d\}$.

    $\{\ \{a\},\{b\},\{c\},\{d\}\ \}$
    $\{\ \{a,b,c,d\}\ \}$
    $\{\ \{a,b\},\{c,d\}\ \}$; $\{\ \{a,c\},\{b,d\}\ \}$; $\{\ \{a,d\},\{b,c\}\ \}$
    $\{\ \{a,b\},\{c\},\{d\}\ \}$; $\{\ \{a,c\},\{b\},\{d\}\ \}$; $\{\ \{a,d\},\{b\},\{c\}\ \}$; $\{\ \{b,c\},\{a\},\{d\}\ \}$; $\{\ \{b,d\},\{a\},\{c\}\ \}$; $\{\ \{c,d\},\{a\},\{b\}\ \}$
    $\{\ \{a,b,c\},\{d\}\ \}$; $\{\ \{a,b,d\},\{c\}\ \}$; $\{\ \{a,c,d\},\{b\}\ \}$; $\{\ \{b,c,d\},\{a\}\ \}$
    There are 15 in all.

16. (a) [BB] The given statement is an implication which concludes "$x - y = x - y$," whereas what is required is a logical argument which concludes "so $\sim$ is reflexive."

    A correct argument is this: For any $(x, y) \in \mathsf{R}^2$, $x - y = x - y$; thus, $(x, y) \sim (x, y)$. Therefore, $\sim$ is reflexive.

    (b) There is confusion between the elements of a binary relation on a set A (which are ordered pairs) and the elements of A which are themselves ordered pairs in this situation. The given statement is correct *provided* each of x and y is understood to be an ordered pair of real numbers, and we understand $\mathcal{R} = \{(x, y) \mid x \sim y\}$ but this is very misleading. Much better is to state symmetry like this:

    if $(x, y) \sim (u, v)$, then $(u, v) \sim (x, y)$.

    (c) The first statement asserts the implication "$x - y = u - v \rightarrow (x, y) \sim (u, v)$" which is the converse of what should have been said. Here is the correct argument:

    If $(x, y) \sim (u, v)$, then $x - y = u - v$, so $u - v = x - y$ and, hence, $(u, v) \sim (x, y)$.

    (d) This suggested answer is utterly confusing. Logical arguments consist of a sequence of implications but here it is not clear where these implications start. Certainly the first sentence is not an implication.

    If $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$ then $x - y = u - v$ and $u - v = w - z$. So $x - y = w - z$ and, hence, $(x, y) \sim (w, z)$.

    (e) $\sim$ defines an equivalence relation on R^2 because it is a reflexive, symmetric and transitive binary relation on R^2.

    (f) The equivalence class of $(0, 0)$ is

    $$\{(x, y) \mid (x, y) \sim (0, 0)\} = \{(x, y) \mid x - y = 0 - 0\} = \{(x, y) \mid y = x\}$$

    which is a straight line of slope 1 in the Cartesian plane passing through the origin. The equivalence class of $(2, 3)$ is

    $$\{(x, y) \mid (x, y) \sim (2, 3)\} = \{(x, y) \mid x - y = 2 - 3 = -1\} = \{(x, y) \mid y = x + 1\}$$

    which is a straight line of slope 1 passing through the point $(2, 3)$.

17. [BB] **Reflexive:** If $(x, y) \in \mathsf{R}^2$, then $x^2 - y^2 = x^2 - y^2$, so $(x, y) \sim (x, y)$.

    Symmetric: If $(x, y) \sim (u, v)$, then $x^2 - y^2 = u^2 - v^2$, so $u^2 - v^2 = x^2 - y^2$ and $(u, v) \sim (x, y)$.

    Transitive: If $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$, then $x^2 - y^2 = u^2 - v^2$ and $u^2 - v^2 = w^2 - z^2$, so $x^2 - y^2 = u^2 - v^2 = w^2 - z^2$; $x^2 - y^2 = w^2 - z^2$ and $(x, y) \sim (w, z)$.

    $$\overline{(0,0)} = \{(x, y) \mid (x, y) \sim (0, 0)\} = \{(x, y) \mid x^2 - y^2 = 0^2 - 0^2 = 0\} = \{(x, y) \mid y = \pm x\}$$

    Thus, the equivalence class of $(0, 0)$ is the pair of lines with equations $y = x$, $y = -x$.

    $$\overline{(1,0)} = \{(x, y) \mid (x, y) \sim (1, 0)\} = \{(x, y) \mid x^2 - y^2 = 1^2 - 0^2 = 1\}.$$

    Thus, the equivalence class of $(1, 0)$ is the hyperbola whose equation is $x^2 - y^2 = 1$.

18. (a) This is an equivalence relation.

    Reflexive: If $(a, b) \in \mathsf{R}^2$, then $a + 2b = a + 2b$, so $(a, b) \sim (a, b)$.

    Symmetric: If $(a, b) \sim (c, d)$, then $a + 2b = c + 2d$, so $c + 2d = a + 2b$ and $(c, d) \sim (a, b)$.

Transitive: If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $a + 2b = c + 2d$ and $c + 2d = e + 2f$, so $a + 2b = e + 2f$ and $(a, b) \sim (e, f)$.

The quotient set is the set of equivalence classes. We have

$$\begin{aligned}\overline{(a,b)} &= \{(x, y) \mid (x, y) \sim (a, b)\} = \{(x, y) \mid x + 2y = a + 2b\} \\ &= \{(x, y) \mid y - b = -\tfrac{1}{2}(x - a)\}\end{aligned}$$

which describes the line through (a, b) with slope $-\frac{1}{2}$. The quotient set is the set of lines with slope $-\frac{1}{2}$.

(b) This is an equivalence relation.

Reflexive: If $(a, b) \in \mathsf{R}^2$, then $ab = ab$, so $(a, b) \sim (a, b)$.

Symmetric: If $(a, b) \sim (c, d)$, then $ab = cd$ so $cd = ab$ and $(c, d) \sim (a, b)$.

Transitive: If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $ab = cd$ and $cd = ef$, so $ab = cd = ef$, $ab = ef$ and $(a, b) \sim (e, f)$.

The quotient set is the set of equivalence classes. We have

$$\overline{(a,b)} = \{(x, y) \mid (x, y) \sim (a, b)\} = \{(x, y) \mid xy = ab\}$$

and consider two cases. If either $a = 0$ or $b = 0$, then $\overline{(a,b)} = \{(x, y) \mid xy = 0\}$; that is, $\{(x, y) \mid x = 0 \text{ or } y = 0\}$. Hence, $\overline{(a,b)}$ is the union of the x-axis and the y-axis. On the other hand, if $a \neq 0$ and $b \neq 0$, then

$$\overline{(a,b)} = \{(x, y) \mid xy = ab\} = \{(x, y) \mid y = \frac{ab}{x}\}$$

since $x \neq 0$ in this case. This time, $\overline{(a,b)}$ is the hyperbola whose equation is $y = ab/x$.

(c) This is not an equivalence relation. We have $(0, 2) \sim (1, 1)$ because $0^2 + 2 = 2 = 1 + 1^2$; however, $(1, 1) \not\sim (0, 2)$ because $1^2 + 1 = 2 \neq 4 = 0 + 2^2$. The relation is not symmetric.

(d) **Reflexive:** For any $(a, b) \in \mathsf{R}^2$, $a = a$, so $(a, b) \sim (a, b)$.

Symmetric: If $(a, b) \sim (c, d)$, then $a = c$, so $c = a$ and, hence, $(c, d) \sim (a, b)$.

Transitive: If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $a = c$ and $c = e$; hence, $a = e$ and so $(a, b) \sim (e, f)$.

Since the relation is reflexive, symmetric and transitive, it is an equivalence relation. The quotient set is the set of equivalence classes. The equivalence class of (a, b) is

$$\{(x, y) \in \mathsf{R}^2 \mid (x, y) \sim (a, b)\} = \{(x, y) \in \mathsf{R}^2 \mid x = a\}.$$

Geometrically, this set is the vertical line with equation $x = a$. The quotient set is the set of vertical lines.

(e) This is not an equivalence relation. For example, it is not reflexive: $(1, 2) \not\sim (1, 2)$ because $1(2) = 2 \neq 1 = 1^2$.

19. (a) "If $\overline{a} \cap \overline{b} = \emptyset$, then $\overline{a} \neq \overline{b}$."

(b) The converse is true. If $\overline{a} \cap \overline{b} = \emptyset$, then $a \in \overline{a}$ but $a \notin \overline{b}$, so $\overline{a} \neq \overline{b}$.

20. Remembering that $\overline{x}$ is just the set of elements equivalent to x, we are given that $a \sim b$, $c \sim d$ and $d \sim b$. By Proposition 2.4.3, $\overline{a} = \overline{b}$, $\overline{c} = \overline{d}$ and $\overline{d} = \overline{b}$. Thus $\overline{a} = \overline{b} = \overline{d} = \overline{c}$.

21. (a) [BB] The ordered pairs defined by $\sim$ are $(1,1)$, $(1,4)$, $(1,9)$, $(2,2)$, $(2,8)$, $(3,3)$, $(4,1)$, $(4,4)$, $(4,9)$, $(5,5)$, $(6,6)$, $(7,7)$, $(8,2)$, $(8,8)$, $(9,1)$, $(9,4)$, $(9,9)$.
    (b) [BB] $\overline{1} = \{1,4,9\} = \overline{4} = \overline{9}$; $\overline{2} = \{2,8\} = \overline{8}$; $\overline{3} = \{3\}$; $\overline{5} = \{5\}$; $\overline{6} = \{6\}$; $\overline{7} = \{7\}$.
    (c) [BB] Since the sets $\{1,4,9\}$, $\{2,8\}$, $\{3\}$, $\{5\}$, $\{6\}$ and $\{7\}$ partition A, they determine an equivalence relation, namely, that equivalence relation in which $a \sim b$ if and only if a and b belong to the same one of these sets. This is the given relation.

22. [BB] **Reflexive:** If $a \in A$, then a^2 is a perfect square, so $a \sim a$.

    Symmetric: If $a \sim b$, then ab is a perfect square. Since $ba = ab$, ba is also a perfect square, so $b \sim a$.

    Transitive: If $a \sim b$ and $b \sim c$, then ab and bc are each perfect squares. Thus $ab = x^2$ and $bc = y^2$ for integers x and y. Now $ab^2c = x^2y^2$, so $ac = \dfrac{x^2y^2}{b^2} = \left(\dfrac{xy}{b}\right)^2$. Because ac is an integer, so also $\dfrac{xy}{b}$ is an integer. Therefore, $a \sim c$.

23. (a) The order pairs of $\sim$ are $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, $(6,6)$, $(7,7)$, $(1,2)$, $(2,1)$, $(1,4)$, $(4,1)$, $(2,4)$, $(4,2)$, $(3,6)$, $(6,3)$.
    (b) $\overline{1} = \{1,2,4\} = \overline{2} = \overline{4}$; $\overline{3} = \{3,6\} = \overline{6}$; $\overline{5} = \{5\}$; $\overline{7} = \{7\}$.
    (c) The sets $\{1,2,4\}$, $\{3,6\}$, $\{5\}$, $\{7\}$ partition A, so the given relation is an equivalence relation.

24. **Reflexive:** If $a \in A$, then $\dfrac{a}{a} = 1 = 2^0$ is a power of 2, so $a \sim a$.

    Symmetric: If $a \sim b$, then $\dfrac{a}{b} = 2^t$, so $\dfrac{b}{a} = 2^{-t}$. Since $-t$ is an integer, $\dfrac{b}{a}$ is also a power of 2, so $b \sim a$.

    Transitive: If $a \sim b$ and $b \sim c$, then $\dfrac{a}{b} = 2^t$ and $\dfrac{b}{c} = 2^s$ for integers t and s. Thus $\dfrac{a}{c} = \dfrac{a}{b}\dfrac{b}{c} = 2^{t+s}$, showing that $a \sim c$.

25. We have to prove that the given sets are disjoint and have union S. For the latter, we note that since $\mathcal{R}$ is reflexive, for any $a \in S$, $(a,a) \in \mathcal{R}$ and so a and a are elements of the same set S_i; that is, $a \in S_i$ for some i. To prove that the sets are disjoint, suppose there is some $x \in S_k \cap S_\ell$. Since $S_k \not\subseteq \bigcup_{j \neq k} S_j$, there exists $y \in S_k$ such that $y \notin S_j$ for any $j \neq k$. Similarly, there exists $z \in S_\ell$ such that $z \notin S_j$ if $j \neq \ell$. Now if $y, x \in S_k$, then $(y,x) \in \mathcal{R}$ and $x, z \in S_\ell$ implies $(x,z) \in \mathcal{R}$. By transitivity, $(y,z) \in \mathcal{R}$, hence, y and z belong to the same set. But the only set to which y belongs is S_k. Since z does not belong to S_k, we have a contradiction: No $x \in S_k \cap S_\ell$ exists.

Exercises 2.5

1. (a) [BB] This defines a partial order.
    Reflexive: For any $a \in \mathsf{R}$, $a \geq a$.
    Antisymmetric: If $a, b \in \mathsf{R}$, $a \geq b$ and $b \geq a$, then $a = b$.
    Transitive: If $a, b, c \in \mathsf{R}$, $a \geq b$ and $b \geq c$, then $a \geq c$.
    This partial order is a total order because for any $a, b \in \mathsf{R}$, either $a \geq b$ or $b \geq a$.
    (b) [BB] This is not a partial order because the relation is not reflexive; for example, $1 < 1$ is not true.

(c) This is not a partial order because the relation is not antisymmetric; for example, $-3 \preceq 3$ because $(-3)^2 \leq 3^2$ and $3 \preceq -3$ because $3^2 \leq (-3)^2$ but $-3 \neq 3$.

(d) This is not a partial order because the relation is not antisymmetric; for example, $(1,4) \preceq (1,8)$ because $1 \leq 1$ and similarly, $(1,8) \preceq (1,4)$, but $(1,4) \neq (1,8)$.

(e) This is a partial order.

Reflexive: For any $(a,b) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (a,b)$ because $a \leq a$ and $b \geq b$.

Antisymmetric: If $(a,b),(c,d) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (c,d)$ and $(c,d) \preceq (a,b)$, then $a \leq c$, $b \geq d$, $c \leq a$ and $d \geq b$. So $a = c$, $b = d$ and, hence, $(a,b) = (c,d)$.

Transitive: If $(a,b),(c,d),(e,f) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (c,d)$ and $(c,d) \preceq (e,f)$, then $a \leq c$, $b \geq d$, $c \leq e$ and $d \geq f$. So $a \leq e$ (because $a \leq c \leq e$) and $b \geq f$ (because $b \geq d \geq f$) and, therefore, $(a,b) \preceq (e,f)$.

This is not a total order; for example, $(1,4)$ and $(2,5)$ are incomparable.

(f) This is reflexive and transitive but not antisymmetric and, hence, not a partial order. For example, *cat* $\preceq$ *dog* and *dog* $\preceq$ *cat* but *dog* $\neq$ *cat.*

2. (a) [BB] 1, 10, 100, 1000, 1001, 101, 1010, 11, 110, 111

   (b) 1, 11, 111, 110, 10, 101, 1010, 100, 1001, 1000

3. (a) [BB] $(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)$; (b) [BB] $(a,b),(c,d)$;
   (c) $(a,b),(a,d),(c,d)$; (d) $(a,b),(a,c),(a,d)$;
   (e) $(a,d),(a,e),(b,e),(b,c),(b,f),(c,f),(d,e)$;
   (f) $(a,f),(b,c),(d,b),(d,c),(d,h),(d,i),(e,c),(e,i),(g,f),(h,i)$.

4. (a) [BB] a is minimal and minimum; d is maximal and maximum.

   (b) [BB] a and c are minimal; b and d are maximal; there are no minimum nor maximum elements.

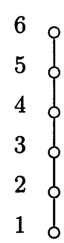
   (c) a and c are minimal; b and d are maximal; there are no minimum nor maximum elements.

   (d) a is minimal and minimum; b, c and d and maximal; there is no maximum element.

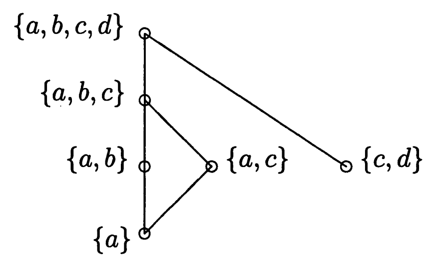
   (e) a and b are minimal; e and f and maximal; there are no minimum nor maximum elements.

   (f) a, d, e and g are minimal; c, f and i are maximal; there are no minimum nor maximum elements.

5. (a)



(b)



6. (a) 1 is minimal and minimum; 6 is maximal and maximum.

   (b) $\{a\}$ and $\{c,d\}$ are minimal; there is no minimum.
   The set $\{a,b,c,d\}$ is maximal and maximum.

7. [BB] $A \subsetneq B$ and the set B contains exactly one more element than A.

8. Helmut Hasse (1898–1979) was one of the more important mathematicians of the twentieth century. He grew up in Berlin and was a member of Germany's navy during the first World War. He received his PhD from the University of Göttingen in 1921 for a thesis in number theory, which was to be the subject of his life's work. He is known for his research with Richard Brauer and Emmy Noether on simple algebras, his proof of the Riemann Hypothesis (one of today's most famous open problems) for zeta functions on elliptic curves, and his work on the arithmetical properties of abelian number fields. Hasse's career started at Kiel and continued at Halle and Marburg. When the Nazis came to power in 1933, all Jewish mathematicians, including eighteen at the University of Göttingen, were summarily dismissed from their jobs. It is hard to know the degree of ambivalence Hasse may have had when he received an offer of employment at Göttingen around this time, but he accepted the position. While some of Hasse's closest research collaborators were Jewish, he nonetheless made no secret of his support for Hitler's policies. In 1945, he was dismissed by the British, lost his right to teach and eventually moved to Berlin. In May 1949, he was appointed professor at the Humboldt University in East Berlin but he moved to Hamburg the next year and worked there until his retirement in 1966.

9. (a) [BB] Let $(A, \preceq)$ be a finite poset and let $a \in A$. If a is not maximal, there is an element a_1 such that $a_1 \succ a$. If a_1 is not maximal, there is an element a_2 such that $a_2 \succ a_1$. Continue. Since A is finite, eventually this process must stop, and it stops at a maximal element. A similar argument shows that $(A, \preceq)$ must also contain minimal elements.

   (b) The result of (a) is not true in general. For example, $(\mathsf{R}, \leq)$ is a poset without maximal elements and without minimal elements.

10. (a) **Reflexive:** For any $\mathsf{a} = (a_1, a_2) \in A$, $\mathsf{a} \leq \mathsf{a}$ because $a_1 \leq a_1$ and $a_1 + a_2 \leq a_1 + a_2$.

    Antisymmetric: If $\mathsf{a} = (a_1, a_2)$ and $\mathsf{b} = (b_1, b_2)$ are in A with $\mathsf{a} \preceq \mathsf{b}$ and $\mathsf{b} \preceq \mathsf{a}$, then $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, and $b_1 \leq a_1$, $b_1 + b_2 \leq a_1 + a_2$. Since $a_1 \leq b_1$ and $b_1 \leq a_1$, we have $a_1 = b_1$. Since $a_1 + a_2 \leq b_1 + b_2$ and $a_1 = b_1$, we have $a_2 \leq b_2$. Similarly, $b_2 \leq a_2$, so $a_2 = b_2$. Thus $\mathsf{a} = \mathsf{b}$.

    Transitive: If $\mathsf{a} = (a_1, a_2)$, $\mathsf{b} = (b_1, b_2)$ and $\mathsf{c} = (c_1, c_2)$ are elements of A with $\mathsf{a} \preceq \mathsf{b}$ and $\mathsf{b} \preceq \mathsf{c}$, then $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, and also $b_1 \leq c_1$, $b_1 + b_2 \leq c_1 + c_2$. Since $a_1 \leq b_1$ and $b_1 \leq c_1$, we have $a_1 \leq c_1$. Since $a_1 + a_2 \leq b_1 + b_2$ and $b_1 + b_2 \leq c_1 + c_2$, we have $a_1 + a_2 \leq c_1 + c_2$, so $\mathsf{a} \preceq \mathsf{c}$.

    This partial order is not a total order: for example, $\mathsf{a} = (0, 0)$ and $\mathsf{b} = (-1, -2)$ are not comparable.

    (b) Let $A = \mathsf{Z}^n$ and for $\mathsf{a} = (a_1, a_2, \dots, a_n)$ and $\mathsf{b} = (b_1, b_2, \dots, b_n)$ in A, define $\mathsf{a} \preceq \mathsf{b}$ if and only if $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, $a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3$, $a_1 + a_2 + a_3 + a_4 \leq b_1 + b_2 + b_3 + b_4$, $\dots$, $a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n$. Then $\preceq$ is a partial order on A.

11. (a) [BB] Suppose that a and b are two maximum elements in a poset $(A, \preceq)$. Then $a \preceq b$ because b is maximum and $b \preceq a$ because a is maximum, so $a = b$ by antisymmetry.

    (b) Suppose that a and b are two minimum elements in a poset $(A, \preceq)$. Then $a \preceq b$ because a is minimum and $b \preceq a$ because b is minimum, so $a = b$ by antisymmetry.

12. (a) [BB] Assuming it exists, the greatest lower bound G of A and B has two properties:

    (1) $G \subseteq A$, $G \subseteq B$;

    (2) if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq G$.

    We must prove that $A \cap B$ has these properties. Note first that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so $A \cap B$ satisfies (1). Also, if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$, so $A \cap B$ satisfies (2) and $A \cap B = A \wedge B$.

(b) Assuming it exists, the least upper bound of A and B has two properties:

(1) $A \subseteq L, B \subseteq L$;
(2) if $A \subseteq C$ and $B \subseteq C$, then $L \subseteq C$.

We must prove that $A \cup B$ has these properties. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $A \cup B$ satisfies (1). Also, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$, so $A \cup B$ satisfies (2) and $A \cup B = A \vee B$.

13. (a) [BB] $a \vee b = b$ and here is why. We are given $a \preceq b$ and have $b \preceq b$ by reflexivity. Thus b is an upper bound for a and b. It is least because if c is any other upper bound, then $a \preceq c$, $b \preceq c$; in particular, $b \preceq c$.

(b) $a \wedge b = a$ and here is why. We are given $a \preceq b$ and have $a \preceq a$ by reflexivity. Thus a is a lower bound for a and b. It is greatest because if c is any other lower bound, then $c \preceq a$, $c \preceq b$; in particular, $c \preceq a$.

14. (a) [BB] Suppose x and y are each glbs of two elements a and b. Then $x \preceq a$, $x \preceq b$ implies $x \preceq y$ because y is a **greatest** lower bound, and $y \preceq a$, $y \preceq b$ implies $y \preceq x$ because x is greatest. So, by antisymmetry, $x = y$.

(b) Suppose x and y are each lubs of two elements a and b. Then $a \preceq x$, $b \preceq x$ implies $y \preceq x$ because y is a **least** upper bound, and $a \preceq y$, $b \preceq y$ implies $x \preceq y$ because x is least. So, by antisymmetry, $x = y$.

15. (a) In a totally ordered set, every two elements are comparable. So given a and b, either $a \preceq b$ or $b \preceq a$; hence, the elements $\max(a, b)$ and $\min(a, b)$ always exist. In a poset which is not totally ordered, they don't necessarily, however. In the two element poset $\{a\}, \{b\}$ with the relation $\subseteq$, for example, $\max(\{a\}, \{b\})$ does not exist because there is no element in the poset containing both $\{a\}$ and $\{b\}$. (Similarly, $\min(\{a\}, \{b\})$ does not exist.)

(b) To prove that a totally ordered set $(A, \preceq)$ is a lattice we must prove that every pair of elements has a glb and lub. We claim that $\text{glb}(a, b) = \min(a, b)$ and $\text{lub}(a, b) = \max(a, b)$.

We show that $\text{glb}(a, b) = \min(a, b)$. (The argument to show that $\text{lub}(a, b) = \max(a, b)$ is very similar.) Let $m = \min(a, b)$. (Note that $m = a$ or $m = b$). Certainly we have $m \preceq a$ and $m \preceq b$ so m is a lower bound. Also, if for some element c we have $c \preceq a$ and $c \preceq b$, then $c \preceq m$ if $m = a$, and $c \preceq m$ if $m = b$. In either case, we have $c \preceq m$, so $\text{glb}(a, b)$ is $\min(a, b)$ as required.

16. (a) [BB] $(\mathcal{P}(S), \subseteq)$ is not totally ordered provided $|S| \geq 2$ (since $\{a\}$ and $\{b\}$ are not comparable if $a \neq b$). But $\emptyset$ is a minimum because $\emptyset$ is a subset of any set and the set S itself is a maximum because any of its subsets is contained in it.

(b) $(\mathsf{Z}, \leq)$ or $(\mathsf{R}, \leq)$ are obvious examples.

17. Suppose a is maximal in a totally ordered set $(A, \preceq)$ and let b be any other element of A. Since A is totally ordered, either $a \preceq b$ or $b \preceq a$. In the first case, $a = b$ because a is maximal so in either case, $b \preceq a$. Thus, a is a maximum.

18. (a) [BB] We have to prove that if $b \preceq a$, then $b = a$. So suppose $b \preceq a$. Since a is minimum, we have also $a \preceq b$. By antisymmetry, $b = a$.

(b) Let b be a minimal element. We claim $b = a$. To see why, note that a minimum implies $a \preceq b$. Then minimality of b says $a = b$.

Chapter 2 Review

1. Since $A = \{1,2,3,4,5,6\}$ and $B = \{3,4,5,6,7\}$, we have $A \oplus B = \{1,2,7\}$ and $(A \oplus B) \smallsetminus C = \{1,7\}$.

2. (a) $A = \{-1,0,1,2\}$, $B = \{-5,-3,-1,1\}$, $C = \{-\frac{2}{3}, -\frac{2}{5}, 0, \pm 1, \pm 2, \pm\frac{1}{5}, \pm\frac{1}{3}\}$;

   (b) $A \cap B = \{\pm 1\}$, so $(A \cap B) \times B = \{(1,-5),(1,-3),(1,-1),(1,1),(-1,-5),(-1,-3),(-1,-1),(-1,1)\}$;

   (c) $B \smallsetminus C = \{-5,-3\}$; (d) $A \oplus C = \{-\frac{2}{3}, -\frac{2}{5}, -2, \pm\frac{1}{5}, \pm\frac{1}{3}\}$.

3. (a) True. ($\longrightarrow$) Suppose $A \cap B = A$ and let $a \in A$. Then $a \in A \cap B$, so $a \in B$. Thus $A \subseteq B$.

   ($\longleftarrow$) Suppose $A \subseteq B$. To prove $A \cap B = A$, we prove each of the two sets, $A \cap B$ and A, is a subset of the other. Let $x \in A \cap B$. By definition of $\cap$, x is in both A and B, in particular, $x \in A$. Thus $A \cap B \subseteq A$. Conversely, let $x \in A$. Since $A \subseteq B$, $x \in B$. Since x is in both A and B, $x \in A \cap B$. Thus $A \subseteq A \cap B$.

   (b) This is false. If $A = \emptyset$, $(A \cap B) \cup C = C$ while $A \cap (B \cup C) = \emptyset$, so any $C \neq \emptyset$, any B, and $A = \emptyset$ provides a counterexample.

   (c) False. Take $A = B = \emptyset$.

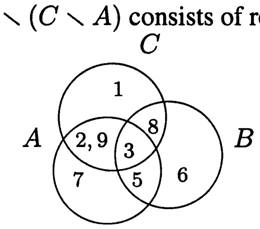
4. Let $b \in B$ and let a be any element of A. Then $(a,b) \in A \times B$, so $(a,b) \in A \times C$. Thus $b \in C$. This shows tht $B \subseteq C$ and a similar argument shows $C \subseteq B$, so $B = C$.

5. (a) Region 2: $(A \cap C) \smallsetminus B$ Region 3: $A \cap B \cap C$ Region 4: $(A \cap B) \smallsetminus C$
   Region 5: $(B \cap C) \smallsetminus A$ Region 6: $B \smallsetminus (A \cup C)$ Region 7: $C \smallsetminus (A \cup B)$

   (b) Region 2, 3, 4, 5, 7 is $(A \cap B) \cup C$; region 2, 3, 4 is $A \cap (B \cup C)$

   (c) $B \smallsetminus (C \smallsetminus A)$ consists of regions 3, 4, and 6. $(B \smallsetminus C) \smallsetminus A$ consists of region 6.

6. (a)



   (b) i. $(A \cup B) \cap C = \{2,3,8,9\}$

   ii. $A \smallsetminus (B \smallsetminus C) = \{2,3,7,9\}$

   iii. $A \oplus B = \{2,6,7,8,9\}$

   iv. $(A \smallsetminus B) \times (B \cap C) = \{(2,3),(2,8),(7,3),(7,8),(9,3),(9,8)\}$

7. $\mathcal{P}(A) = \{\emptyset, A\}$, so $\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{A\}, \{\emptyset, A\}\}$.

8. (a) Take $A = B = C = \{3\}$. Then $B \smallsetminus C = \emptyset$ so $A \oplus (B \smallsetminus C) = A$. On the other hand, $A \oplus B = \emptyset$ so $(A \oplus B) \smallsetminus C = \emptyset \neq A$.

   (b) Let $(a,b) \in A \times B$. Then $a \in A$ and $b \in B$. Since $a \in A$ and $A \subseteq C$, $a \in C$. Since $b \in B$ and $B \subseteq D$, $b \in D$. So $(a,b) \in C \times D$. Hence $A \times B \subseteq C \times D$.

9. Take $B = C = \emptyset$, $A = \{1\} = D$. Then $A \times B = \emptyset = C \times D$, so $A \times B \subseteq C \times D$. On the other hand, $A \not\subseteq C$.

10. This follows quickly from one of the laws of De Morgan and the identity $X \setminus Y = X \cap Y^c$.

$$A \setminus (B \cap C) = A \cap (B \cap C)^c = A \cap (B^c \cup C^c) \stackrel{\downarrow}{=} (A \cap B^c) \cup (A \cap C^c) = (A \setminus B) \cup (A \setminus C),$$

using (3), p. 62 at the spot marked with the arrow.

11. (a) A *binary relation* on A is a subset of $A \times A$.

    (b) If A has 10 elements, $A \times A$ has 100 elements, so there are 2^{100} binary relations on A.

12. **Reflexive**: For any a with $|a| \leq 1$, we have $a^2 = |a^2| = |a||a| \leq |a|$, thus $(a, a) \in \mathcal{R}$.

    Symmetric by definition.

    Not antisymmetric because $(\frac{1}{2}, \frac{1}{4})$ is in $\mathcal{R}$ $((\frac{1}{2})^2 \leq \frac{1}{4}$ and $(\frac{1}{4})^2 \leq \frac{1}{2})$ but $\frac{1}{2} \neq \frac{1}{4}$.

    Not transitive. We have $(\frac{1}{2}, \frac{1}{3}) \in \mathcal{R}$ because $(\frac{1}{2})^2 \leq \frac{1}{3}$ and $(\frac{1}{3})^2 \leq \frac{1}{2}$ and $(\frac{1}{3}, \frac{1}{5}) \in \mathcal{R}$ because $(\frac{1}{3})^2 \leq \frac{1}{5}$ and $(\frac{1}{5})^2 \leq \frac{1}{3}$, but $(\frac{1}{2}, \frac{1}{5})$ is not in $\mathcal{R}$ because $(\frac{1}{2})^2 \not\leq \frac{1}{5}$.

13. (a) **Reflexive:** For any natural number a, we have $a \leq 2a$, so $a \sim a$.

    Not symmetric: $2 \sim 5$ because $2 \leq 2(5)$, but $5 \not\sim 2$ because $5 \not\leq 2(2)$.

    Not antisymmetric: Let $a = 1$ and $b = 2$. Then $a \sim b$ because $1 \leq 2(2)$ and also $b \sim a$ because $2 \leq 2(1)$.

    Not transitive: Let $a = 3$, $b = 2$, $c = 1$. Then $a \sim b$ because $3 \leq 2(2)$ and $b \sim c$ because $2 \leq 2(1)$. However, $a \not\sim c$ because $3 \not\leq 2(1)$.

    Since the relation is not transitive, it is not an equivalence relation and it is not a partial order.

    (b) **Not reflexive:** $(1, 2) \not\sim (1, 2)$ because $1 \leq 2$ and $2 \not\leq 1$.

    Not symmetric: $(1, 2) \sim (4, 3)$ because $1 \leq 2$ and $3 \leq 4$, but $(4, 3) \not\sim (1, 2)$ because $4 \not\leq 3$.

    Not antisymmetric: $(1, 1) \sim (2, 2)$ because $1 \leq 1$ and $2 \leq 2$, and $(2, 2) \sim (1, 1)$ for the same reason, but $(1, 1) \neq (2, 2)$.

    Transitive: if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $a \leq b$, $d \leq c$, $c \leq d$ and $f \leq e$, so $a \leq b$ and $f \leq e$ which implies $(a, b) \sim (e, f)$.

    This is not an equivalence relation because it's not reflexive (or symmetric).

    This is not a partial order because it's not reflexive (or antisymmetric).

14. We must determine whether or not $a < b$ and $b < a$ implies $a = b$. Since the hypothesis is always false, this implication is true. The relation is antisymmetric.

15. **Reflexive:** For any $a \in \mathsf{Z}$, $4a + a = 5a$ is a multiple of 5.

    Symmetric: If $a\mathcal{R}b$, then $4a + b$ is a multiple of 5, so $4b + a = 5(a + b) - (4a + b)$ is also a multiple of 5, that is, $b\mathcal{R}a$.

    Transitive: If $a\mathcal{R}b$ and $b\mathcal{R}c$, then both $4a + b$ and $4b + c$ are multiples of 5, hence so is their sum, $4a + 5b + c$. It follows that $(4a + 5b + c) - 5b = 4a + c$ is also a multiple of 5, so $a\mathcal{R}c$.

16. (a) **Reflexive:** For any $a \in \mathsf{Z}$, $a\mathcal{R}a$ because $2a + 5a = 7a$ is a multiple of 7.

    Symmetric: If $a, b \in \mathsf{Z}$ and $a\mathcal{R}b$, then $2a + 5b = 7k$ for some integer k, so $5a + 2b = 7(a + b) - (2a + 5b)$ is the difference of multiples of 7, hence also a multiple of 7. Thus $b\mathcal{R}a$.

    Transitive: If $a, b, c \in \mathsf{Z}$ with $a\mathcal{R}b$ and $b\mathcal{R}c$, then $2a + 5b = 7k$ for some integer k and $2b + 5c = 7\ell$ for some integer ℓ. Thus $(2a + 5b) + (2b + 5c) = 2a + 7b + 5c = 7(k + \ell)$ and $2a + 5c = 7(k + \ell) - 7b$ is the difference of multiples of 7, hence a multiple of 7. Thus $a\mathcal{R}c$.

(b) We have $0\mathcal{R}7$ and $7\mathcal{R}0$, yet $0 \neq 7$. The relation is not antisymmetric, so it is not a partial order.

17. ($\longrightarrow$) Assume $x \sim a$. Let $t \in \overline{x}$. Then $t \sim x$ so $t \sim a$ by transitivity. Thus $t \in \overline{a}$. This proves $\overline{x} \subseteq \overline{a}$. Similarly, $\overline{a} \subseteq \overline{x}$, so $\overline{x} = \overline{a}$.

($\longleftarrow$) Assume $\overline{x} = \overline{a}$. Since $x \in \overline{x}$, by symmetry, $x \in \overline{a}$. Thus $x \sim a$.

18. Since $a \in \overline{b}$, $a \sim b$ and hence $\overline{a} = \overline{b}$ by Proposition 2.4.3. Similarly $d \in \overline{b}$ implies $\overline{d} = \overline{b}$, hence $\overline{a} = \overline{d}$. Now $d \notin \overline{c}$ implies $\overline{d} \neq \overline{c}$, so $\overline{c} \cap \overline{d} = \emptyset$ by Proposition 2.4.4. Since $\overline{a} = \overline{d}$, so also $\overline{a} \cap \overline{c} = \emptyset$.

19. (a) We must show that $\preceq$ is reflexive, antisymmetric and transitive on A. The relation is reflexive: For any $(a, b) \in A$, $(a, b) \sim (a, b)$ because $a \leq a$ and $b \leq b$.

It is antisymmetric: If $(a, b), (c, d) \in A$, with $(a, b) \sim (c, d)$ and $(c, d) \sim (a, b)$, then $a \leq c$, $d \leq b$, $c \leq a$ and $b \leq d$. So $a = c$, $b = d$, hence, $(a, b) = (c, d)$.

It is transitive: If $(a, b), (c, d), (e, f) \in A$ with $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $a \leq c$, $d \leq b$, $c \leq e$ and $f \leq d$. So $a \leq e$ (because $a \leq c \leq e$) and $f \leq b$ (because $f \leq d \leq b$), so $(a, b) \sim (e, f)$.

(b) $(A, \preceq)$ is not totally ordered since, for example, $(1, 4)$ and $(2, 5)$ are not comparable: $(1, 4) \not\sim (2, 5)$ because $5 \not\leq 4$ and $(2, 5) \not\sim (1, 4)$ because $2 \not\leq 1$.

20. (a) **Reflexive:** For any $p \in A$, $p \sim p$ since $p = p$.

Symmetric: If $p \sim q$ then either $p = q$ (so $q = p$) or the line through p and q passes through the origin (in which case the line through q and p passes through the origin). Thus $q \sim p$.

Transitive: Suppose $p \sim q$ and $q \sim r$. If the points p, q, r are different, then the line through p and q passes through the origin, as does the line through q and r. Since the line through the origin and q is unique, p and r lie on this line, $p \sim r$. If $p = r$, then $p \sim r$. If $p = q \neq r$, then the line through q and r passes through the origin, so the line through p and r passes through the origin; thus $p \sim r$. The remaining case, $p \neq q = r$ is similar.

(b) The equivalence class of a point p is the line through the origin and p. The equivalence classes are lines through the origin.

21. **Reflexive:** For any $A \in \mathcal{P}(Z)$, $A \subseteq A$.

Antisymmetric: If $A, B \subseteq \mathcal{P}(Z)$ with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

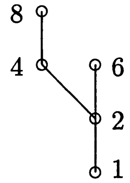
Transitive: If $A, B, C \subseteq \mathcal{P}(Z)$ with $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$.

22. (a) **Reflexive:** For any $a \in A$, $\dfrac{a}{a} = 1$ is an integer, so $a \preceq a$.

Antisymmetric: If $a \preceq b$ and $b \preceq a$, then both $\dfrac{b}{a}$ and $\dfrac{a}{b}$ are (necessarily positive) integers. The only positive integer whose reciprocal is also an integer is 1, so $a = b$.

Transitive: If $a \preceq b$ and $b \preceq c$, then $\dfrac{b}{a}$ and $\dfrac{c}{b}$ are both integers. Thus $\dfrac{b}{a}\dfrac{c}{b} = \dfrac{c}{a}$ is an integer. So $a \preceq c$.

(b)



(c) 1 is minimal and minimum; 6 and 8 are maximal. There is no maximum element.

(d) $(A, \preceq)$ is not totally ordered; for example, 4 and 6 are not comparable.

23. Two elements of a poset can have at most one least upper bound and here's why. Let ℓ_1, ℓ_2 each be least upper bounds for elements a and b. Then ℓ_1 is an upper bound for a and b, so $\ell_2 \preceq \ell_1$ because ℓ_2 is **least**. Interchanging ℓ_1, ℓ_2 in the preceding statement gives $\ell_1 \preceq \ell_2$. So $\ell_1 = \ell_2$ by antisymmetry.