Instructor's Solutions Manual

Numerical Analysis

Third Edition

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CHAPTER 1
Solving Equations

EXERCISES 1.1 The Bisection Method

1 (a) Check that $f(x) = x^3 - 9$ satisfies $f(2) = -1$ and $f(3) = 27 - 9 = 18$. By the Intermediate Value Theorem, $f(2)f(3) < 0$ implies the existence of a root between $x = 2$ and $x = 3$.
1 (b) Define $f(x) = 3x^3 + x^2 - x - 5$. Check that $f(1) = -2$ and $f(2) = 21$, so there is a root in $[1, 2]$.
1 (c) Define $f(x) = \cos^2 x - x + 6$. Check that $f(6) > 0$ and $f(7) < 0$. There is a root in $[6, 7]$.

2 (a) $[0, 1]$
2 (b) $[-1, 0]$
2 (c) $[1, 2]$

3 (a) Start with $f(x) = x^3 + 9$ on $[2, 3]$, where $f(2) < 0$ and $f(3) > 0$. The first step is to evaluate $f(\frac{5}{2}) = \frac{53}{8} > 0$, which implies the new interval is $[2, \frac{5}{2}]$. The second step is to evaluate $f(\frac{9}{4}) = \frac{729}{64} - 9 > 0$, giving the interval $[2, \frac{9}{4}]$. The best estimate is the midpoint $x_c = \frac{17}{8}$.
3 (b) Start with $f(x) = 3x^3 + x^2 - x - 5$ on $[1, 2]$, where $f(1) > 0$ and $f(2) < 0$. Since $f(\frac{3}{2}) > 0$, the second interval is $[1, \frac{3}{2}]$. Since $f(\frac{5}{4}) > 0$, the third interval is $[1, \frac{5}{4}]$. The best estimate is the endpoint $x_c = \frac{9}{8}$.
3 (c) Start with $f(x) = \cos^2 x + 6 - x$ on $[6, 7]$, where $f(6) > 0$ and $f(7) < 0$. Since $f(6.5) > 0$, the second interval is $[6.5, 7]$. Since $f(6.75) > 0$, the third interval is $[6.75, 7]$. The best estimate is the midpoint $x_c = 6.875$.

4 (a) 0.875
4 (b) -0.875
4 (c) 1.625

5 (a) Setting $f(x) = x^4 - x^3 - 10$, check that $f(2) = -2$ and $f(3) = 44$, so there is a root in $[2, 3]$.
5 (b) According to (1.1), the error after n steps is less than $(3-2)/2^{n+1}$. Ensuring that the error is less than 10^{-10} requires $\left(\frac{1}{2}\right)^{n+1} < 10^{-10}$, or $2^{n+1} > 10^{10}$, which yields $n > 10/\log_{10}(2) - 1 \approx 32.2$. Therefore 33 steps are required.

6 Bisection Method converges to 0, but 0 is not a root.

COMPUTER PROBLEMS 1.1

1 (a) There is a root in $[2,3]$ (see Exercise 1.1.1). In MATLAB , use the textbook's Program 1.1, `bisect.m`. Six correct decimal places corresponds to error tolerances 5×10^{-7}, according to Def. 1.3. The calling sequence

```
>> f=@(x) x^3-9;
>> xc=bisect(f,2,3,5e-7)
```

returns the approximate root 2.080083.

1 (b) Similar to (a), on interval $[1,2]$. The command

```
>> xc=bisect(@(x) 3*x^3+x^2-x-5,1,2,5e-7)
```

returns the approximate root 1.169726.

1 (c) Similar to (a), on interval $[6,7]$. The command

```
>> xc=bisect(@(x) cos(x)^2+6-x,6,7,5e-7)
```

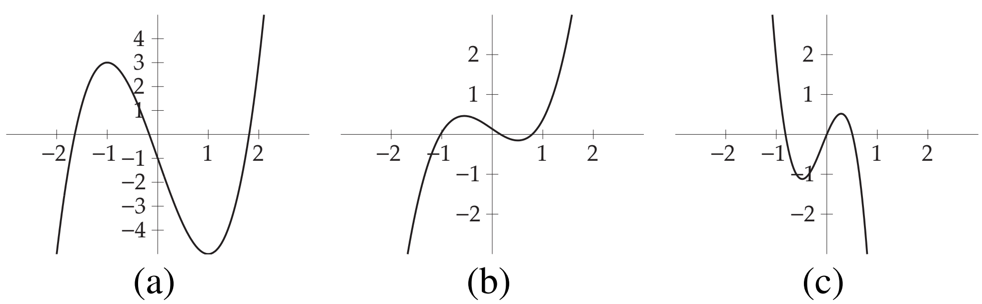
returns the approximate root 6.776092.

2 (a) 0.75487767

2 (b) -0.97089892

2 (c) 1.59214294

3 (a) Plots for parts (a) - (c) are:



In part (a), it is clear from the graph that there is a root in each of the three intervals $[-2,-1]$, $[-1,0]$, and $[1,2]$. The command

```
>> bisect(@(x) 2*x^3-6*x-1,-2,-1,5e-7)
```

yields the first approximate root -1.641783. Repeating for the next two intervals gives the approximate roots -0.168254 and 1.810038.

(b) There are roots in $[-2,-1]$, $[-0.5,0.5]$, and $[0.5,1.5]$. Using `bisect` as in part (a) yields the approximate roots -1.023482, 0.163823, and 0.788942.

(c) There are roots in $[-1.7,-0.7]$, $[-0.7,0.3]$, and $[0.3,1.3]$. Using `bisect` as in part (a) yields the approximate roots -0.818094, 0, and 0.506308.

4 (a) $[1, 2]$, 27 steps, 1.41421356
4 (b) $[1, 2]$, 27 steps, 1.73205081
4 (c) $[2, 3]$, 27 steps, 2.23606798

5 (a) There is a root in the interval $[1, 2]$. Eight decimal place accuracy implies an error tolerance of 5×10^{-9}. The command

```
>> bisect(@(x) x^3-2,1,2,5e-9)
```

yields the approximate cube root 1.25992105 in 27 steps.
5 (b) There is a root in the interval $[1, 2]$. Using `bisect` as in (a) gives the approximate cube root 1.44224957 in 27 steps.
5 (c) There is a root in the interval $[1, 2]$. Using `bisect` as in (a) gives the approximate cube root 1.70997595 in 27 steps.

6 0.785398

7 Trial and error, or a plot of $f(x) = \det(A) - 1000$, shows that $f(-18)f(-17) < 0$ and $f(9)f(10) < 0$. Applying `bisect` to $f(x)$ yields the roots -17.188498 and 9.708299. The backward errors of the roots are $|f(-17.188498)| = 0.0018$ and $|f(9.708299)| = 0.00014$.

8 2.948011

9 The desired height is the root of the function $f(H) = \pi H^2(1 - \frac{1}{3}H) - 1$. Using

```
>> bisect(@(H) pi*H^2*(1-H/3)-1,0,1,0.001)
```

gives the solution 636 mm.

EXERCISES 1.2 Fixed-Point Iteration

1 (a) $\frac{3}{x} = x \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$
1 (b) $x^2 - 2x + 2 = x \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$
1 (c) $x^2 - 4x + 2 = x \Rightarrow x^2 - 5x + 2 = 0 \Rightarrow x = \dfrac{5 \pm \sqrt{17}}{2}$

2 (a) $-1, 2$
2 (b) 2
2 (c) $-1, 0, 1$

3 (a) Check by substitution. For example, $\dfrac{1^3 + 1 - 6}{6(1) - 10} = 1$.
3 (b) Check by substitution.

4 (a) Check by substitution.
4 (b) Check by substitution.

5 (a) No, $g(\sqrt{3}) \neq \sqrt{3}$.
5 (b) Yes, $g(\sqrt{3}) = \dfrac{2\sqrt{3}}{3} + \dfrac{1}{\sqrt{3}} = \sqrt{3}$.
5 (c) No, $g(\sqrt{3}) \neq \sqrt{3}$.
5 (d) Yes, $g(\sqrt{3}) = 1 + \dfrac{2}{\sqrt{3}+1} = \sqrt{3}$.

6 (a) Yes.
6 (b) Yes.
6 (c) No.
6 (d) Yes.

7 (a) $g'(x) = \frac{2}{3}(2x-1)^{-\frac{2}{3}}$, and $|g'(1)| = \frac{2}{3} < 1$. Theorem 1.6 implies that FPI is locally convergent to $r = 1$.
7 (b) $g'(x) = \frac{3}{2}x^2$, and $|g'(1)| = \frac{3}{2} > 1$; FPI diverges from $r = 1$.
7 (c) $g'(x) = \cos x + 1$, and $|g'(0)| = 2 > 1$; FPI diverges from $r = 0$.

8 (a) locally convergent
8 (b) locally convergent
8 (c) divergent

9 (a) Solve $\frac{1}{2}x^2 + \frac{1}{2}x = x$ to find the fixed points $r = 0, 1$. The derivative $g'(x) = x + \frac{1}{2}$. By Theorem 1.6, $|g'(0)| = \frac{1}{2} < 1$ implies that FPI converges to $r = 0$, and $|g'(1)| = \frac{3}{2} > 1$ implies that FPI diverges from $r = 1$.
9 (b) Solve $x^2 - \frac{1}{4}x + \frac{3}{8} = x$ to find the fixed points $r = \frac{1}{2}, \frac{3}{4}$. The derivative $g'(x) = 2x - \frac{1}{4}$. $|g'(\frac{1}{2})| = \frac{3}{4} < 1$ implies that FPI is locally convergent to $r = \frac{1}{2}$. $|g'(\frac{3}{4})| = \frac{5}{4} > 1$ implies that FPI diverges from $r = \frac{3}{4}$.

10 (a) FPI diverges from $3/2$, while 1 is locally convergent
10 (b) FPI diverges from 1, while $-1/2$ is locally convergent

11 (a) There is a variety of answers, obtained by rearranging the equation $x^3 - x + e^x = 0$ to isolate x. For example, $x = x^3 + e^x$, $x = \sqrt[3]{x - e^x}$, $x = \ln(x - x^3)$.
11 (b) As in (a), rearrange $3x^{-2} + 9x^3 = x^2$ to isolate x. For example, $x = \dfrac{3}{x^3} + 9x^2$, $x = \dfrac{1}{9} - \dfrac{1}{3x^4}$, $x = \dfrac{x^5 - 9x^6}{3}$.

12 (a) Faster than Bisection Method
12 (b) FPI diverges from the fixed point 1.2

13 (a) Solving the fixed point equation$x = g(x) = 0.39 - x^2$ yields the fixed points $r = 0.3$ and -1.3.
13 (b) $g'(x) = -2x$ so $|g'(0.3)| = 0.6$ and $|g'(-1.3)| = 2.6$. By Theorem 1.6, Fixed Point Iteration is locally convergent to $r = 0.3$.
13 (c) Convergence by FPI is at the rate $e_{i+1} \approx 0.6e_i$, which is slower than the Bisection Method.

14 All converge to $\sqrt{2}$, from fastest to slowest: (A), (B), (C).

15 Check that $\sqrt{5}$ is a fixed point for each iteration. Then calculate convergence rates for the three iterations. (A) $g'(x) = \dfrac{4}{5} - \dfrac{1}{x^2}, g'(\sqrt{5}) = \dfrac{4}{5} - \dfrac{1}{\sqrt{5}^2} = \dfrac{3}{5}$.
(B) $g'(x) = \dfrac{1}{2} + \dfrac{5}{2}\left(-\dfrac{1}{x^2}\right), g'(\sqrt{5}) = \dfrac{1}{2} - \dfrac{1}{2} = 0$.
(C) $g'(x) = -\dfrac{4}{(x+1)^2}, g'(\sqrt{5}) = -\dfrac{4}{(\sqrt{5}+1)^2} \approx -0.382$.
From fastest to slowest: (B), (C), (A).

16 All converge to $4^{1/3}$, from fastest to slowest: (C), (B), (A).

17 Solving $x^2 = \dfrac{1-x}{2}$ for x results in the two separate equations $g_1(x) = \sqrt{\dfrac{1-x}{2}}$ and $g_2(x) = -\sqrt{\dfrac{1-x}{2}}$. First notice that $g_1(x)$ returns only positive numbers, and $g_2(x)$ only negative. Therefore -1 cannot be a fixed point of $g_1(x)$, and $\frac{1}{2}$ cannot be a fixed point of $g_2(x)$. Check that $g_1(\frac{1}{2}) = \frac{1}{2}$ and $g_1'(x) = -\dfrac{1}{2\sqrt{2-2x}}$. $|g_1'(\frac{1}{2})| = \frac{1}{2} < 1$ confirms that FPI with $g_1(x)$ is locally convergent to $r = \frac{1}{2}$. Likewise, $g_2(-1) = -1$, $g_2'(x) = \dfrac{1}{2\sqrt{2-2x}}$ and $|g_2'(-1)| = \frac{1}{4}$ implies that FPI with $g_2(x)$ is locally convergent to $r = -1$.

18 For a positive number A, consider applying Fixed Point Iteration to $g(x) = (x + A/x)/2$. Note that $g'(\sqrt{A}) = 0$, so FPI is locally convergent to $\sqrt{A}$ by Theorem 1.6. A simple sketch of $y = g(x)$ shows that FPI converges to $\sqrt{A}$ for all positive initial guesses.

19 Define $g(x) = (x + A/x^2)/2$. Since $|g'(\sqrt[3]{A})| = \frac{1}{2} < 1$, FPI is locally convergent to the cube root $\sqrt[3]{A}$.

20 $w = 2/3$

21 (a) Substitute roots and check.
21 (b) $g'(x) = -5 + 15x - \frac{15}{2}x^2$. FPI diverges from all three roots, because $|g'(1 - \sqrt{3/5})| = |g'(1 + \sqrt{3/5})| = 2$ and $|g'(1)| = 2.5$.

22 Initial guesses $0, 1$ and 2 all lead to $r = 1$. Neaby initial guesses cause FPI to move away from the divergent fixed point 1 and oscillate chaotically.

23 The slopes of g at r_1 and r_3 imply that the graph of $y = g(x)$ must pass through the line $y = x$ at $x = r_2$ from below the line to above the line. Therefore $g'(r_2)$ must belong to the interval $(1, \infty)$.

24 $g'(1) = 1$

25 Let x belong to $[a, b]$. By the Mean Value Theorem, $|g(x_0) - r| \leq B|x_0 - r| < |x_0 - r|$. Since r belongs to $[a, b]$, $x_1 = g(x_0)$ does also, and by extension, so does x_2, x_3, etc. Similarly, $|x_1 - r| \leq B|x_0 - r|$ extends to $|x_i - r| \leq B^i|x_0 - r|$, which converges to zero as $i \to \infty$.

26 If $x_1 = g(x_1)$ and $x_2 = g(x_2)$ are both fixed points, then by the Mean Value Theorem, there exists c between x_1 and x_2 for which $x_2 - x_1 = g(x_2) - g(x_1) = g'(c)(x_2 - x_1)$, which implies $g'(c) = 1$, a contradiction.

27 (a) Solving $x - x^3 = x$ yields $x^3 = 0$, or $x = 0$.
27 (b) Assume $0 < x_0 < 1$. Then $x_0^3 < x_0$, and so $0 < x_1 = x_0 - x_0^3 < x_0 < 1$. The same argument implies by induction that $x_0 > x_1 > x_2 > ... > 0$.
27 (c) The limit $L = \lim_{i\to\infty} x_i$ exists because the x_i form a bounded monotonic sequence. Since $g(x)$ is continuous, $g(L) = g(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} g(x_i) = \lim_{i\to\infty} x_{i+1} = L$, so L is a fixed point, and by (a), $L = 0$.

28 (a) $x = x + x^3$ implies $x = 0$
28 (b) If $0 < x_i$, then $x_{i+1} = x_i + x_i^3 = x_i(1 + x_i^2) > x_i$.
28 (c) $g'(0) = 1$, but the x_i move away from $r = 0$.

29 (a) Set $g(x) = \dfrac{x^3 + (c+1)x - 2}{c}$. Then $g'(x) = \dfrac{3x^2 + (c+1)}{c}$, and $|g'(1)| = |\dfrac{4+c}{c}| < 1$ for $c < -2$. By Theorem 1.6, FPI is locally convergent to $r = 1$ if $c < -2$.
29 (b) $g'(1) = 0$ if $c = -4$.

30 By Taylor's Theorem, $g(x_i) = g(r) + g'(r)(x_i - r) + g''(c)(x - r)^2/2$, where c is between x_i and r. Thus $e_{i+1} = |r - x_{i+1}| = |g''(c)|(r - x_i)^2/2 = |g''(c)|e_i^2/2$. In the limit, c converges to r.

31 By factoring or the quadratic formula, the roots of the equation are $-\frac{5}{4}$ and $\frac{1}{4}$. Set $g(x) = \frac{5}{16} - x^2$. Using the cobweb diagram of $g(x)$, it is clear that initial guesses in $(-\frac{5}{4}, \frac{5}{4})$ converge to $r_2 = \frac{1}{4}$, and initial guesses in $(-\infty, -\frac{5}{4}) \cup (\frac{5}{4}, \infty)$ diverge to $-\infty$ under FPI. Initial guesses $-\frac{5}{4}$ and $\frac{5}{4}$ limit on $-\frac{5}{4}$.

32 The open interval $(-4/3, 4/3)$ of initial guesses converge to the fixed point $1/3$; the two initial guesses $-4/3, 4/3$ lead to $-4/3$.

33 (a) Choose $a = 0$ and $|b| < 1$, c arbitrary. Since $a = 0$, $r = 0$ is a fixed point, and $g'(x) = b + 2cx$ implies $|g'(0)| = |b| < 1$, so FPI is locally convergent to 0 by Theorem 1.6.
33 (b) Choose $a = 0$ and $|b| > 1$ to make initial guesses move away from the fixed point 0.

COMPUTER PROBLEMS 1.2

1 (a) Define $g(x) = (2x + 2)^{\frac{1}{3}}$, for example. Using the `fpi` code, the command

```
>> x=fpi(@(x) (2*x+2)^(1/3),1/2,20)
```

yields the solution 1.76929235 to 8 correct decimal places.
1 (b) Define $g(x) = \ln(7 - x)$. Using `fpi` as in part (a) returns the solution 1.67282170 to 8 correct decimal places.
1 (c) Define $g(x) = \ln(4 - \sin x)$. Using `fpi` as in part (a) returns the solution 1.12998050 to 8 correct decimal places.

2 (a) 0.75487767
2 (b) -0.97089892
2 (c) 1.59214294

3 (a) Iterate $g(x) = (x + 3/x)/2$ with starting guess 1. After 4 steps of FPI, the results is 1.73205081 to 8 correct places.
3 (b) Iterate $g(x) = (x + 5/x)/2$ with starting guess 1. After 5 steps of FPI, the results is 2.23606798 to 8 correct places.

4 (a) 1.25992105
4 (b) 1.44224957
4 (c) 1.70997595

5 Iterating $g(x) = \cos^2 x$ with initial guess $x_0 = 1$ results in 0.641714 to six correct places after 350 steps. Checking $|g'(0.641714)| \approx 0.96$ verifies that FPI is locally convergent by Theorem 1.6.

6 (a) $-1.641784, -0.168254, 1.810038$
6 (b) $-1.023482, 0.163822, 0.788941$
6 (c) $-0.818094, 0, 0.506308$.

7 (a) Almost all numbers between 0 and 1.
7 (b) Almost all numbers between 1 and 2.
7 (c) Any number greater than 3 or less than -1 will work.

EXERCISES 1.3 Limits of Accuracy

1 (a) The forward error is $|r - x_c| = |0.75 - 0.74| = 0.01$. The backward error is $|f(x_c)| = |4(0.74) - 3| = 0.04$.
1 (b) $FE = |r - x_c| = 0.01$ as in (a). $BE = |f(0.74)| = (0.04)^2 = 0.0016$.
1 (c) $FE = |r - x_c| = 0.01$ as in (a). $BE = |f(0.74)| = (0.04)^3 = 0.000064$.
1 (d) $FE = |r - x_c| = 0.01$ as in (a). $BE = |f(0.74)| = (0.04)^{\frac{1}{3}} = 0.342$.

2 (a) FE $= 0.00003$, BE $= 10^{-4}$
2 (b) FE $= 0.00003$, BE $= 10^{-8}$
2 (c) FE $= 0.00003$, BE $= 10^{-12}$
2 (d) FE $= 0.00003$, BE $= 0.0464$

3 (a) Check derivatives: $f(0) = f'(0) = 0$, $f''(0) = \cos 0 = 1$. The multiplicity of the root $r = 0$ is 2.
3 (b) The forward error is $|r - x_c| = |0 - 0.0001| = 0.0001$. The backward error is $|f(x_c)| = |1 - \cos 0.0001| \approx 5 \times 10^{-9}$.

4 (a) 4
4 (b) FE $= 10^{-2}$, BE $= 10^{-8}$

5 The root of $f(x) = ax - b$ is $r = b/a$. If x_c is an approximate root, the forward error is $FE = |b/a - x_c|$ while the backward error is $BE = |f(x_c)| = |ax_c - b| = |a||\frac{b}{a} - x_c| = |a|FE$. Therefore the backward error is a factor of $|a|$ larger than the forward error.

6 (a) 1
6 (b) Let ϵ be the backward error. By the Sensitivity Formula, the forward error Δr is $\epsilon / f'(A^{1/n}) = \epsilon/(nA^{(n-1)/n})$.

7 (a) $W'(x) = (x-2)\cdots(x-20) + (x-1)(x-3)\cdots(x-20) + \ldots + (x-1)\cdots(x-19)$, so $W'(16) = (16-1)(16-2)\cdots(16-15)(16-17)(16-18)(16-19)(16-20) = 15!4!$
7 (b) For a general integer j between 1 and 20,
$W'(j) = (j-1)(j-2)\cdots(1)(-1)(-2)\cdots(j-20) = (-1)^j(j-1)!(20-j)!$

8 (a) Predicted root $a + \Delta r = a - \epsilon a$
8 (b) Actual root $a/(1+\epsilon) = a - \epsilon a + \epsilon^2 a - \epsilon^3 a + \ldots$

COMPUTER PROBLEMS 1.3

1 (a) Check the derivatives of $f(x) = \sin x - x$ to see that $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) = -\cos 0 = -1$, giving multiplicity 3.
1 (b) `fzero` returns $x_c = -2.0735 \times 10^{-8}$. The forward error is 2.0735×10^{-8} and MATLAB

reports the backward error to be $|f(x_c)| = 0$. This means the true backward error is likely less than machine epsilon.

2 (a) $m = 9$
2 (b) $x_c =$ FE $= 0.0014$, BE $= 0$

3 (a) The MATLAB command

```
>> xc=fzero(@(x) 2*x*cos(x)-2*x+sin(x^3),[-0.1,0.2])
```

returns $x_c = 0.00016881$. The forward error is $|x_c - r| = 0.00016881$ and the backward error is reported by MATLAB as $|f(x_c)| = 0$.
3 (b) The bisection method with starting interval $[-0.1, 0.2]$ stops after 13 steps, giving $x_c = -0.00006103$. Neither method can determine the root $r = 0$ to more than about 3 correct decimal places.

4 (a) $r + \Delta r = 3 - 2.7\epsilon$
4 (b) Predicted root $= 3 - 0.0027 = 2.9973$, actual root $= 2.9973029$

5 To use (1.21), set $f(x) = (x-1)(x-2)(x-3)(x-4)$, $\epsilon = -10^{-6}$ and $g(x) = x^6$. Then near the root $r = 4$, $\Delta r \approx -\epsilon g(r)/f'(r) = 4^6/6 \approx 0.00068267$. According to (1.22), the error magnification factor is $|g(r)|/|rf'(r)| = 4^6/24 \approx 170.7$. `fzero` returns the approximate root 4.00068251, close to the guess 4.00068267 given by (1.21).

6 Actual root $x_c = 14.856$, predicted root $= r + \Delta r = 15 - 0.14 = 14.86$

EXERCISES 1.4 Newton's Method

1 (a) $x_1 = x_0 - (x_0^3 + x_0 - 2)/(3x_0^2 + 1) = 0 - (-2)/(1) = 2$; $x_2 = 2 - (2^3 + 2 - 2)/(3(2^2) + 1) = 18/13$.
1 (b) $x_1 = x_0 - (x_0^4 - x_0^2 + x_0 - 1)/(4x_0^3 - 2x_0 + 1) = 1$; $x_2 = 1$.
1 (c) $x_1 = x_0 - (x_0^2 - x_0 - 1)/(2x_0 - 1) = -1$; $x_2 = -\frac{2}{3}$.

2 (a) $x_1 = 0.8, x_2 = 0.756818$
2 (b) $x_1 = -0.2, x_2 = 0.180856$
2 (c) $x_1 = x_2 = 2$

3 (a) According to Theorem 1.11, $f'(-1) = 8$ implies that convergence to $r = -1$ is quadratic, with $e_{i+1} \approx |f''(-1)/(2f'(-1))|e_i^2 = |-40/(2)(8)|e_i^2 = 2.5e_i^2$; $f'(0) = -1$ implies convergence to $r = 0$ is quadratic, $e_{i+1} \approx 2e_i^2$; $f'(1) = f''(1) = 0$ and $f'''(1) = 12$ implies that convergence to $r = 1$ is linear, $e_{i+1} \approx \frac{2}{3}e_i$.
3 (b) $f'(-\frac{1}{2}) = -27/4$ implies that convergence to $r = -\frac{1}{2}$ is quadratic, with error relationship $e_{i+1} \approx |27/2(-\frac{27}{4})|e_i^2 = 2e_i^2$; $f'(1) = f''(1) = 0$ and $f'''(1) = 18$ implies that convergence to $r = 1$ is linear, $e_{i+1} \approx \frac{2}{3}e_i$.

4 (a) $r = -1/2, e_{i+1} = 1.6e_i^2; r = 3/4, e_{i+1} = \frac{1}{2}e_i$
4 (b) $r = -1, e_{i+1} = \frac{1}{2}e_i; r = 3, e_{i+1} = \frac{1}{2}e_i^2$

5 Convergence to $r = 0$ is quadratic since $f'(0) = -1 \neq 0$, so Newton's Method converges faster than the Bisection Method. Convergence to $r = \frac{1}{2}$ is linear since $f'(\frac{1}{2}) = f''(\frac{1}{2}) = 0$ and $f'''(\frac{1}{2}) = 24$, with $e_{i+1} \approx \frac{2}{3}e_i$. Since $S = \frac{2}{3} > \frac{1}{2}$, Newton's Method will converge to $r = \frac{1}{2}$ slower than the Bisection Method.

6 Many possible answers; for example, $f(x) = xe^{-x}$ with initial guess greater than 1.

7 Computing derivatives, $f'(2) = f''(2) = 0$ and $f'''(2) = 6$ implies that $r = 2$ is a triple root. Therefore Newton's Method does not converge quadratically, but converges linearly and $e_{i+1}/e_i \to \frac{2}{3}$ according to Theorem 1.12.

8 $x_1 = x_0 - (ax_0 + b)/a = -b/a$

9 Since $f'(x) = 2x$, Newton's Method is

$$x_{i+1} = x_i - \frac{x_i^2 - A}{2x_i} = \frac{x_i}{2} + \frac{A}{2x_i} = \frac{x_i + A/x_i}{2}.$$

10 $x_{i+1} = (2x_i + A/x_i^2)/3$

11 The n^{th} root of A is the real root of $f(x) = x^n - A = 0$. Newton's Method applied to the equation is

$$x_{i+1} = x_i - \frac{x_i^n - A}{nx_i^{n-1}} = \frac{n-1}{n}x_i + \frac{A}{nx_i^{n-1}} = \frac{(n-1)x_i + A/x_i^{n-1}}{n}.$$

Since $f'(A^{\frac{1}{n}}) = nA^{\frac{n-1}{n}}$, Theorem 1.11 implies that Newton's Method converges quadratically as long as $A \neq 0$.

12 $x_{50} = 2^{50}$

13 (a) Newton's Method converges quadratically to $r = 2$ since $f'(2) = 8 \neq 0$, and $e_5 \approx f''(2)/(2f'(2))e_4^2 = \frac{3}{4}(10^{-6})^2 = 0.75 \times 10^{-12}$.
13 (b) Since $f'(0) = -4$ and $f''(0) = 0$, Theorem 1.11 implies that $\lim_{i\to\infty} e_{i+1}/e_i^2 = 0$, and no useful estimate of e_5 follows. Essentially, convergence is faster than quadratic. Reverting to the definition of Newton's Method, $x_{i+1} = x_i - \dfrac{x_i^3 - 4x_i}{3x_i^2 - 4} = \dfrac{2x_i^3}{3x_i^2 - 4}$, and because $r = 0$, $e_{i+1} = \left|\dfrac{2e_i^3}{3e_i^2 - 4}\right|$. Substituting $e_4 = 10^{-6}$ yields $e_5 = \left|\dfrac{2 \times 10^{-18}}{3 \times 10^{-12} - 4}\right| \approx 0.5 \times 10^{-18}$.

COMPUTER PROBLEMS 1.4

1 (a) Newton's Method is $x_{i+1} = x_i - (x_i^3 - 2x_i - 2)/(3x_i^2 - 2)$. Setting $x_0 = 1$ yields $x_7 = 1.76929235$ to eight decimal places.
1 (b) Applying Newton's Method with $x_0 = 1$ yields $x_5 = 1.67282170$ to eight places.
1 (c) Applying Newton's Method with $x_0 = 1$ yields $x_3 = 1.12998050$ to eight places.

2 (a) 0.75487767
2 (b) -0.97089892
2 (c) 1.59214294

3 (a) Newton's Method converges linearly to $x_c = -0.6666648$. Subtracting x_c from x_i shows error ratios $|x_{i+1} - x_c|/|x_i - x_c| \approx \frac{2}{3}$, implying a multiplicity 3 root. Applying Modified Newton's Method with $m = 3$ and $x_0 = 0.5$ converges to $x_c = -\frac{2}{3}$.
3 (b) Newton's Method converges linearly to $x_c = 0.166666669$. The error ratios $|x_{i+1} - x_c|/|x_i - x_c| \approx \frac{1}{2}$, implying a multiplicity 2 root. Applying Modified Newton's Method with $m = 2$ and $x_0 = 1$ converges quadratically to $0.166666667 \approx \frac{1}{6}$. In fact, one checks by direct substitution that the root is $r = \frac{1}{6}$.

4 (a) $r = 1, m = 3$
4 (b) $r = 2, m = 2$

5 The volume of the silo is $400 = 10\pi r^2 + \frac{2}{3}\pi r^3$. Solving for r by Newton's Method yields 3.2362 meters.

6 $r = 2.0201$ cm

7 Newton's Method converges quadratically to -1.197624 and 1.530134, and converges linearly to the root 0. The error ratio is $|x_{i+1} - 0|/|x_i - 0| \approx \frac{3}{4}$, implying that $r = 0$ is a multiplicity 4 root. This can be confirmed by evaluating the first four derivatives.

8 0.841069, quadratic convergence; $\pi/3 \approx 1.047198$, linear convergence, $m = 3$; 2.300524, quadratic convergence

9 Newton's Method converges quadratically to 0.8571428571 with quadratic error ratio $M = \lim_{i\to\infty} e_{i+1}/e_i^2 \approx 2.4$, and converges linearly to the root 2 with error ratio $S = \lim_{i\to\infty} e_{i+1}/e_i \approx \dfrac{2}{3}$.

10 -1.381298, quadratic convergence; $-2/3$, linear convergence, $m = 2$; 0.205183, quadratic convergence; $1/2$, quadratic convergence; 1.176116, quadratic convergence

11 Solving the ideal gas law for an initial approximation gives $V_0 = nRT/P = 1.75$. Applying Newton's Method to the non-ideal gas Van der Waal's equation with initial guess $V_0 = 1.75$ converges to $V = 1.701$.

12 initial guess $V_0 = 2.87$, solution $V = 2.66$ L

13 (a) The equation is equivalent to $1 - 3/(4x) = 0$, and has the root $r = \frac{3}{4}$.
13 (b) Newton's Method applied to $f(x) = (1 - 3/(4x))^{\frac{1}{3}}$ does not converge.
13 (c) $f(x)$ is not differentiable at 0.

14 (a) Assume that $h(c) = c, g(c) \neq c$, and $f''(c) = 0$. First note that

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2},$$

implying that $g'(c) = 0$, and therefore that $h'(c) = g'(g(c))g'(c) = 0$. By Theorem 1.6, the fixed point iteration h is locally convergent to c.
14 (b) Define $f(x) = 4x^4 - 6x^2 - 11/4$; then $f'(x) = 16x^3 - 12x$. Set $c = 1/2$. Then

$$g(1/2) = \frac{1}{2} - \frac{f(1/2)}{f'(1/2)} = \frac{1}{2} - \frac{-4}{-4} = -\frac{1}{2}$$

and likewise $g(-1/2) = 1/2$. Now we can verify that $h(1/2) = g(g(1/2)) = g(-1/2) = 1/2$, that $g(1/2) \neq 1/2$, and that $f''(1/2) = 48(1/2)^2 - 12 = 0$, as required.

15 0.6355

16 (a) 0.578 and 1.670
16 (b) 0.644 and 1.767

17 (a) 0.866% per year
17 (b) 0.576% per year

18 (a) 0.02098

EXERCISES 1.5 Root-Finding without Derivatives

1 (a) Applying the Secant Method with $x_0 = 1$ and $x_1 = 2$ yields $x_2 = x_1 - \dfrac{(x_1 - x_0)f(x_1)}{f(x_1) - f(x_0)} = \dfrac{8}{5}$ and $x_3 \approx 1.742268$.
1 (b) Using the Secant Method formula with $x_0 = 1$ and $x_1 = 2$ as in (a) returns $x_2 \approx 1.578707$ and $x_3 \approx 1.660160$.
1 (c) The Secant Method yields $x_2 \approx 1.092907$ and $x_3 \approx 1.119357$.

2 (a) $x_2 = 8/5, x_3 = 1.742268$
2 (b) $x_2 = 1.578707, x_3 = 1.66016$
2 (c) $x_2 = 1.092907, x_3 = 1.119357$

3 (a) Applying IQI with $x_0 = 1$, $x_1 = 2$ and $x_2 = 0$ yields $x_3 = -\frac{1}{5}$ and $x_4 \approx -0.11996018$ from formula (1.37).
3 (b) Applying the IQI formula gives $x_3 \approx 1.75771279$ and $x_4 \approx 1.66253117$.
3 (c) Applying IQI as in (a) and (b) yields $x_3 \approx 1.13948155$ and $x_4 \approx 1.12927246$.

4 10.25 m

5 Setting $A = f(a)$, $B = f(b)$, $C = f(c)$, and $y = 0$ in (1.35) gives

$$\begin{aligned}
P(0) &= \frac{af(b)f(c)}{(f(a)-f(b))(f(a)-f(c))} + \frac{bf(a)f(c)}{(f(b)-f(a))(f(b)-f(c))} \\
&+ \frac{cf(a)f(b)}{(f(c)-f(a))(f(c)-f(b))} \\
&= \frac{a\frac{f(b)-f(c)}{f(a)} + b\frac{f(c)-f(a)}{f(b)} + c\frac{f(a)-f(b)}{f(c)}}{(1-\frac{f(b)}{f(a)})(\frac{f(a)}{f(c)}-1)(1-\frac{f(c)}{f(b)})} \\
&= \frac{as(1-qs)+bqs(r-q)+c(q-1)}{(q-1)(r-1)(s-1)} \\
&= c + \frac{as(1-r)+br(r-q)-c(r^2-qr-rs+s)}{(q-1)(r-1)(s-1)} \\
&= c - \frac{(c-b)r(r-q)+(c-a)s(1-r)}{(q-1)(r-1)(s-1)}.
\end{aligned}$$

7 (a) (A) is the Bisection Method, which cuts uncertainty in half on each step.
(B) Check that $f(2^{1/4}) = 0$ and $f'(2^{1/4}) = (4)2^{3/4} \neq 0$. Therefore the Secant Method converges superlinearly.
(C) $2^{1/4}$ is a fixed point because $g(2^{1/4}) = \dfrac{2^{1/4}}{2} + \dfrac{1}{2^{3/4}} = \dfrac{2^{1/4}+2^{1/4}}{2} = 2^{1/4}$.
Note that $g'(x) = \dfrac{1}{2} - \dfrac{3}{x^4} \Rightarrow g'(2^{1/4}) = \dfrac{1}{2} - \dfrac{3}{(2^{1/4})^4} = \dfrac{1}{2} - \dfrac{3}{2} = -1$.
(D) $2^{1/4}$ is a fixed point because $g(2^{1/4}) = \dfrac{2^{1/4}}{3} + \dfrac{1}{(3)2^{3/4}} = \dfrac{2+1}{(3)2^{3/4}} = 2^{1/4}$.
Note that $g'(x) = \dfrac{1}{3} - \dfrac{3}{3x^4} \Rightarrow g'(2^{1/4}) = \dfrac{1}{3} - \dfrac{1}{(2^{1/4})^4} = \dfrac{1}{3} - \dfrac{1}{2} = -1/6$.
Fastest to slowest: (B), (D), (A); (C) does not converge to $2^{1/4}$.
7 (b) Newton's Method will converge faster than the four above choices.

COMPUTER PROBLEMS 1.5

1 (a) Applying the Secant Method formula shows convergence to the root 1.76929235
1 (b) 1.67282170
1 (c) 1.12998050.

2 (a) 1.76929235
2 (b) 1.67282170
2 (c) 1.12998050

3 (a) Applying formula (1.37) for Inverse Quadratic Interpolation shows convergence to 1.76929235.
3 (b) Similar to part (a). Converges to 1.67282170
3 (c) Similar to part (a). Converges to 1.129998050.

4 -1.381298, superlinear; $-2/3$, linear; 0.205183, superlinear; $1/2$, superlinear; 1.176116, superlinear

5 The MATLAB command

```
>> fzero(@(x) 1/x,[-2,1])
```

converges to zero, although there is no root there.

6 `fzero` fails in both cases because the functions never cross zero