# A FIRST COURSE IN DIFFERENTIAL GEOMETRY
## Woodward and Bolton

# Solutions to exercises

## Chapter 1

**1.1.** A sketch of the astroid is given in Figure 1(a). It is clear that all points in the image of $\boldsymbol{\alpha}$ satisfy the equation of the astroid. Conversely, if $x^{2/3} + y^{2/3} = 1$, then there exists $u \in \mathbb{R}$ such that $(x^{1/3}, y^{1/3}) = (\cos u, \sin u)$. Thus every point of the astroid is in the image of $\boldsymbol{\alpha}$.

Trigonometric identities may be used to show that $\boldsymbol{\alpha}' = (3/2)\sin 2u(-\cos u, \sin u)$, which is zero only when $u$ is an integer multiple of $\pi/2$. The corresponding points of the astroid are the cusps in Figure 1(a).

The required length is

$$\frac{3}{2}\int_0^{\pi/2} \sin 2u \, du = \frac{3}{2} \; .$$


(a) Astroid (b) An epicycloid

Figure 1

**1.2.** A sketch of the trace of an epicycloid is given in Figure 1(b). Trigonometric identities may be used to show that

$$\boldsymbol{\alpha}' = 4r\sin(u/2)\big(\sin(3u/2), \cos(3u/2)\big) \; .$$

So, for $0 \leq u \leq 2\pi$, $|\boldsymbol{\alpha}'| = 4r\sin(u/2)$, and required length is $4r\int_0^{2\pi}\sin(u/2)du = 16r$.

**1.3.** When $r = 1$, a calculation shows that $\boldsymbol{\alpha}' = \tanh u \; \operatorname{sech} u(\sinh u, -1)$, so that, for $u \geq 0$, $\boldsymbol{t} = \operatorname{sech} u(\sinh u, -1)$. It follows that $\boldsymbol{\alpha} + \boldsymbol{t} = (u, 0)$. A sketch of the trace of a tractrix is given in Figure 2(a).

**1.4.** Here, $|\boldsymbol{\alpha}'| = (1 + g'^2)^{1/2}$ and $\boldsymbol{t} = (1 + g'^2)^{-1/2}(1, g')$. Hence $\boldsymbol{n} = (1 + g'^2)^{-1/2}(-g', 1)$. A calculation shows that $\boldsymbol{t}' = g''(1 + g'^2)^{-3/2}(-g', 1)$, so that

$$\frac{d\boldsymbol{t}}{ds} = \frac{1}{|\boldsymbol{\alpha}'|}\boldsymbol{t}' = \frac{g''}{(1 + g'^2)^{3/2}}\boldsymbol{n} \; .$$

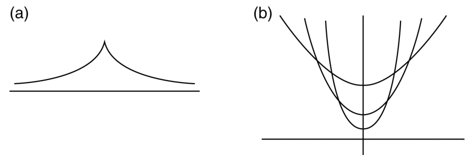


Figure 2: (a) shows a tractrix and (b) shows three catenaries

Hence $\kappa = g''(1+g'^2)^{-3/2}$.

Taking $x(u) = u$, $y(u) = g(u)$ in the formula given in Exercise 1.8 gives the same formula for $\kappa$.

**1.5.** Use the method of Example 2 of §1.3. For $u \geq 0$, $|\boldsymbol{\alpha}'| = \tanh u$ and $\boldsymbol{t} = (\tanh u, -\operatorname{sech} u)$. It follows that $d\boldsymbol{t}/ds = (|\boldsymbol{\alpha}'|)^{-1}\boldsymbol{t}' = \boldsymbol{n}/\sinh u$. Hence $\kappa = \operatorname{cosech} u$.

**1.6.** EITHER: use Exercise 1.4 to show that the curvature of the catenary $\boldsymbol{\alpha}(u) = (u, \cosh u)$ is given by $\kappa = \operatorname{sech}^2 u$,
OR: use the method of Example 2 of §1.3, and proceed as follows:-
$\boldsymbol{\alpha}' = (1, \sinh u)$, so that $|\boldsymbol{\alpha}'| = \cosh u$ and $\boldsymbol{t} = (\operatorname{sech} u, \tanh u)$. Hence $\boldsymbol{n} = (-\tanh u, \operatorname{sech} u)$, and

$$\frac{d\boldsymbol{t}}{ds} = \frac{1}{|\boldsymbol{\alpha}'|}\boldsymbol{t}' = \frac{1}{\cosh^2 u}(-\tanh u, \operatorname{sech} u) = \frac{1}{\cosh^2 u}\boldsymbol{n} \ .$$

Hence $\kappa = \operatorname{sech}^2 u$. A sketch of the traces of three catenaries is given in Figure 2(b).

**1.7.** Differentiating with respect to $u$, we see that, using Serret-Frenet,

$$\boldsymbol{\alpha}_\ell{}' = \boldsymbol{\alpha}' + \ell\boldsymbol{n}' = |\boldsymbol{\alpha}'|(\boldsymbol{t} - \kappa\ell\boldsymbol{t}) = |\boldsymbol{\alpha}'|(1-\kappa\ell)\boldsymbol{t} \ .$$

It follows that $|\boldsymbol{\alpha}_\ell{}'| = |\boldsymbol{\alpha}'|\,|1-\kappa\ell|$ and $\boldsymbol{t}_\ell = \epsilon\boldsymbol{t}$, where $\epsilon = (1-\kappa\ell)/|1-\kappa\ell|$. Hence $\boldsymbol{n}_\ell = \epsilon\boldsymbol{n}$, so, if $s_\ell$ denotes arc length along $\boldsymbol{\alpha}_\ell$, we have

$$\frac{d\boldsymbol{t}_\ell}{ds_\ell} = \frac{1}{|\boldsymbol{\alpha}'|\,|1-\kappa\ell|}\boldsymbol{t}'_\ell = \frac{\epsilon}{|\boldsymbol{\alpha}'|\,|1-\kappa\ell|}\boldsymbol{t}' \ .$$

Using Serret-Frenet, $\boldsymbol{t}' = |\boldsymbol{\alpha}'|\kappa\boldsymbol{n} = |\boldsymbol{\alpha}'|\kappa\epsilon\boldsymbol{n}_\ell$, from which the result follows.

**1.8.** Since $\boldsymbol{\alpha}' = (x', y')$, we have that $|\boldsymbol{\alpha}'| = (x'^2 + y'^2)^{1/2}$. Hence $\boldsymbol{t} = (x', y')/(x'^2+y'^2)^{1/2}$ and $\boldsymbol{n} = (-y', x')/(x'^2+y'^2)^{1/2}$. Hence

$$\boldsymbol{t}' = \frac{\boldsymbol{\alpha}''}{(x'^2+y'^2)^{1/2}} - \frac{\boldsymbol{\alpha}'(x'x''+y'y'')}{(x'^2+y'^2)^{3/2}} \ ,$$

and a short calculation shows that

$$\frac{d\boldsymbol{t}}{ds} = \frac{1}{(x'^2+y'^2)^2}\Big(y'(x''y'-x'y''), x'(x'y''-x''y')\Big)$$
$$= \frac{x'y''-x''y'}{(x'^2+y'^2)^{3/2}}\boldsymbol{n} \ ,$$

and the result follows.

**1.9.** (i) Let $s_{\boldsymbol{\alpha}}$ be arc length along $\boldsymbol{\alpha}$ measured from $u=0$. Since $\boldsymbol{\alpha}' = (1, \sinh u)$ we see that $ds_{\boldsymbol{\alpha}}/du = |\boldsymbol{\alpha}'| = \cosh u$. Hence $s_{\boldsymbol{\alpha}}(u) = \sinh u$ and $\boldsymbol{t}_{\boldsymbol{\alpha}} = (\text{sech } u, \tanh u)$. The result follows from formula (1.9) for the involute.
(ii) The evolute of $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\beta} = \boldsymbol{\alpha} + \frac{1}{\kappa_{\boldsymbol{\alpha}}}\boldsymbol{n}_{\boldsymbol{\alpha}} \ .$$

Here, we have (from Exercise 1.6) that $\kappa_{\boldsymbol{\alpha}} = \text{sech}^2 u$ and $\boldsymbol{n}_{\boldsymbol{\alpha}} = (-\tanh u, \text{sech } u)$. A direct substitution gives the result.

A short calculation shows that $\boldsymbol{\beta}' = 0$ if and only if $u=0$, so this gives the only singular point of $\boldsymbol{\beta}$ (where the curve $\boldsymbol{\beta}$ has a cusp). A sketch of the traces of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is given in Figure 3.

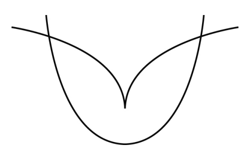

Figure 3: A catenary and its evolute

**1.10.** Let $s_{\boldsymbol{\alpha}}$ denote arc length along $\boldsymbol{\alpha}$ starting at $u=u_0$. Then, using (1.9) and the notation used there, we see that

$$\boldsymbol{\beta}' = \boldsymbol{\alpha}' - s_{\boldsymbol{\alpha}}{}'\boldsymbol{t}_{\boldsymbol{\alpha}} - s_{\boldsymbol{\alpha}}\boldsymbol{t}_{\boldsymbol{\alpha}}{}' = -s_{\boldsymbol{\alpha}}\boldsymbol{t}_{\boldsymbol{\alpha}}{}' \ .$$

It follows that $\boldsymbol{\beta}' = -s_{\boldsymbol{\alpha}}|\boldsymbol{\alpha}'|\kappa_{\boldsymbol{\alpha}}\boldsymbol{n}_{\boldsymbol{\alpha}}$, so the only singular point of $\boldsymbol{\beta}$ is when $s_{\boldsymbol{\alpha}} = 0$, that is at $u = u_0$.

**1.11.** For ease, assume that $\kappa_{\boldsymbol{\alpha}} > 0$, and restrict attention to $u_0 < u_1 < u$. Then, from (1.12), we have that $\kappa_0 = 1/s_0$ and $\kappa_1 = 1/s_1$.

Let $\ell$ be the length of $\boldsymbol{\alpha}$ measured from $\boldsymbol{\alpha}(u_0)$ to $\boldsymbol{\alpha}(u_1)$. Then $\ell = s_0 - s_1 > 0$, so the definition of involute gives that $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_0 + (s_0 - s_1)\boldsymbol{t}_{\boldsymbol{\alpha}} = \boldsymbol{\beta}_0 + \ell\boldsymbol{n}_0$. Hence $\boldsymbol{\beta}_1$ is a parallel curve to $\boldsymbol{\beta}_0$, and

$$\frac{\kappa_0}{|1-\kappa_0\ell|} = \frac{1/s_0}{|1-\ell/s_0|} = \frac{1}{|s_0-\ell|} = \frac{1}{s_1} = \kappa_1 \ .$$

**1.12.** A sketch of the trace of $\boldsymbol{\alpha}$ is given in Figure 4.

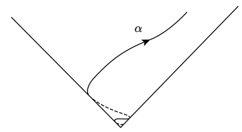


Figure 4: The curve in Exercise 1.12

A short calculation shows that $|\boldsymbol{\alpha}'| = \sqrt{3}e^u$ and

$$\boldsymbol{t} = \frac{1}{\sqrt{3}}(\cos u - \sin u, \sin u + \cos u, 1) \ .$$

When $z = \lambda_0$ we have that $u = \log \lambda_0$, and when $z = \lambda_1$ we have that $u = \log \lambda_1$. So, required length is

$$\sqrt{3}\int_{\log \lambda_0}^{\log \lambda_1} e^u du = \sqrt{3}(\lambda_1 - \lambda_0) \ .$$

Also, using the method of Example 2 of §1.5, we find that

$$\frac{d\boldsymbol{t}}{ds} = \frac{1}{3e^u}(-\sin u - \cos u, \cos u - \sin u, 0) \ ,$$

so that $\kappa = \sqrt{2}/(3e^u)$ and

$$\boldsymbol{n} = \frac{1}{\sqrt{2}}(-\sin u - \cos u, \cos u - \sin u, 0) \ .$$

It then follows that

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} = \frac{1}{\sqrt{6}}(\sin u - \cos u, -\cos u - \sin u, 2) \ ,$$

so that

$$\frac{d\boldsymbol{b}}{ds} = \frac{1}{3\sqrt{2}e^u}(\sin u + \cos u, -\cos u + \sin u, 0) \ .$$

Hence $\tau = -1/(3e^u)$.

**1.13.** Calculations similar to those of Example 2 of §1.5 show that $|\boldsymbol{\alpha}'| = \sqrt{2}\cosh u$, $\boldsymbol{t} = (\tanh u, 1, \operatorname{sech} u)/\sqrt{2}$, $\kappa = (1/2)\operatorname{sech}^2 u$, $\boldsymbol{n} = (\operatorname{sech} u, 0, -\tanh u)$, and $\boldsymbol{b} = (-\tanh u, 1, -\operatorname{sech} u)/\sqrt{2}$. Differentiating one more time, we find that $d\boldsymbol{b}/ds = (1/2)\operatorname{sech}^2 u(-\operatorname{sech} u, 0, \tanh u)$, so that $\tau = -(1/2)\operatorname{sech}^2 u$.

**1.14.** Assume that $\boldsymbol{\alpha}(s)$ is a smooth curve in $\mathbb{R}^3$ parametrised by arc length. If $\boldsymbol{\alpha}$ has zero curvature then $d\boldsymbol{t}/ds = 0$, so that $\boldsymbol{t}$ is a constant unit vector $\boldsymbol{t}_0$, say. Since $d\boldsymbol{\alpha}/ds = \boldsymbol{t}_0$, it follows that $\boldsymbol{\alpha}(s) = s\boldsymbol{t}_0 + \boldsymbol{v}_0$, for some constant vector $\boldsymbol{v}_0$. Hence $\boldsymbol{\alpha}$ is a line. Conversely, if $\boldsymbol{\alpha}$ is the line through $\boldsymbol{v}_0$ in direction of unit vector $\boldsymbol{t}_0$, say, then $\boldsymbol{\alpha}$ may be parametrised as $\boldsymbol{\alpha}(s) = s\boldsymbol{t}_0 + \boldsymbol{v}_0$, and it quickly follows that $\kappa = 0$.

**1.15.** Using Serret-Frenet, we have

$$\boldsymbol{\alpha}' = |\boldsymbol{\alpha}'|\boldsymbol{t} \ , \quad \boldsymbol{\alpha}'' = \kappa|\boldsymbol{\alpha}'|^2\boldsymbol{n} + |\boldsymbol{\alpha}'|'\boldsymbol{t} \ ,$$

so that $\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'' = \kappa|\boldsymbol{\alpha}'|^3\boldsymbol{b}$. Equating the lengths of both sides, we obtain the required formula for $\kappa$. Differentiating one more time, we find that

$$\boldsymbol{\alpha}''' = -\kappa\tau|\boldsymbol{\alpha}'|^3\boldsymbol{b} + \text{ terms involving } \boldsymbol{t} \text{ and } \boldsymbol{n} \ ,$$

so that $(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'').\boldsymbol{\alpha}''' = -\kappa^2\tau|\boldsymbol{\alpha}'|^6$. Using the expression we have just found for $\kappa$ now gives the required expression for $\tau$.

**1.16.** (i) Assume first that $\boldsymbol{\alpha}(u) = (a\cos u, a\sin u, bu)$. The tangent vector to $\boldsymbol{\alpha}(u)$ makes angle $\phi$ with the generating lines where $\cos\phi = b/\sqrt{a^2+b^2}$. Hence $\phi$ is a constant different from 0 and $\pi/2$. We saw in Example 2 of §1.5 that $\boldsymbol{\alpha}(u)$ has non-zero torsion, so it follows that $\boldsymbol{\alpha}(u)$ is a helix with value $(a,0,0)$ when $u=0$.

Conversely, assume that $\boldsymbol{\alpha}(v) = (a\cos\theta(v), a\sin\theta(v), v+c)$ describes a helix with value $(a,0,0)$ when $v=0$. Then $c=0$ and, denoting differentiation with respect to $v$ by $'$, we have

$$\boldsymbol{\alpha}' = (-a\theta'\sin\theta, a\theta'\cos\theta, 1) \ ,$$

so the angle $\phi$ of the tangent vector to the $z$-axis is given by $\cos\phi = (1 + a^2\theta'^2)^{-1/2}$.

Hence, our assumption that $\boldsymbol{\alpha}(v)$ is a helix implies that $\theta'$ is a non-zero constant, so that $\theta = c_1 v + c_2$, where $c_1 \neq 0$ and $c_2$ are constants. The initial condition $\boldsymbol{\alpha}(0) = (a,0,0)$ shows that $c_2$ is an integer multiple of $2\pi$, so if we let $u = c_1 v$ then $\boldsymbol{\alpha}(u) = (a\cos u, a\sin u, u/c_1)$, which is of the required form.

(ii) Assume that $\boldsymbol{\alpha}(s)$ has constant $\kappa$ and $\tau$. It follows immediately from Serret-Frenet that the derivative of $\tau\boldsymbol{t} - \kappa\boldsymbol{b}$ is zero, so that $\tau\boldsymbol{t} - \kappa\boldsymbol{b}$ is constant, $\boldsymbol{X}_0$, say.

Let $a = \kappa/(\kappa^2+\tau^2)$. Then, using Serret-Frenet,

$$\begin{aligned}\frac{d}{ds}(\boldsymbol{\alpha} + a\boldsymbol{n}) &= \boldsymbol{t} + \frac{\kappa}{\kappa^2+\tau^2}(-\kappa\boldsymbol{t} - \tau\boldsymbol{b}) \\ &= \frac{\tau}{\kappa^2+\tau^2}(\tau\boldsymbol{t} - \kappa\boldsymbol{b}) \ .\end{aligned}$$

Hence $\boldsymbol{\alpha} + a\boldsymbol{n}$ has constant rate of change $\tau\boldsymbol{X}_0/(\kappa^2+\tau^2)$, so that

$$\boldsymbol{\alpha} + a\boldsymbol{n} = \boldsymbol{Y}_0 + s\frac{\tau}{\kappa^2+\tau^2}\boldsymbol{X}_0 \ ,$$

for some constant vector $\boldsymbol{Y}_0$. Since $\boldsymbol{n}$ is perpendicular to $\boldsymbol{t}$ and to $\boldsymbol{X}_0$, it follows that $\boldsymbol{\alpha}$ lies on the cylinder $S$ of radius $a$ whose axis of rotation is the line through $\boldsymbol{Y}_0$ in direction $\boldsymbol{X}_0$. Finally, we note that $\boldsymbol{t}.\boldsymbol{X}_0 = \tau$, which is constant, so that $\boldsymbol{\alpha}$ is a helix on $S$.

**1.17.** Assume there is a unit vector $\boldsymbol{X}_0$ such that $\boldsymbol{t}.\boldsymbol{X}_0 = c$, a constant. Then $\boldsymbol{n}.\boldsymbol{X}_0 = 0$, so that $\boldsymbol{X}_0 = c\boldsymbol{t} + c_1\boldsymbol{b}$ for some constant $c_1$. Then $0 = \boldsymbol{X}_0{}' = |\boldsymbol{\alpha}'|(c\kappa + c_1\tau)\boldsymbol{n}$, so that $\kappa/\tau = -c_1/c$ which is constant.

Conversely, if $\kappa/\tau = k$, a constant, the Serret-Frenet formulae show that

$$(\boldsymbol{t} - k\boldsymbol{b})' = |\boldsymbol{\alpha}'|\big(\kappa\boldsymbol{n} - (\kappa/\tau)\tau\boldsymbol{n}\big) = 0 \ ,$$

so that $\boldsymbol{t} - k\boldsymbol{b}$ is constant. The result follows since $\boldsymbol{t}.(\boldsymbol{t} - k\boldsymbol{b}) = 1$.

**1.18.** The assumption on $\boldsymbol{\alpha}$ implies the existence of a smooth function $r(u)$ such that $\boldsymbol{\alpha} + r\boldsymbol{n} = p$. If we differentiate this expression, use Serret-Frenet, and then equate the coefficients of $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ to zero, we find that $r' = 0$ (so that $r$ is a non-zero constant), $\kappa = 1/r$, and $\tau = 0$. The result now follows from Lemma 1 of §1.5 and Example 8 of §1.3.

**1.19.** The given information implies that $\boldsymbol{n}_{\boldsymbol{\alpha}} = \pm\boldsymbol{n}_{\boldsymbol{\beta}}$.
(i) Differentiating $\boldsymbol{t}_{\boldsymbol{\alpha}}.\boldsymbol{t}_{\boldsymbol{\beta}}$ (with respect to $u$), and using Serret-Frenet, we find that

$$(\boldsymbol{t}_{\boldsymbol{\alpha}}.\boldsymbol{t}_{\boldsymbol{\beta}})' = |\boldsymbol{\alpha}'|\kappa_{\boldsymbol{\alpha}}\boldsymbol{n}_{\boldsymbol{\alpha}}.\boldsymbol{t}_{\boldsymbol{\beta}} + |\boldsymbol{\beta}'|\kappa_{\boldsymbol{\beta}}\boldsymbol{t}_{\boldsymbol{\alpha}}.\boldsymbol{n}_{\boldsymbol{\beta}} = 0 \ ,$$

so that $\boldsymbol{t}_{\boldsymbol{\alpha}}.\boldsymbol{t}_{\boldsymbol{\beta}}$ is constant.
(ii) The given information implies the existence of a smooth function $r(u)$ such that $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\boldsymbol{n}_{\boldsymbol{\alpha}}$. Differentiating this, using Serret-Frenet, and taking inner product with $\boldsymbol{n}_{\boldsymbol{\alpha}}$ gives that $r' = 0$. Hence result.

**1.20.** Here, $|\boldsymbol{\alpha}'| = \sqrt{a^2+b^2}$ and $\boldsymbol{t}_{\boldsymbol{\alpha}} = (-a\sin u, a\cos u, b)/\sqrt{a^2+b^2}$. Hence $s_{\boldsymbol{\alpha}}(u) = u\sqrt{a^2+b^2}$, from which it follows that

$$\boldsymbol{\beta}(u) = (a\cos u + au\sin u, a\sin u - au\cos u, 0) \ ,$$

so, in particular, the third component of $\boldsymbol{\beta}$ is zero. The circle of intersection of the plane $z = 0$ with the cylinder $x^2 + y^2 = a^2$ may be parametrised as $\boldsymbol{\gamma}(u) = (a\cos u, a\sin u, 0)$, and the exercise may now be completed using the formula for the involute by noting that $\boldsymbol{\gamma}' = (-a\sin u, a\cos u, 0)$, so that $\boldsymbol{t}_{\boldsymbol{\gamma}} = (-\sin u, \cos u, 0)$ and $s_{\boldsymbol{\gamma}}(u) = au$.

**1.21.** First assume that $\boldsymbol{\alpha}(s)$ lies on a sphere with centre $\boldsymbol{p}$ and radius $r$, or, equivalently, that $(\boldsymbol{\alpha} - \boldsymbol{p}).(\boldsymbol{\alpha} - \boldsymbol{p}) = r^2$. We shall differentiate repeatedly to find an expression for $\boldsymbol{\alpha} - \boldsymbol{p}$ in terms of $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$.

So, differentiate once to find that $(\boldsymbol{\alpha} - \boldsymbol{p}).\boldsymbol{t} = 0$. Differentiating again and using Serret-Frenet, we obtain $(\boldsymbol{\alpha} - \boldsymbol{p}).\boldsymbol{n} = -1/\kappa$. Differentiating this and using Serret-Frenet gives $(\boldsymbol{\alpha} - \boldsymbol{p}).\boldsymbol{b} = -\kappa'/(\tau\kappa^2)$. It now follows that

$$\boldsymbol{\alpha} - \boldsymbol{p} = -\frac{1}{\kappa}\boldsymbol{n} - \frac{\kappa'}{\tau\kappa^2}\boldsymbol{b} \ .$$

The derivative of the left hand side, and hence of the right hand side, of the above equation is $\boldsymbol{t}$. In particular, the coefficient of $\boldsymbol{b}$ of the derivative of the right hand side is zero, which gives the desired relation between $\kappa$ and $\tau$.

Conversely, assume that $\kappa$ and $\tau$ for a regular curve $\boldsymbol{\alpha}(s)$ are related as in the given formula, and let

$$\boldsymbol{p}(s) = \boldsymbol{\alpha} + \frac{1}{\kappa}\boldsymbol{n} + \frac{\kappa'}{\tau\kappa^2}\boldsymbol{b} \ .$$

The given relation between $\kappa$ and $\tau$ may be used to show that $\boldsymbol{p}' = 0$ so that $\boldsymbol{p}$ is constant. It now follows (again by differentiating) that $(\boldsymbol{\alpha}-\boldsymbol{p}).(\boldsymbol{\alpha}-\boldsymbol{p})$ is also constant. Since $\boldsymbol{\alpha}$ is not constant we have that $|\boldsymbol{\alpha}-\boldsymbol{p}| = r$ for some positive constant $r$, so that $\boldsymbol{\alpha}$ lies on the sphere with centre $\boldsymbol{p}$ and radius $r$.

## Chapter 2

**2.1.** The line through $(u, v, 0)$ and $(0, 0, 1)$ may be parametrised by $\boldsymbol{\alpha}(t) = t(u, v, 0) + (1-t)(0, 0, 1)$. This line intersects $S^2(1)$ when $\left|\big(tu, tv, (1-t)\big)\right|^2 = 1$, and a short calculation gives that $t = 0$ or $t = 2/(u^2 + v^2 + 1)$. Since $t = 0$ corresponds to $(0, 0, 1)$, we quickly see that $\boldsymbol{x}(u, v)$ is as claimed.

The formula for $\boldsymbol{F}$ follows from consideration of similar triangles, OR we may use the following method which is similar to the one used in the earlier part of the solution to this exercise. The line through $(x, y, z)$ and $(0, 0, 1)$ may be parametrised by $\boldsymbol{\beta}(t) = t(x, y, z) + (1-t)(0, 0, 1)$. For $z \neq 1$, this line cuts the $xy$-plane when $t = (1-z)^{-1}$, which gives the point $(1-z)^{-1}(x, y, 0)$ on the line $\boldsymbol{\beta}$. The formula for $\boldsymbol{F}$ now follows.

That $\boldsymbol{F}\boldsymbol{x}(u, v) = (u, v)$ is a routine calculation (and also follows from the geometrical construction). That $\boldsymbol{x}$ is a local parametrisation as claimed is now immediate from conditions (S1) and (S2), taking $U = \mathbb{R}^2$ and $W = \mathbb{R}^3 \setminus P$.

**2.2.** Let $V$ be an open subset of $\mathbb{R}^n$ such that $X = V \cap S$ is non-empty. If $p \in X$, let $\boldsymbol{x} : U \to \mathbb{R}^n$ be a local parametrisation of $S$ whose image contains $p$. Then $\boldsymbol{x}^{-1}(V) = \{(u, v) \in U : \boldsymbol{x}(u, v) \in V\}$ is an open subset of $\mathbb{R}^2$ and the restriction of $\boldsymbol{x}$ to $\boldsymbol{x}^{-1}(V)$ is a local parametrisation of $X$ whose image contains $p$. To see this, if $\boldsymbol{F} : W \to \mathbb{R}^2$ is a map satisfying condition (S2) for $\boldsymbol{x}$ then consider the restriction of $\boldsymbol{F}$ to $V \cap W$.

**2.3(a).** (i) There are many ways. The one which perhaps is closest to that given in Example 4 of §2.1 is to cover the cylinder by four local parametrisations. Firstly, let $U = \{(u, v) \in \mathbb{R}^2 : -1 < u < 1,\ v \in \mathbb{R}\}$ and let $\boldsymbol{x}^+ : U \to \mathbb{R}^3$ be given by $\boldsymbol{x}^+(u, v) = (\sqrt{1-u^2}, u, v)$. If we let $W = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ and let $\boldsymbol{F} : W \to \mathbb{R}^2$ be given by $\boldsymbol{F}(x, y, z) = (y, z)$, then conditions (S1) and (S2) are satisfied. Hence $\boldsymbol{x}^+$ is a local parametrisation of the given cylinder $S$, and the corresponding coordinate neighbourhood is shown in the left hand picture of Figure 5. The whole of $S$ may be covered by $\boldsymbol{x}^+$ and an additional three local parametrisations with domain $U$ given by

$$\boldsymbol{x}^-(u, v) = (-\sqrt{1-u^2}, u, v) \ ,$$