

CHAPTER 2 LIMITS

Limits provide the foundation for all the key ideas of calculus (differentiation, integration, and infinite series, to name a few). This chapter supplies the tools that your students will use to understand the important concepts of calculus.



2.1 The Idea of Limits

Overview

In this opening section, we introduce the idea of a limit through an investigation of the relationship between instantaneous velocity and tangent lines. The intent is to provide students with motivation for the entire chapter and to give an intuitive sense of how limits work. We feel this material is essential even though tangent lines and instantaneous velocity are treated in more detail in Chapter 3.

Lecture Support Notes

Limits arise naturally when we define instantaneous velocity and the line tangent to a curve.

- Ask students to compute an average velocity for a simple situation (e.g. a car travels 110 mi in 2 hr; what is its average velocity?). Increase the complexity of the problem by providing a plausible position function for an object, and ask students to compute the average velocity over some time interval. Students should be able to come up with the formula

$$\text{average velocity} = (\text{change in position})/(\text{change in time}).$$
- Point out that average velocities are just slopes of secant lines on a position curve (Example 1; Figures 2.1–2.3).
- Show how average velocity can be used to approximate instantaneous velocity.
- Then show how shrinking the time interval leads to better approximations of instantaneous velocity.
- Introduce the idea of a limit in moving from average to instantaneous velocity (Example 2).

This scheme can be repeated to introduce the notion of a tangent line as the limit of approaching secant lines. Students can compute the slopes of secant lines over smaller intervals to arrive at a preliminary definition for the slope of a tangent line. Emphasize ideas rather than details; the details will come in later sections.

Interactive Figures

- Figures 2.1–2.2 display the position and average velocity of a rock for $0 \leq t \leq 3$.
- Figure 2.4 illustrates calculating instantaneous velocity via limits.
- Figure 2.5–2.6 display the tangent and secant lines, while the latter also displays average and instantaneous velocity.

Connections



This section lays the foundation for a formal discussion of tangent lines and instantaneous velocity found in Chapter 3. Examples 1 and 3 of Section 3.1 build upon the ideas introduced in Examples 1 and 2 of Section 2.1.

- Section 1.2 includes a review of linear functions and the formula for the slope of a line through two points.
- Section 3.5 continues the discussion of the derivative as a rate of change (instantaneous velocity in particular).
- Remind students to bring calculators or laptops to the next class if you intend to call upon them to produce the graphs and tables necessary to determine limits.

Additional Activities



Suggested Guided Project: *Local Linearity* is a guided project that fits well with this section; it extends the idea introduced in the margin note on page 53 of the text. Part 1 of this project could be used in class as a launching point for a discussion about tangent lines (the remainder could be assigned for completion outside of class.)



Section 2.2 Definitions of Limits

Overview

This section gives a standard treatment of left-hand, right-hand, and two-sided limits, and shows how to compute them (informally) with graphs and numerical methods.

Lecture Support Notes

A preliminary (informal) definition of a limit is adequate for understanding and computing most of the limits encountered in the text.

- Present the informal definition of a limit.
- At this stage, computing the value of a limit is most easily carried out with a graph (Example 1). Point out that the limit of a function f at the point a (if it exists) does not depend upon the value of $f(a)$.
- The value of a limit at a point can also be investigated by tabulating function values near that point (Example 2).
- Introduce left- and right-hand limits (Example 3) and their relationship to the corresponding two-sided limit.
- Show that limits may fail to exist, either because the left- and right-hand limits do not agree (Example 4), or because the values of a function do not approach a single number (Example 5).

Interactive Figures

- Figures 2.7–2.10 illustrate how to find a limit of a function.
- Figure 2.11 illustrates how computer generated plots often do not plot where the function is undefined.
- Figure 2.12 displays limits from the left and right.
- Figure 2.14 zooms in on $y = \cos(1/x)$ near $x = 0$.

Connections

Most of the exercises in this section rely upon a graph or a table. For those exercises where students must produce a graph, we expect them to rely on technology at this point in the course—be prepared to help with various calculators and software packages. That said, students should begin the work of mastering the basic graphs of all the familiar functions (presented in Sections 1.2–1.4).

Additional Activities

Cover Example 3 (or use a function similar to Exercise 15 as a suitable replacement with easier factoring) for an easy-to-implement, in-class activity. Assign various values of x in Table 2.3 to rows of students in your classroom, and ask them to compute $f(x)$ on a calculator. Solicit responses from each row and fill in a table at the board, from which the class can determine the limit. This is just the preliminary work for the intended activity: Explore the algebraic reasons behind the value of the limit (alluding to techniques introduced in

Section 2.3) by showing that $\frac{x^3 - 8}{4(x - 2)} = \frac{x^2 + 2x + 4}{4}$ ($x \neq 2$). The graph of the function on the right side of the

equation is the parabola $y = \frac{1}{4}(x + 1)^2 + \frac{3}{4}$. Ask students to verify that the graph of the function on the left side of the equation has the same graph except at $x = 2$, and that the values of $y = \frac{1}{4}(x + 1)^2 + \frac{3}{4}$ approach 3 as x approaches 2. With this activity, you can review factoring formulas (difference of perfect cubes), completing the square, the standard form of a parabola ($y = a(x - h)^2 + k$, whose vertex is (h, k)), and graphing parabolas by hand.

Section 2.2 Quick Quiz

Answer the following multiple choice questions by circling the correct response.

1. If $f(x)$ approaches 8 as x approaches 5 from the left, then you can be certain that

(a) $\lim_{x \rightarrow 5^+} f(x) = 8.$	(b) $\lim_{x \rightarrow 5^-} f(x) = 8.$	(c) $\lim_{x \rightarrow 5} f(x) = 8.$
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2. If $\lim_{x \rightarrow 2^-} f(x) = 5$, $\lim_{x \rightarrow 2^+} f(x) = 5$, and $f(2) = 6$, then

(a) $\lim_{x \rightarrow 2} f(x) = 5.$	(b) $\lim_{x \rightarrow 2} f(x) = 6.$	(c) $\lim_{x \rightarrow 2} f(x)$ does not exist.
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3. If $\lim_{x \rightarrow 2} f(x) = 5$, then

(a) $f(2) = 5.$	(b) $f(2)$ is undefined.	(c) not enough information is given to determine $f(2)$.
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4. If $f(2) = 5$, then

(a) $\lim_{x \rightarrow 2} f(x) = 5.$	(b) $\lim_{x \rightarrow 2} f(x) \neq 5.$	(c) not enough information is given to determine $\lim_{x \rightarrow 2} f(x)$.
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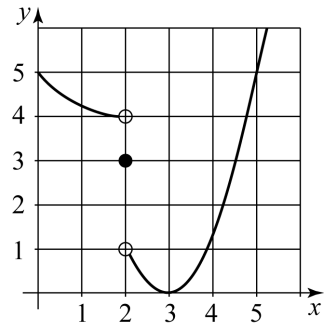
5. If $\lim_{x \rightarrow 1^-} f(x) = 6$ and $\lim_{x \rightarrow 1^+} f(x) = 7$, then

(a) $\lim_{x \rightarrow 1} f(x) = 6.$	(b) $\lim_{x \rightarrow 1} f(x) = 7.$	(c) $\lim_{x \rightarrow 1} f(x)$ does not exist.
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6. The values $f(0.1) = 7.9$, $f(0.01) = 7.99$, $f(.001) = 7.999$, and $f(.0001) = 7.9999$ lead us to conjecture that

(a) $\lim_{x \rightarrow 0^-} f(x) = 8.$	(b) $\lim_{x \rightarrow 0^+} f(x) = 8.$	(c) $\lim_{x \rightarrow 0} f(x) = 8.$
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Refer to the accompanying figure for Exercises 7–10.



7. $\lim_{x \rightarrow 2^-} g(x)$ equals

(a) 1.	(b) 3.	(c) 4.
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8. $\lim_{x \rightarrow 2^+} g(x)$ equals

(a) 1.	(b) 3.	(c) 4.
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9. $\lim_{x \rightarrow 2} g(x)$

(a) equals 1.	(b) equals 4.	(c) does not exist.
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10. $\lim_{x \rightarrow 5} g(x)$

(a) equals 1.	(b) equals 5.	(c) does not exist.
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Section 2.3 Techniques for Computing Limits

Overview

Analytical methods for evaluating limits are presented.

Lecture Support Notes

Limits of polynomial, rational, and algebraic functions can usually be evaluated by direct substitution, provided the function is defined at the limit point (exceptions include piecewise functions; see Example 5). Limit laws, algebraic manipulation, and the Squeeze Theorem provide tools for evaluating more challenging limits.

- Pack in as many examples as class time allows, using graphs to illustrate the fact that $\lim_{x \rightarrow a} f(x) = f(a)$ for many standard functions. At this point we have neither the tools nor the concepts—namely, the ε - δ definition of a limit and continuity—to prove Theorems 2.2–2.4 rigorously, so it is important to provide visual evidence that these theorems are true.
- Limits that cannot be evaluated by direct substitution can often be transformed into limits that yield to direct substitution via factoring and multiplication by the conjugate. In some cases, the Squeeze Theorem is an essential tool.
- Consider beginning the task of identifying standard limit forms in this section. For instance, you could point out that the limits in Example 6 are of the form $0/0$, one of several indeterminate forms to be treated in Section 4.7. Students will have an easier time deciding which technique to apply to a particular limit if they can identify the form of a limit.
- Cover Example 7; it is the first piece of the puzzle for establishing the derivative of e^x (see more in *Connections* below), and it revisits the idea of a tangent line.

Interactive Figures

- Figures 2.15 and 2.19 display limits and slopes of the tangent line from the left and right, respectively.
- Figure 2.20 illustrates the squeeze theorem. Figures 2.21–2.22 illustrate the squeeze theorem for particular trigonometric functions.

Connections

- If you assigned Exercise 28 from Section 2.2, you may use it to segue into Example 7.
- The justification for the method of direct substitution, continuity, is presented in Section 2.6.

It is worth pausing for a moment to compare and contrast two standard paths through the first semester calculus curriculum: the *Early Transcendental* (ET) and the *Late Transcendental* (LT) paths.

In the LT path, we put off discussion of the calculus-related properties of logarithmic, exponential, and inverse functions (inverse trigonometric functions in particular). Only after the Fundamental Theorem of

Calculus is stated does it make sense to define the natural logarithm as $\ln x = \int_1^x (1/t)dt$. From there, we can show that $\ln x$ is continuous on its domain (because it is differentiable), that it is one-to-one (its derivative is

positive), and therefore it has an inverse. After developing the properties of inverse functions in general (when $f(x)$ is continuous on its domain, $f^{-1}(x)$ is continuous on the range of f), the natural exponential function comes onto stage, followed by the inverse trigonometric functions. This is the chain of logic that allows us to establish the continuity and differentiability of e^x . Typically, these transcendental functions are not introduced (due to time considerations) until the beginning of second semester calculus.

In the ET path, the idea is to introduce the important transcendental functions early so that students who require only one semester of calculus are exposed to the exponential, logarithmic and inverse trigonometric functions. This comes at the expense of rigor: we cannot easily establish the continuity and differentiability of exponential functions (and thus logarithmic functions) without relying upon numerical estimates of a few key limits. It all boils down to the assumption that e^x is defined and continuous on the entire real number line—this assumption (Sections 1.3 and 2.6) is used to prove that e^x has a well-defined derivative (Section 3.2).

Thus the logic behind labeling Example 7 “An Important Limit”: It helps to establish the derivative of e^x . In addition, it reinforces the concept of a tangent line, which is thoroughly explored in Chapter 3. Note that we do not intend for instructors to explain all these theoretical connections at this stage. It will, however, help to have them in mind if you intend to circle back to these topics in Section 6.7, where the gaps in theory that are part of the ET curriculum are filled in.

Additional Activities

This is the first section in our study of calculus where students will be required to draw fully on their knowledge of algebra (in particular, factoring polynomials, simplifying rational functions, and using the technique of multiplying by a conjugate). The following selection of exercises can be used as a “math minute” activity, where students have one minute (or longer if you so choose) to complete as many problems as possible. Five minutes is a more reasonable amount of time if you expect students to complete all of the problems.

Factor:

1. $x^2 - 1$

2. $x^2 - 2x - 3$

3. $3t^2 - 7t + 2$

4. $(x+b)^7 + (x+b)^{10}$

5. $(2x-1)^2 - 9$

6. $x^2 - 2cx + c^2$

Simplify:

7. $\frac{x^2 - 16}{4 - x}$

8. $\frac{3t^2 - 7t + 2}{2 - t}$

9. $\frac{\frac{1}{5+h} - \frac{1}{5}}{h}$

10. $\frac{(5+h)^2 - 25}{h}$

11. $\frac{\sqrt{x} - 3}{x - 9}$

12. $\frac{1}{x-2} - \frac{2}{x^2 - 2x}$

To motivate your students, point out that these drill problems are linked directly to the exercise set (Exercises 37–40, 42–45, and 60–62). If they can get through these quickly, evaluating the limits that are associated with them won't seem so bad. Provide answers at the end of the activity so that students can assess whether they need more practice (given the pressures associated with completing the problems at a fast pace, this exercise is not intended to count toward a student's grade; rather, it's meant to be a fun way to engage your students and help them identify their weak areas). If you have not already done so, this would be a good time in the semester to advertise the resources available to students at MyMathLab.

ANSWERS:

1. $(x-1)(x+1)$

2. $(x-3)(x+1)$

3. $(3t-1)(t-2)$

4. $(x+b)^7(x+b+1)((x+b)^2 - (x+b) + 1)$

5. $4(x-2)(x+1)$

6. $(x-c)^2$

7. $-(x+4)$

8. $-(3t-1)$

9. $-1/(5(5+h))$

10. $10+h$

11. $1/(\sqrt{x}+3)$

12. $1/x$

Section 2.3 Quick Quiz

Answer the following multiple choice questions by circling the correct response.

- The limit $\lim_{x \rightarrow 3} \frac{(x-1)(x-3)}{x-3}$
(a) does not exist. (b) cannot be determined. (c) equals 2.
- If $f(x) = g(x)$ for all values of x except possibly for $x = 2$ and $\lim_{x \rightarrow 2} g(x) = 5$, then
(a) $f(2) = 5$. (b) $\lim_{x \rightarrow 2} f(x)$ may not equal 5. (c) $\lim_{x \rightarrow 2} f(x) = 5$.
- If $f(x) = \begin{cases} 4x+1 & \text{if } x \neq 0 \\ x+b & \text{if } x = 0 \end{cases}$ and $\lim_{x \rightarrow 0} f(x) = f(0)$, then
(a) $b = 0$. (b) the value of b cannot be determined. (c) $b = 1$.
- If $f(x) = \begin{cases} 4 & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$, then
(a) $\lim_{x \rightarrow 2} f(x) = 5$. (b) $\lim_{x \rightarrow 2} f(x) = 4$. (c) $\lim_{x \rightarrow 2} f(x)$ does not exist.
- If $f(5)$ does not exist, then $\lim_{x \rightarrow 5} f(x)$
(a) equals 5. (b) does not exist. (c) might exist.
- If $f(5) = 0$, then $\lim_{x \rightarrow 5} f(x)$
(a) equals 5. (b) might not exist. (c) equals 0.
- If $f(x) = \begin{cases} g(x) & \text{if } x < 0 \\ 5 & \text{if } x \geq 0 \end{cases}$ and $\lim_{x \rightarrow 0} g(x) = 2$, then
(a) $\lim_{x \rightarrow 0^+} f(x) = 2$. (b) $\lim_{x \rightarrow 0^+} f(x) = 5$.
(c) $\lim_{x \rightarrow 0^+} f(x)$ exists, but its precise value cannot be determined.
- If $-4x + 12 \leq g(x) \leq 8 - x^2$ for all x , then
(a) $\lim_{x \rightarrow 2} g(x) = 4$. (b) $\lim_{x \rightarrow 2} g(x)$ exists, but its precise value cannot be determined.
(c) $\lim_{x \rightarrow 2} g(x)$ does not exist.
- If p is a polynomial, then $\lim_{x \rightarrow a} p(x)$
(a) may not exist. (b) equals a . (c) equals $p(a)$.
- If $f(x) = \sqrt{x^2 - 4}$, then
(a) $\lim_{x \rightarrow 2^-} f(x)$ exists, but $\lim_{x \rightarrow 2} f(x)$ does not exist.
(b) $\lim_{x \rightarrow 2^+} f(x)$ exists, but $\lim_{x \rightarrow 2} f(x)$ does not exist. (c) $\lim_{x \rightarrow 2} f(x)$ exists.



Section 2.4 Infinite Limits

Overview

Infinite limits are introduced (initially alongside a limit at infinity to help students distinguish between the two scenarios), and their connection to vertical asymptotes is explained.

Lecture Support Notes

- Make the distinction between infinite limits and limits at infinity so that students understand their similarities and differences before moving onto Section 2.5.
- It is important that students understand that an infinite limit does not exist; we only use the symbol ∞ as a convenience to indicate that the function attains arbitrarily large values. It doesn't hurt to explain this once more after one-sided infinite limits are introduced: We say a two-sided limit is ∞ (or $-\infty$) only when the left- and right-hand limits “agree,” despite the fact that neither the left nor right-hand limit exists.
- The graph of f has the vertical asymptote $x = a$ whenever the limit of f (left, right, or two-sided) is infinite in magnitude at a .
- Consider employing the notation “ $L/0^+$ ” and “ $L/0^-$ ” (where L is a nonzero constant) for those limits where the numerator approaches L and the denominator approaches 0, either from above or below. This notation sees widespread use, and certainly has value—the authors decided against formally introducing it in the text for various reasons. If you spend the time in class to develop and explain the notation, it gives students another tool for getting an intuitive feel for the value of a limit. Example 5 provides a good illustration of the different behaviors that can arise when the denominator of a function approaches 0.

Interactive Figures

- The figures accompanying Tables 2.6–2.7 illustrate infinite limits.
- Figure 2.28 illustrates how computer generated plots often do not include undefined points and make vertical asymptotes appear as a solid line (rather than the dotted line that is seen in the text).

Connections

- Vertical asymptotes make a brief appearance in Chapter 1 (Section 1.2 and Section 1.4 in connection with trigonometric functions). They are also treated in Section 4.3 in conjunction with graphing functions.
- Vertical tangent lines are later discussed on pp. 130–1; see also Exercises 61–64 in Section 3.1.

Additional Activities

Chapter 2 opens with a brief discussion of tangent lines; the tangent problem is then set aside for most of the remainder of the chapter. (We pick it up again in Chapter 3.) If you'd like to continue to develop tangent lines alongside the discussion of limits, here's an activity that incorporates infinite limits and vertical tangent lines. The idea is to find the slope of the line tangent to the unit circle at $(1, 0)$ and $(-1, 0)$.

- Let $P(x, \sqrt{1-x^2})$ denote a variable point on the upper half of the unit circle, with $0 \leq x < 1$. The slope of the secant line between P and $(1, 0)$ is $m_{\text{sec}} = \frac{\sqrt{1-x^2}}{x-1}$.

- The slope of the tangent line at $(1, 0)$ is $m_{\text{tan}} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1-x^2}}{x-1}$.

- This limit can be investigated numerically by choosing values of $x < 1$ that get closer to 1, and/or by algebraic manipulation:

$$m_{\text{tan}} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1-x^2}}{x-1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1-x}\sqrt{1+x}}{-(1-x)} = -\lim_{x \rightarrow 1^-} \frac{\sqrt{1+x}}{\sqrt{1-x}},$$

where the limit on the right is of the form $-\sqrt{2}/0^+ = -\infty$.

- Interpret this limit to show that the graph of the unit circle has a vertical tangent line at $(1, 0)$.
- A similar argument shows that the unit circle also has a vertical tangent line at $(-1, 0)$, though the limit in this case is ∞ (assuming as before that P is a point on the upper half of the unit circle).

This activity could be teacher-directed (walk through the above steps with your students, supplying additional details where necessary), or modified to be given as an out of class project. One could also analyze the minor

differences that appear when the point P is selected on the lower half of the circle. Note that $\lim_{x \rightarrow 1^-} \frac{\sqrt{1-x^2}}{x-1}$ is of

the form $0/0$: If you introduced this notation to your class in earlier sections (e.g. Example 6 in Section 2.3), this activity helps to drive home the reason why we label “ $0/0$ ” as an indeterminate form. Though students aren't quite ready to hear it (because we have not yet introduced the derivative), it's worth noting the difference

between the limits investigated in this activity and the more standard limits of this section. After derivatives have been covered, come back to this activity and point out that when $\lim_{x \rightarrow a^+} f'(x) = \pm\infty$ (or $\lim_{x \rightarrow a^-} f'(x) = \pm\infty$),

there is a vertical tangent at $x = a$ (assuming that a is in the domain of f). When $\lim_{x \rightarrow a^+} f(x) = \pm\infty$, we

conclude that the graph of f has a vertical asymptote at $x = a$.

Section 2.4 Quick Quiz

Answer the following multiple choice questions by circling the correct response.

1. Which of the following statements is false if $\lim_{x \rightarrow a^+} f(x) = \infty$?
- (a) As $x \rightarrow a^+$, $f(x)$ gets closer to the real number ∞ .
 - (b) The values of f grow arbitrarily large as x approaches a from the right.
 - (c) The line $x = a$ is a vertical asymptote of $f(x)$.

2. If $f(1.1) = -500$, $f(1.01) = -5000$, $f(1.001) = -50000$, $f(1.0001) = -500000$ and so forth, then the most plausible conclusion is that

- (a) $\lim_{x \rightarrow 1^+} f(x) = -\infty$.
- (b) $\lim_{x \rightarrow 1^+} f(x) = \infty$.
- (c) $\lim_{x \rightarrow -1} f(x) = -\infty$.

3. For the function $y = g(x)$ in the accompanying figure,

- (a) $\lim_{x \rightarrow 1^+} g(x) = \infty$.
- (b) $\lim_{x \rightarrow 1^+} g(x) = 0$.
- (c) $\lim_{x \rightarrow 1^+} g(x) = -\infty$.

4. For the function $y = g(x)$ in the accompanying figure,

- (a) $\lim_{x \rightarrow 1^-} g(x) = \infty$.
- (b) $\lim_{x \rightarrow 1^-} g(x) = 0$.
- (c) $\lim_{x \rightarrow 1^-} g(x) = -\infty$.

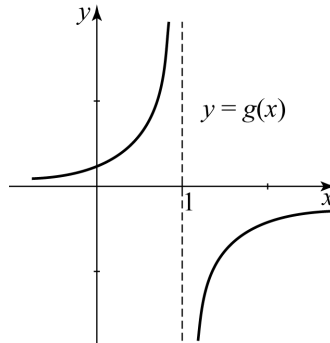


Figure for Exercises 3 and 4

5. If $\lim_{x \rightarrow 1} f(x) = 10$ and $\lim_{x \rightarrow 1} g(x) = 0$, then

- (a) $\lim_{x \rightarrow 1} \frac{f(x)}{[g(x)]^2} = 0$.
- (b) $\lim_{x \rightarrow 1} \frac{f(x)}{[g(x)]^2} = -\infty$.
- (c) $\lim_{x \rightarrow 1} \frac{f(x)}{[g(x)]^2} = \infty$.

6. Suppose $g(x)$ is negative and approaches 0 as x approaches 5. Then

- (a) $\lim_{x \rightarrow 5} \frac{1}{g(x)} = \infty$.
- (b) $\lim_{x \rightarrow 5} \frac{1}{g(x)} = 0$.
- (c) $\lim_{x \rightarrow 5} \frac{1}{g(x)} = -\infty$.

7. As $x \rightarrow 3^-$, suppose $f(x) \rightarrow -5$, $g(x) \rightarrow 2$, and $h(x)$ is negative and approaches 0. Then

- (a) $\lim_{x \rightarrow 3^-} \frac{f(x)g(x)}{h(x)} = \infty$.
- (b) $\lim_{x \rightarrow 3^-} \frac{f(x)g(x)}{h(x)} = -\infty$.
- (c) $\lim_{x \rightarrow 3^-} \frac{f(x)g(x)}{h(x)} = 0$.

8. $\lim_{x \rightarrow 1} \frac{(x-3)(x+1)}{(x-1)^2}$ equals

- (a) $-\infty$.
- (b) ∞ .
- (c) 0.



Section 2.5 Limits at Infinity

Overview

Limits at infinity determine the end behavior of a function, detect the presence of horizontal asymptotes, and reveal whether a system attains a steady state.

Lecture Support Notes

- When they exist, limits at infinity indicate a horizontal asymptote.
- As with Section 2.4, it's worth pointing out that infinite limits at infinity do not exist; contrast this situation with $\lim_{x \rightarrow \infty} \cos x$, which also does not exist, but for different reasons.
- Include examples from all the standard families of functions so that students get a feel for the end behavior of the various functions they will encounter throughout the text. Theorems 2.6, 2.7 and 2.8 are particularly important as they allow students to quickly evaluate many limits at infinity.
- Be cautious with slant asymptotes. For instance, in Example 3c, it's easy for students to apply this faulty logic:

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 6x - 2}{x + 1} = \lim_{x \rightarrow \infty} \frac{2x + 6 - \frac{2}{x}}{1 + \frac{1}{x}} = \lim_{x \rightarrow \infty} (2x + 6),$$

which implies that $y = 2x + 6$ and $h(x) = \frac{2x^2 + 6x - 2}{x + 1}$ behave the same way as x gets large, and therefore $y = 2x + 6$ is an asymptote for the graph of h (this is incorrect). The proper method for finding a slant asymptote is to perform long division.

- This section offers another opportunity to include informal notation for limits that can be very useful for students. For example, limits where the numerator approaches any constant L and where the denominator approaches $\pm\infty$ can be labeled " L/∞ ," the value of which is 0. The indeterminate form ∞/∞ also makes an appearance in this section; we leave it to you to decide whether to draw attention to this form at this point in the course.

Interactive Figures

- Figures 2.31–2.33 illustrate horizontal asymptotes.
- Figures 2.34–2.35 illustrate infinite limits as x approaches infinity.
- Figures 2.36–2.38 illustrate the behavior of rational functions as x approaches infinity.
- Figure 2.39 illustrates the behavior of an algebraic function as x approaches infinity.

Connections

- Horizontal asymptotes are briefly mentioned in Section 1.2.
- It may be helpful to introduce notation that will be used later in the text (Section 4.7, L'Hôpital's Rule); there are ample opportunities in this section to point out the indeterminate form ∞/∞ .
- The absolute value function (and its piecewise definition) is covered in Section 1.2; the subtleties of Example 4 require a good understanding of $|x|$.
- Exercises 50 and 51 introduce the hyperbolic sine and cosine functions. Assign these if you intend to cover hyperbolic functions later in the course. Note that hyperbolic functions are covered in the Guided Projects and in Appendix C.

Additional Activities

Guided Project: *Fixed Point Iteration* explores the idea of the limit of a sequence. It provides (and asks for) illustrations of a limit at infinity in the context of solving equations, and is particularly well suited to visual learners.

Here is another math minute activity that is intended for use after students have mastered the material in this section and Section 2.4. If you introduced the notations $L/0^\pm$, L/∞ , and ∞/∞ , encourage students to identify the form of each limit before coming to an answer.

Evaluate the following limits by inspection:

$$1. \lim_{x \rightarrow 3^+} \frac{x+2}{x-3}$$

$$2. \lim_{x \rightarrow \infty} \frac{x+2}{x-3}$$

$$3. \lim_{x \rightarrow \infty} \frac{x^2+2}{x-3}$$

$$4. \lim_{x \rightarrow \infty} \frac{2}{x-3}$$

$$3. \lim_{x \rightarrow \infty} \frac{4}{e^{-x}}$$

$$6. \lim_{x \rightarrow 0} \frac{e^x}{1 - \cos x}$$

$$7. \lim_{x \rightarrow 0^-} \frac{e^x}{x}$$

$$8. \lim_{x \rightarrow \infty} (4 - 3x^3)$$

$$9. \lim_{x \rightarrow \infty} \frac{3x^3 + 2x^2 - 5}{4x^3 + 3x}$$

ANSWERS:

$$1. \infty$$

$$2. 1$$

$$3. \infty$$

$$4. 0$$

$$5. \infty$$

$$6. \infty$$

$$7. -\infty$$

$$8. -\infty$$

$$9. 3/4$$

Section 2.5 Quick Quiz

Answer the following multiple choice questions by circling the correct response.

- If $f(10) = 4.9$, $f(100) = 4.99$, $f(1000) = 4.999$, $f(10000) = 4.9999$ and so forth, then the most plausible conclusion is that
 - $\lim_{x \rightarrow 5} f(x) = \infty$.
 - $\lim_{x \rightarrow \infty} f(x) = 5$.
 - $\lim_{x \rightarrow -\infty} f(x) = 5$.
- Evaluate $\lim_{x \rightarrow \infty} \frac{2x}{x+1}$.
 - 2
 - $\frac{1}{2}$
 - 0
- Evaluate $\lim_{x \rightarrow \infty} \frac{2x}{x^2+1}$.
 - 2
 - $\frac{1}{2}$
 - 0
- As $x \rightarrow \infty$, suppose $f(x) \rightarrow 5$, $g(x) \rightarrow -2$, and $h(x) \rightarrow \infty$. Then
 - $\lim_{x \rightarrow \infty} \frac{f(x)g(x)}{h(x)} = \infty$.
 - $\lim_{x \rightarrow \infty} \frac{f(x)g(x)}{h(x)} = -\infty$.
 - $\lim_{x \rightarrow \infty} \frac{f(x)g(x)}{h(x)} = 0$.
- The vertical and horizontal asymptotes of $f(x) = \frac{x^2}{2x+1}$ are
 - $x = -\frac{1}{2}$ and $y = \frac{1}{2}$.
 - $x = -\frac{1}{2}$ and no horizontal asymptote.
 - no vertical asymptote and $y = \frac{1}{2}$.
- Evaluate $\lim_{x \rightarrow \infty} 2^{-x}$.
 - 0
 - ∞
 - 2
- Evaluate $\lim_{x \rightarrow \infty} (2 \ln x)$.
 - 0
 - ∞
 - $\ln 2$
- The end behavior of the function $f(x) = \tan^{-1} x$ may be summarized as follows: f has
 - no horizontal asymptotes.
 - one horizontal asymptote.
 - two horizontal asymptotes.
- The population of a culture of bacteria is given by $p(t) = \frac{1200e^{0.05t}}{2e^{0.05t} + 100}$. What can you conclude?
 - The population grows without bound.
 - The population reaches a steady state level of 0.
 - The population reaches a steady state level of 600.
- As $x \rightarrow \infty$, the function $f(x) = e^{-2x} \sin 2x$
 - approaches 1.
 - approaches 0.
 - grows without bound.



Section 2.6 Continuity

Overview

A standard treatment of continuity is offered, with the important Intermediate Value Theorem given at the end of the section.

Lecture Support Notes

The authors made a conscious decision to avoid defining the phrase “continuous function,” which is usually taken to mean a function continuous on its domain. Rather, we are careful to claim that a function is continuous either at a point, or on an interval (occasionally specified only as the domain of the function in question, which of course could be a collection of intervals). The reason behind this decision: It is correct to say that $f(x) = 1/x$ is a continuous function, and yet it has a discontinuity at $x = 0$. Avoiding this apparent inconsistency in terminology is easier on students who are encountering continuity for the first time.

- Show that the definition of continuity allows the use of direct substitution when evaluating $\lim_{x \rightarrow a} f(x)$, provided f is continuous at a .
- Figure 2.44 (and its corresponding function) is useful to illustrate why we need a precise definition for continuity.
- Establish the fact that all the standard families of functions (polynomial, rational, trigonometric, etc.) are continuous on their domains.
- Cover Theorem 2.11; it is used to evaluate limits of the composition of functions.

Interactive Figures

- Figure 2.44 zooms in on a piecewise function to illustrate continuity at a point.
- Figure 2.47 illustrates left-continuity and right-continuity.
- Figure 2.48 illustrates intervals of continuity.
- Figure 2.49 illustrates continuity near roots.
- Figures 2.53–2.54 illustrate the Intermediate Value Theorem.

Connections

- We assume that the exponential functions (and thus logarithmic functions) are continuous on their domains; the details supporting this assumption are given in Section 6.7.
- Theorem 2.13 is important in the early transcendental curriculum (see discussion under *Connections* in Section 2.3). It is used to establish that the inverse of a differentiable (and invertible) function is also differentiable (Section 3.9). This, in turn, helps to establish the derivative of the logarithmic functions.
- The Intermediate Value Theorem is used in subsequent sections to prove additional results (e.g. the Mean Value Theorem for Integrals in Section 5.4).
- The property of continuity is the justification for the method of substitution used in previous sections to evaluate limits.

Additional Activities

The Intermediate Value Theorem can be used to zero in on a root of an equation using the bisection method. For example, consider the function $f(x) = e^x + x$, which is continuous on $(-\infty, \infty)$. Because $f(-1) < 0$, and $f(0) > 0$, we know $f(x) = 0$ has at least one solution on $[-1, 0]$, though we can't solve the equation using standard techniques from algebra. A graph of the function shows that there is only one solution on $[-1, 0]$ (this fact can be proved by noting that $f'(x) > 0$ on $[-1, 0]$ —in fact, on $(-\infty, \infty)$ —but students aren't ready for this argument at the moment, and therefore a graph is the best we can do). One strategy for finding the root is to evaluate f with a calculator at the midpoint of $[-1, 0]$, and then use that information to determine a new interval on which $f(x) = 0$ is guaranteed to have a solution. Because $f(-0.5) > 0$, we know that the solution lies in $[-1, -0.5]$. Continuing in this manner, the solution can be determined up to the accuracy of the calculating device used, albeit with a great deal of tedium. A better approach is to discuss with your students how the strategy might be implemented, and then to carry it out by writing a program (on a TI-83, using *Excel*, *Mathematica*, or whatever technology you prefer).

The following table can be used to find the root of $f(x) = 0$ using an *Excel* spreadsheet. If you intend for your students to understand the algorithm, you will need to explain how the *Excel* function IF behaves (alternatively, you could explain that you've created the code necessary to carry out the bisection method). Select cells A3 through E3, drag your mouse down to highlight as many rows as you like, and then choose *Edit* > *Fill* > *Down* (we assume some familiarity with *Excel*; it doesn't take much to learn it well enough to include this activity in class).

	A	B	C	D	E
1	Left endpoint	Right endpoint	New midpoint	$f(\text{left endpoint})$	$f(\text{right endpoint})$
2	-1	0	$=(A2+B2)/2$	$=EXP(A2)+A2$	$=EXP(B2)+B2$
3	$=IF((EXP(C2)+C2)>0,A2,C2)$	$=IF((EXP(C2)+C2)>0,C2,B2)$	$=(A3+B3)/2$	$=EXP(A3)+A3$	$=EXP(B3)+B3$

The values in columns A and B converge to the root of $f(x) = 0$. Columns D and E aren't entirely necessary, though they provide a check on whether the values of f at the endpoints are as they should be with regard to sign and value (column D should contain negative values, column E positive values, and both should appear to converge to 0).

The next spreadsheet solves $g(x) = \cos x - x = 0$ on the interval $[0, 1]$. It assumes we already know $g(0) > 0$ and $g(1) < 0$, and that there is only one solution on this interval (this can be proved by looking at $g'(x)$, or by showing students a graph).

	A	B	C	D	E
1	Left endpoint	Right endpoint	New midpoint	$g(\text{left endpoint})$	$g(\text{right endpoint})$
2	0	1	$=(A2+B2)/2$	$=COS(A2)-A2$	$=COS(B2)-B2$
3	$=IF((COS(C2)-C2)>0,C2,A2)$	$=IF((COS(C2)-C2)>0,B2,C2)$	$=(A3+B3)/2$	$=COS(A3)-A3$	$=COS(B3)-B3$

Notice the subtle difference between the formula in cell A3 given here, and the formula in cell A3 from the preceding example: In the IF statement, the order of C2 and A2 has been switched (this transposition also occurs in cell B3). This is necessary because f is increasing on $[0, 1]$, whereas g is decreasing on $[0, 1]$. These facts affect how the new endpoints should be chosen for the next step.

These two examples alone provide ample material for students to wrestle with, particularly if you ask them to create a spreadsheet (or write a program) on their own.

Section 2.6 Quick Quiz

Answer the following multiple choice questions by circling the correct response.

- The graph of $y = f(x)$ is continuous on
 (a) $[0,1]$. (b) $[0,1)$. (c) $(0,1]$.
- The graph of $y = f(x)$ is continuous on
 (a) $(1, 2)$. (b) $[1, 2)$. (c) $(1, 2]$.
- The graph of $y = f(x)$ is continuous on
 (a) $(3,5]$. (b) $[3,5)$. (c) $[3,5]$.

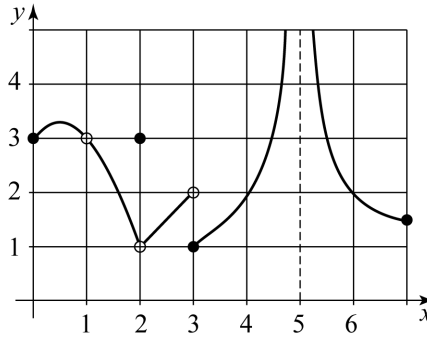


Figure for Exercises 1–3

- If $\lim_{x \rightarrow 2^-} f(x) = 4$, $\lim_{x \rightarrow 2^+} f(x) = 5$, and $f(2) = 5$, then f is continuous
 (a) at $x = 2$. (b) from the left at $x = 2$. (c) from the right at $x = 2$.
- The function $\frac{x}{x^2 - 1}$ is continuous for all values of x except
 (a) -1 . (b) 1 . (c) -1 and 1 .
- If $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 3x - 2 & \text{if } x \geq 1 \end{cases}$, then f is continuous
 (a) at $x = 1$. (b) from the left at $x = 1$. (c) from the right at $x = 1$.
- If $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ a & \text{if } x \geq 1 \end{cases}$, then f is continuous at $x = 1$ if
 (a) $a = 2$. (b) $a = 1$. (c) $a = 3$.
- If $\lim_{x \rightarrow 5^-} f(x) = 10$, $\lim_{x \rightarrow 5^+} f(x) = 12$, and f is continuous from the right at $x = 5$, then
 (a) $f(5) = 10$. (b) $f(5) = 12$. (c) $f(5)$ is undefined.
- The largest interval on which the function $\sqrt{25 - x^2}$ is continuous is
 (a) $[-10, 10]$. (b) $[-5, 0]$. (c) $[-5, 5]$.
- Which of the following functions is continuous for all values of x ?
 (a) $\sqrt{3x^2 - 2}$. (b) $\frac{5x}{|x^8 - 1|}$. (c) $\frac{5x}{x^8 + 1}$.



Section 2.7 Precise Definitions of Limits

Overview

The ε - δ definition of a limit is presented, allowing rigorous proofs of limits and limit laws.

Lecture Support Notes

This optional section is challenging for students. Depending on your students, you might want to keep the technical details to a minimum by focusing on limits of linear functions or piecewise linear functions (Examples 1, 3, and 4 provide a nice progression from the idea of a rigorous limit to the proof of a limit). Once students grasp the concept of a limit proof, prove at least one of the limit laws (Example 5). Assigning Exercise 28 helps to put the concept of continuity on a solid foundation.

Interactive Figures

- Figures 2.57–2.59 introduce how δ and ε relate in moving toward a precise definition of the limit.
- Figure 2.60 illustrates how δ and ε relate in the precise definition of the limit.
- Figures 2.64–2.65 illustrate how to find symmetric intervals.
- Figure 2.66 illustrates the precise definition of an infinite limit.

Connections

If you intend to introduce more formality into the idea of a limit later in the calculus curriculum (e.g. Sections 8.2 and 12.3), it may be worth covering at least the basic ideas of this section.

Additional Activities

If you want to add more material to this section, Exercises 50–53 deal with limits at infinity (both finite and infinite). Using the definition given in Exercises 50–51, discuss the proof of $\lim_{x \rightarrow \infty} \frac{x+1}{x} = 1$ (choose $N = 1/\varepsilon$) with your students to prepare them for the exercises in the text.

Section 2.7 Quick Quiz

Answer the following multiple choice questions by circling the correct response.

- If $0 < |x - 1| < 4$, then $x \neq 1$ and
 - $3 < x < 6$.
 - $-3 < x < 5$.
 - $-5 < x < 3$.
- If $x \neq 9$ and $8 < x < 10$, then
 - $0 < |x - 10| < 1$.
 - $0 < |x - 8| < 1$.
 - $0 < |x - 9| < 1$.
- For all $\varepsilon > 0$, suppose there is a $\delta > 0$ such that $|f(x) - 3| < \varepsilon$ whenever $0 < |x - 1| < \delta$. Then
 - $\lim_{x \rightarrow 3} f(x) = 1$.
 - $\lim_{x \rightarrow 1} f(x) = 3$.
 - $\lim_{x \rightarrow 3} f(x) = f(1)$.
- For all $\varepsilon > 0$, suppose there is a $\delta > 0$ such that $|f(x) - 5| < \varepsilon$ whenever $0 < |x - 2| < \delta$. Then
 - $\lim_{x \rightarrow 5} f(x) = 2$.
 - $\lim_{x \rightarrow 5} f(x) = f(5)$.
 - $\lim_{x \rightarrow 2} f(x) = 5$.
- Suppose $|f(x) - 3| < 1$ whenever $0 < |x - 5| < \frac{1}{2}$. Then it is also true that $|f(x) - 3| < 1$ whenever $0 < |x - 5| < \delta$ for which of the following values of δ ?
 - $\frac{2}{3}$
 - $\frac{3}{4}$
 - $\frac{1}{3}$
- Suppose $\lim_{x \rightarrow 10} f(x) = 7$ and $|f(x) - 7| < 0.25$ whenever $9.75 < x < 11$. Then $|f(x) - 7| < 0.25$ whenever $|x - 10| < \delta$ if δ equals
 - 0.25.
 - 0.5.
 - .75.
- Suppose $\lim_{x \rightarrow 2} f(x) = \infty$. Then for every integer $N > 0$ there is a $\delta > 0$ such that
 - $f(x) > N$ if $x - 2 < \delta$.
 - $|x - 2| < \delta$ if $f(x) > N$.
 - $f(x) > N$ if $0 < |x - 2| < \delta$.

Chapter 2 Key Terms and Concepts

Informal limit definitions (p. 56)
 Relationship between one-sided and two-sided limits (Theorem 2.1)(p. 59)
 Limits of linear functions (Theorem 2.2) (p. 65)
 Limit laws (Theorem 2.3) (p. 66)
 Limits of polynomials and rational functions (Theorem 2.4) (p. 67)
 Squeeze Theorem (Theorem 2.5) (p. 71)
 Definition of various infinite limits (p. 77)
 Vertical asymptotes (p. 78)
 Definition of various limits at infinity (p. 85)
 End behavior (p. 86)
 End behavior of polynomials (Theorem 2.6) (p. 87)
 End behavior of rational functions (Theorem 2.7) (p. 89)
 End behavior of logarithmic and exponential functions (Theorem 2.8) (p. 91)
 Continuity at a point (p. 95)
 Continuity check list (p. 95)
 Continuity of sums, products, quotients (Theorem 2.9) (p. 96)
 Common continuous functions (Theorem 2.10) (p. 97)
 Continuity of composite functions (Theorem 2.11) (p. 97)
 Left-continuous and right-continuous functions (p. 98)
 Continuity on an interval (p. 98)
 Continuity of functions with roots (Theorem 2.12) (p. 99)
 Continuity of inverse functions (Theorem 2.13) (p. 100)
 Continuity of trigonometric, logarithmic, and exponential functions (Theorem 2.14) (p. 101)
 Intermediate Value Theorem (Theorem 2.15) (p. 102)
 Precise definition of a limit (p. 109)
 Precise definitions of two-sided infinite limits (p. 113)

Chapter 2 Review Questions

1. Give an informal explanation of the meaning of $\lim_{x \rightarrow a} f(x) = L$.
2. State the formal definition of $\lim_{x \rightarrow a} f(x) = L$.
3. Suppose $\lim_{x \rightarrow 2} f(x) = 5$. Suppose a value of $\delta = \frac{1}{4}$ satisfies $|f(x) - 5| < \frac{1}{2}$ whenever $0 < |x - 2| < \delta$. Describe other values of δ that satisfy this condition.
4. If $\lim_{x \rightarrow a} f(x)$ exists, what is the relationship between $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$?
5. Explain why $\lim_{x \rightarrow 1^+} \sqrt{x-1}$ exists but $\lim_{x \rightarrow 1^-} \sqrt{x-1}$ does not.
6. Explain why $\lim_{x \rightarrow 1} \frac{x^2 - 7x + 6}{x - 1} = \lim_{x \rightarrow 1} (x - 6)$ even though $\frac{x^2 - 7x + 6}{x - 1} \neq x - 6$ when $x = 1$.
7. Are $x = 1$ and $x = 3$ both vertical asymptotes of $\frac{x-1}{x^2 - 4x + 3}$? Explain.

8. For a rational function $\frac{p(x)}{q(x)}$, where p and q are polynomials, how do you determine the location of the vertical asymptotes?
9. How do you calculate the value of $\lim_{x \rightarrow a} p(x)$ if p is a polynomial?
10. Use limits to explain what it means for a function f to be continuous from the left at $x = a$ and then continuous from the right at $x = a$.
11. Give an informal explanation of the meaning of $\lim_{x \rightarrow a} f(x) = \infty$.
12. For a particular function f , assume $f(1) = -1$ and $f(3) = 1$, but suppose there is no value of c in the interval $[1, 3]$ for which $f(c) = 0$. Is f continuous on $[1, 3]$? Explain your reasoning and support your answer with a possible graph of $y = f(x)$.
13. Suppose $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \infty$ and $\lim_{x \rightarrow a} f(x) = -3$. Find $\lim_{x \rightarrow a} g(x)$.

Chapter 2 Test Bank Exercises

1–4. Evaluating limits Evaluate the following limits.

$$1. \lim_{x \rightarrow 1} 5(3x^2 + 6x + 1)$$

$$2. \lim_{x \rightarrow -2} [(x^3 + 5x)(4x + 3)]$$

$$3. \lim_{x \rightarrow -3} \frac{4x}{x + 9}$$

$$4. \lim_{w \rightarrow -2} \frac{w^3 - w}{2w^2 + 3w}$$

5–11. When direct substitution fails Evaluate the following limits.

$$5. \lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a}$$

$$6. \lim_{x \rightarrow a} \frac{x^2 - a^2}{a - x}$$

$$7. \lim_{x \rightarrow a} \frac{x^2 - a^2 + (x - a)^{100}}{x - a}$$

$$8. \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x - x^3}$$

$$9. \lim_{x \rightarrow 3} \frac{\frac{1}{x+2} - \frac{1}{5}}{x - 3}$$

$$10. \lim_{x \rightarrow 100} \frac{10 - \sqrt{x}}{x - 100}$$

$$11. \lim_{x \rightarrow 3} \frac{(-2x + 4)^2 - 4}{3 - x}$$

12–13. One-sided limits

$$12. \text{ Let } g(x) = \begin{cases} 4x - 10 & \text{if } x < 5 \\ \frac{x^2 - 5}{x - 5} & \text{if } x > 5 \end{cases}. \text{ Compute the following limits when they exist.}$$

$$a. \lim_{x \rightarrow 5^-} g(x)$$

$$b. \lim_{x \rightarrow 5^+} g(x)$$

$$c. \lim_{x \rightarrow 5} g(x)$$

$$13. h(x) = \begin{cases} x^2 - 3x - 6 & \text{if } x < -2 \\ \frac{4 - x^2}{x + 2} & \text{if } x > -2 \end{cases}. \text{ Compute the following limits when they exist.}$$

$$a. \lim_{x \rightarrow -2^-} h(x)$$

$$b. \lim_{x \rightarrow -2^+} h(x)$$

$$c. \lim_{x \rightarrow -2} h(x)$$

14–17. Evaluating limits Evaluate the following limits.

$$14. \lim_{w \rightarrow 4} \frac{16 - w^2}{w^3 - 7w^2 + 12w}$$

$$15. \lim_{t \rightarrow 2} \left(\frac{t}{t^2 - 4} - \frac{2}{t^2 - 4} \right)$$

$$16. \lim_{x \rightarrow 3} \frac{\sqrt{3x - 3}}{x - 3}$$

17. Limit of the radius of a sphere A sphere with a volume of V has a radius given by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Find

$$\lim_{V \rightarrow 0^+} r(V) \text{ and interpret your result.}$$

18–19. Determining infinite limits analytically Determine the following limits, if possible.

$$18. \lim_{x \rightarrow 5^-} \frac{x-6}{x-5}$$

$$19. \lim_{x \rightarrow -2^+} \frac{x^2 + x - 2}{x + 2}$$

20–21. Horizontal asymptotes Evaluate $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$. State the horizontal asymptote(s) of f .

$$20. f(x) = \frac{6000x^9 - 9x + 8}{-6x^{10} + 2x}$$

$$21. f(x) = \frac{16x^5 + 13x^3 + 1}{\sqrt{64x^{10} + 10x^8 + 2x^4}}$$

22–23. Horizontal and Vertical Asymptotes

a. Find the horizontal and vertical asymptotes of the following functions. Evaluate the appropriate limits to justify your answers.

b. Use a graphing utility to graph the function. Then sketch a graph of the function by hand, correcting any errors appearing in the computer-generated graph.

$$22. f(x) = \frac{x^2}{\sqrt{x^4 + 1}}$$

$$23. f(x) = \frac{x^2 + 3x}{3x^3 + 5x - 8}$$

24–25. Determining continuity at a point Determine whether the following functions are continuous at $x = a$ using the Continuity Check List to justify your answer.

$$24. f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{if } x \neq 1; \\ 2 & \text{if } x = 1 \end{cases}; \quad a = 1$$

$$25. f(x) = \begin{cases} \frac{x^2 - 4x + 3}{x - 3} & \text{if } x \neq 3; \\ 2 & \text{if } x = 3 \end{cases}; \quad a = 3$$

26. Determining an unknown constant Determine the value of the constant a for which the function

$$g(x) = \begin{cases} x^2 + 5x + 2 & \text{if } x \leq 1 \\ 3x + a & \text{if } x > 1 \end{cases} \text{ is continuous at } x = 1.$$

$$27. \text{Determining unknown constants} \text{ Let } g(x) = \begin{cases} x^2 - 2x + a & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ bx + 2 & \text{if } x > 1 \end{cases}$$

a. Determine values of a and b for which g is continuous at $x = 1$.

b. For what values of a and b is g continuous from the left at $x = 1$, but not continuous from the right at $x = 1$?

c. For what values of a and b is g continuous from the right at $x = 1$, but not continuous from the left at $x = 1$?



Chapter 2 Answer Section

Quick Quiz 2.1 Answers:

1a 2c 3b 4a 5a 6a

Quick Quiz 2.2 Answers:

1b 2a 3c 4c 5c 6b 7c 8a 9c 10b

Quick Quiz 2.3 Answers:

1c 2c 3c 4b 5c 6b 7b 8a 9c 10b

Quick Quiz 2.4 Answers:

1a 2a 3c 4a 5c 6c 7a 8a

Quick Quiz 2.5 Answers:

1b 2a 3c 4c 5b 6a 7b 8c 9c 10b

Quick Quiz 2.6 Answers:

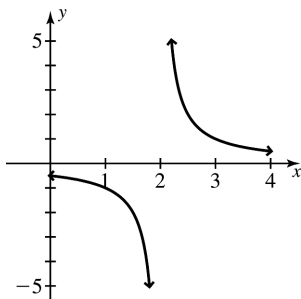
1b 2a 3c 4c 5c 6c 7a 8b 9c 10c

Quick Quiz 2.7 Answers:

1b 2c 3b 4c 5c 6a 7c

Chapter 2 Review Question Answers

- The value of the function tends to L as x tends to a .
- Let f be a function defined on an open interval containing a ; f need not be defined at the point a . The limit $\lim_{x \rightarrow a} f(x) = L$ means that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
- Any positive value of δ will satisfy this condition provided that it is less than $\frac{1}{4}$.
- They must exist and be equal.
- The function $\sqrt{x-1}$ is defined only when $x \geq 1$, which occurs only in the right-hand limit.
- This is because the limit asks how the function behaves as x gets near 1, not when $x = 1$.
- No; the left- and right-hand limits of $\frac{x-1}{x^2-4x+3}$ as x approaches 1 do not approach $\pm\infty$ (a requirement for a vertical asymptote), whereas the limits associated with $x = 3$ do approach $\pm\infty$.
- The vertical asymptotes are given by the roots of q that are not also roots of p .
- Evaluate p at $x = a$.
- Continuity from the left means $\lim_{x \rightarrow a^-} f(x) = f(a)$. Continuity from the right means $\lim_{x \rightarrow a^+} f(x) = f(a)$.
- We can make $f(x)$ as large as we wish by making x sufficiently close to a .
- No, if $f(x)$ were continuous on $[1, 3]$ then $f(x)$ would attain every value between -1 and 1 .



- As x goes to a , $g(x)$ gets closer to 0 from the negative side.

Chapter 2 Test Bank Answers

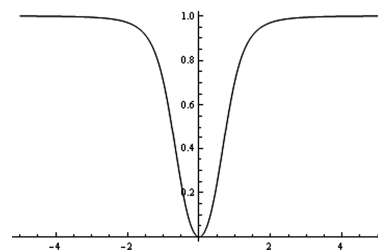
1. 50 2. 90 3. -2 4. -3
5. $2a$ 6. $-2a$ 7. $2a$ 8. $\frac{1}{2}$
9. $-\frac{1}{25}$ 10. $-\frac{1}{20}$ 11. -8
12. a. 10 b. 10 c. 10
13. a. 4 b. 4 c. 4
14. -2 15. $\frac{1}{4}$ 16. $\frac{1}{2}$

17. $\lim_{V \rightarrow 0^+} \sqrt[3]{\frac{3V}{4\pi}} = 0$; This result means that as the volume of a sphere gets arbitrarily small, then the radius of the same sphere also approaches zero.

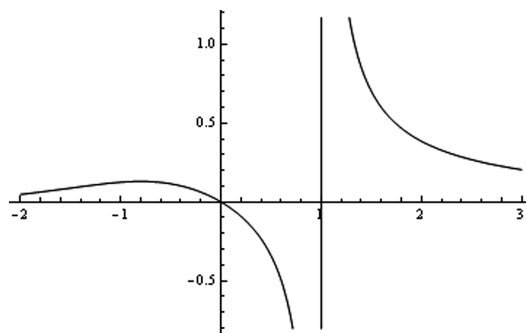
18. ∞ 19. -3 20. $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$; horizontal asymptote at $y = 0$

21. $\lim_{x \rightarrow \infty} f(x) = 2$ and $\lim_{x \rightarrow -\infty} f(x) = -2$; horizontal asymptotes at $y = \pm 2$

22. a. no vertical asymptotes, horizontal asymptote at $y = 1$
b.



23. a. vertical asymptote at $x = 1$, horizontal asymptote at $y = 0$
b. The hand-drawn sketch should show the vertical asymptote at $x = 1$ as a dashed line (rather than the solid line shown in the figure).



24. continuous
25. continuous
26. $a = 5$
27. a. $a = 4, b = 1$
b. $a = 4, b \neq 1$
c. $a \neq 4, b = 1$