**Chapter 2. Single-degree-of-freedom systems**

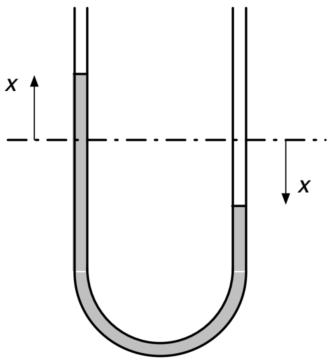
**Topics**

Equation of motion; undamped free vibrations (Q1-3)
Damped free vibrations (Q4-6)
Step load response; harmonic response; response to irregular dynamic loads (Q7-10)
Earthquake response spectra (Q11)
Laplace transform methods (Q12)

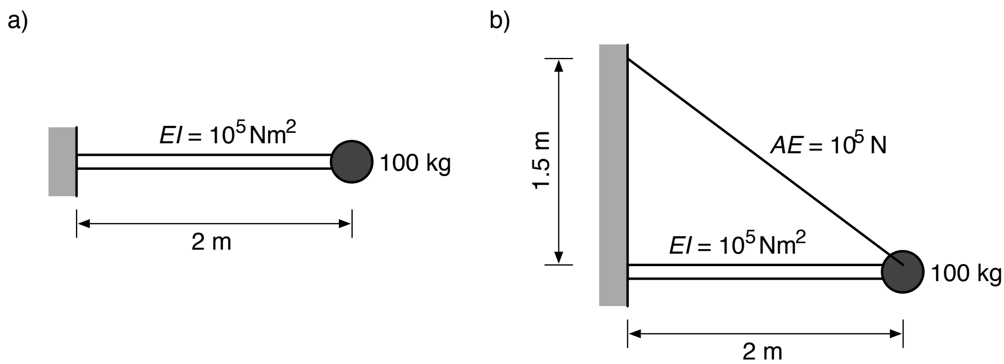
**Problems**

1. You are given a lumped mass of unknown magnitude and a spring of unknown stiffness. When the spring is hung vertically from a rigid support and the mass attached to its bottom end, it stretches by 25 mm. You then give it a small disturbance from this equilibrium position and count the number of cycles of vibration occurring in 10 seconds. How many cycles will you count?

2. The U-tube shown contains a length $L$ of fluid. An initial offset between the heights of fluid in the two vertical parts results in an oscillatory flow. Write the equation of motion for the oscillations of the fluid, and find the natural frequency. Assume no energy losses.

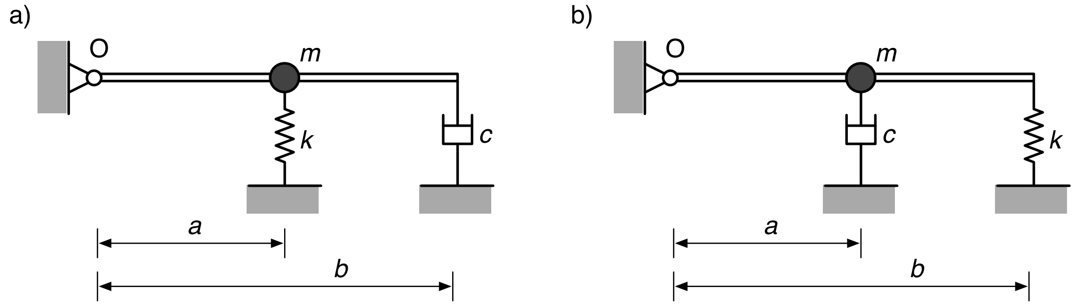



3. Find the natural period of vertical vibrations of the light cantilever supporting a mass at its tip, as shown in diagram a). The stiffness is increased by adding an inclined cable, as shown in b). Find the new natural period.



4. In the system shown in a) below, the bar is rigid and weightless, and pivots about O. Write the equation of motion in terms of the rotation θ about O, and determine the undamped and damped natural fequencies, and the value of c required to critically damp the system.

   Repeat with the positions of the spring and damper reversed, as shown in b).




5. A damped SDOF system has mass 50 kg and stiffness 6000 N/m. When set in motion, its displacement amplitude reduces from 30 mm to 27.5 mm in one cycle. Find the damping ratio ξ and dashpot coefficient *c*.

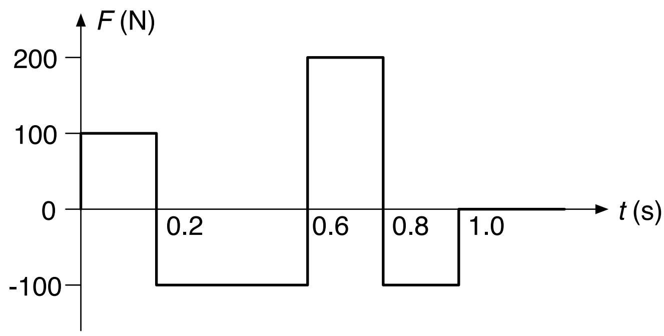
6. A SDOF system has mass 50 kg, undamped natural period 0.3 s and a damping ratio of 10% of critical. It undergoes free vibration subject to an initial displacement of 0.05 m and an initial velocity of 2.0 m/s. Find an expression for the variation of displacement with time. Find also the time at which the maximum displacement is achieved, the value of this displacement and the time taken for the amplitude of motion to reduce to 1% of this value.

7. An undamped SDOF system has mass 150 kg and natural frequency 1.8 Hz. It is loaded by a harmonic force $F = F_0\sin(2\pi ft)$ where $F_0 = 1.25$ kN and $f = 2.0$ Hz. a) Find the amplitude of the undamped steady state displacement response. b) Find the damping ratio that would be required to reduce the amplitude to 100 mm.

8. A SDOF system has mass 100 kg, natural frequency 1.0 Hz and a damping ratio of 0.05. It is loaded by a harmonic force $F = F_0\sin(2\pi ft)$ where $F_0 = 3.948$ kN. Consider cases where the forcing frequency $f$ takes the value a) 0.8 Hz, b) 1.0 Hz, c) 1.5 Hz. For each case, find an expression for the displacement response as a function of time and draw phasor diagrams showing the applied force together with the inertia, damping and stiffness forces.

9. A SDOF system has mass 10 kg, natural frequency 2.0 Hz and damping ratio 0.05. It is loaded by a sequence of six equal impulses of magnitude 10 Ns, regularly spaced at 0.5 s intervals. Neglecting any distinction between damped and undamped natural frequency, find an expression for the variation of displacement with time, for times after the last impulse.

10. Find and plot the response of an undamped SDOF system of stifness 10 kN/m and natural period 0.4 s to the loading function shown below.

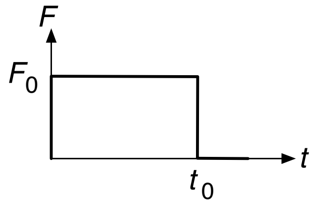


11. A SDOF structure has mass 1800 kg, 5% damping and natural period 0.5 s. Assume it is subjected to an earthquake defined by the EC8 response spectra shown in Fig. 2.22. Calculate the peak force experienced by the system and the peak displacement of the mass when the system is subjected to an earthquake with peak ground acceleration 0.3g if the structure is founded on: a) rock (ground type A), or b) stiff soil (ground type C).

    In a second structure, the mass is doubled and the stiffness halved. Calculate the new peak force and displacement in each case.

12. Find the Laplace transform of the load function plotted below.

    Taking $F_0$ = 10 N, $t_0$ = 0.6 s, find the response to the load of a SDOF system of mass $m$ = 2 kg, stiffness $k$ = 50 N/m and 5% damping, with zero initial displacement and velocity.



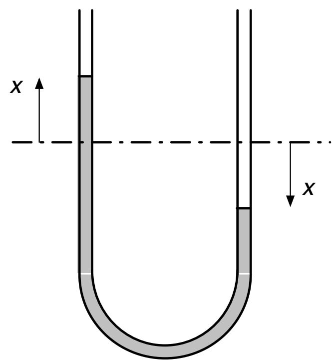
**Solutions**

1. You are given a lumped mass of unknown magnitude and a spring of unknown stiffness. When the spring is hung vertically from a rigid support and the mass attached to its bottom end, it stretches by 25 mm. You then give it a small disturbance from this equilibrium position and count the number of cycles of vibration occurring in 10 seconds. How many cycles will you count?

Static deflection: $\dfrac{mg}{k} = 0.025 \quad \rightarrow \quad \dfrac{k}{m} = \dfrac{g}{0.025} = 392.4$

Number of cycles in 10 s = $10 f_n = \dfrac{10}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{5}{\pi}\sqrt{392.4} = 31.5 \text{ cycles}$

2. The U-tube shown contains a length $L$ of fluid. An initial offset between the heights of fluid in the two vertical parts results in an oscillatory flow. Write the equation of motion for the oscillations of the fluid, and find the natural frequency. Assume no energy losses.




Let fluid have density ρ, tube cross-sectional area $A$. With a displacement $x$ from the equilibrium position, the weight of out-of-balance fluid is

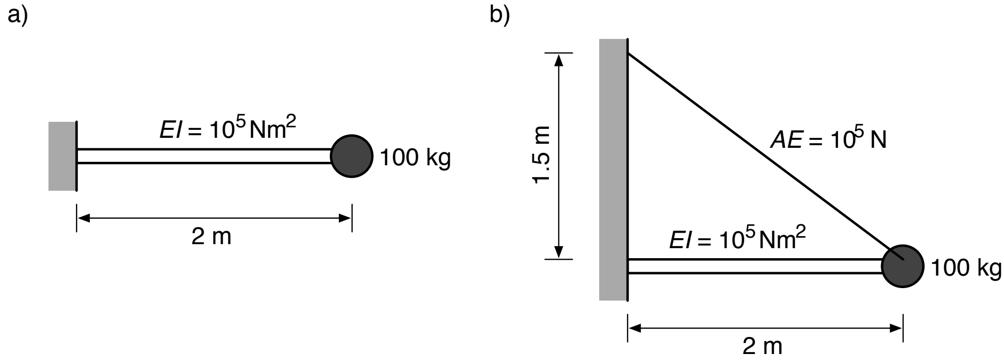
$F = \rho A.2x.g$

The total mass of fluid is $m = \rho AL$

Newton's second law: $2\rho Axg = \rho AL.-\ddot{x} \quad \rightarrow \quad \ddot{x} + \dfrac{2g}{L}x = 0$

This has the form $\ddot{x} + \omega_n^2 x = 0$. Hence $\omega_n = \sqrt{\dfrac{2g}{L}}, \quad f_n = \dfrac{\omega_n}{2\pi} = \dfrac{1}{\pi}\sqrt{\dfrac{g}{2L}}$

3. Find the natural period of vertical vibrations of the light cantilever supporting a mass at its tip, as shown in diagram a). The stiffness is increased by adding an inclined cable, as shown in b). Find the new natural period.



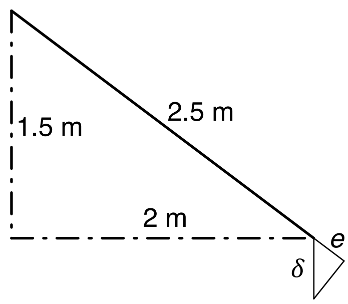

a) We calculated the stiffness of configuration a) in Chapter 1, Problem 6a): $k$ = 37,500 N/m.

Hence natural period $T_n = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{100}{37{,}500}} = 0.32 \text{ s}$

b) To find vertical stiffness of cable, first find its axial stiffness:

$$k_A = \frac{AE}{L} = \frac{10^5}{2.5} = 40{,}000 \text{ N/m}$$

Now consider a vertical deflection δ at the tip of the cantilever, drawn to an exaggerated scale.




Movement consists of a cable extension $e$, and a perpendicular rotation. By similar triangles, $e = 0.6\delta$. The force in the cable is $T = k_A e = 0.6k_A\delta$ and the vertical component of this force is $F = 0.6T = 0.36k_A\delta$. Therefore the vertical stiffness is:

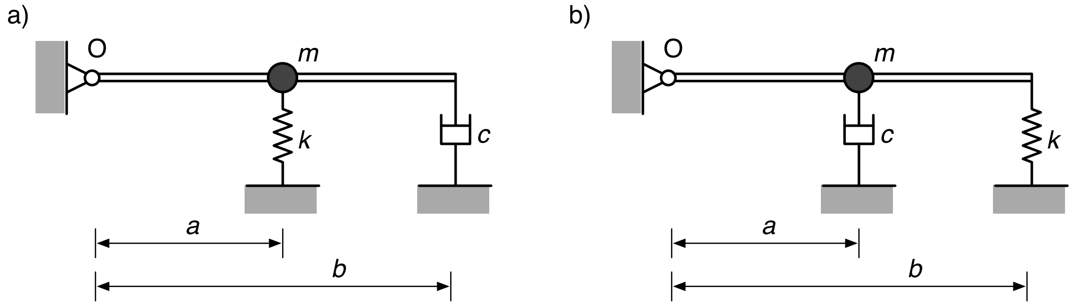
$$k_V = \frac{F}{\delta} = 0.36k_A = 0.36 \times 40{,}000 = 14{,}400 \text{ N/m}$$

Total vertical stiffness of beam plus cable = 51,900 N/m, and hence new period is

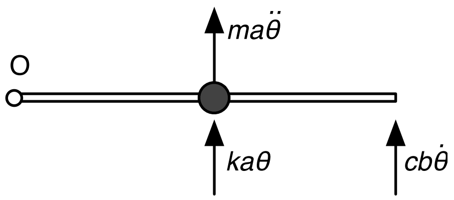
$$T_n = 2\pi\sqrt{\frac{100}{51{,}900}} = 0.28 \text{ s}$$

4. In the system shown in a) below, the bar is rigid and weightless, and pivots about O. Write the equation of motion in terms of the rotation θ about O, and determine the undamped and damped natural fequencies, and the value of c required to critically damp the system.

   Repeat with the positions of the spring and damper reversed, as shown in b).




a) For a small rotation θ about O, the vertical displacements at distances *a* and *b* are *a*θ and *b*θ. Expressing the inertia force using d'Alembert's principle, the forces on the bar are:




Moments about O: $ma^2\ddot{\theta} + cb^2\dot{\theta} + ka^2\theta = 0$

Hence: $\ddot{\theta} + \dfrac{cb^2}{ma^2}\dot{\theta} + \dfrac{k}{m}\theta = 0$

Comparing to Eq (2.14): $\ddot{\theta} + 2\xi\omega_n\dot{\theta} + \omega_n^2\theta = 0$, we see that:

$$\omega_n = \sqrt{\frac{k}{m}}, \quad 2\xi\omega_n = \frac{cb^2}{ma^2} \quad \rightarrow \quad \xi = \frac{cb^2}{2a^2\sqrt{km}},$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \left[\frac{k}{m} - \left(\frac{cb^2}{2ma^2}\right)^2\right]^{1/2}$$

To find the critical damping coefficient, put ξ = 1, giving

$$c_{crit} = 2\sqrt{km}\,\frac{a^2}{b^2}$$

b) Solution is very similar. With spring and damper swapped the governing equation becomes

$$ma^2\ddot{\theta} + ca^2\dot{\theta} + kb^2\theta = 0$$

Hence: $\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{kb^2}{ma^2}\theta = 0$

Comparing to Eq (2.14): $\ddot{\theta} + 2\xi\omega_n\dot{\theta} + \omega_n^2\theta = 0$, we see that:

$$\omega_n = \frac{b}{a}\sqrt{\frac{k}{m}}, \quad 2\xi\omega_n = \frac{c}{m} \quad \rightarrow \quad \xi = \frac{ca}{2b\sqrt{km}},$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \left[\frac{k}{m}\left(\frac{b}{a}\right)^2 - \left(\frac{c}{2m}\right)^2\right]^{1/2}$$

To find the critical damping coefficient, put ξ = 1, giving

$$c_{crit} = 2\sqrt{km}\frac{b}{a}$$

5. A damped SDOF system has mass 50 kg and stiffness 6000 N/m. When set in motion, its displacement amplitude reduces from 30 mm to 27.5 mm in one cycle. Find the damping ratio ξ and dashpot coefficient *c*.

The log decrement $\delta = \ln\left(\frac{30}{27.5}\right) = 0.087$

From Eq. 2.22 $\xi = \frac{\delta}{2\pi} = 0.014$

(Strictly, this is an approximation. A more precise solution can be obtained by noting that

$$\delta = \frac{\xi\omega_n.2\pi}{\omega_d} = \frac{\xi\omega_n.2\pi}{\omega_n\sqrt{1-\xi^2}} \quad \rightarrow \quad \xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$

However, for this low level of damping, this gives the same answer to four decimal places.)

The dashpot coefficient $c = \xi.2\sqrt{km} = 15.2$ Ns/m

6. A SDOF system has mass 50 kg, undamped natural period 0.3 s and a damping ratio of 10% of critical. It undergoes free vibration subject to an initial displacement of 0.05 m and an initial velocity of 2.0 m/s. Find an expression for the variation of displacement with time. Find also the time at which the maximum displacement is achieved, the value of this displacement and the time taken for the amplitude of motion to reduce to 1% of this value.

Eqs 2.16 and 2.18 give the general solutions for displacement and velocity:

$$x = Ae^{-\xi\omega_n t}\sin(\omega_d t + \phi)$$

$$x = Ae^{-\xi\omega_n t}[-\xi\omega_n \sin(\omega_d t + \phi) + \omega_d \cos(\omega_d t + \phi)]$$

where $\xi = 0.1$, $\omega_n = \dfrac{2\pi}{0.3} = 20.94$ rad/s, $\omega_d = \omega_n\sqrt{1-\xi^2} = 20.84$ rad/s

Applying the initial conditions:

(1) $0.05 = A\sin\phi$

(2) $2 = A[-\xi\omega_n \sin\phi + \omega_d \cos\phi]$

Sub. from (1) into (2) and plug in values for $\xi$, $\omega_n$, $\omega_d$ to give $A\cos\phi = 0.101$

Hence $A = \sqrt{0.05^2 + 0.101^2} = 0.113$ m, $\phi = \tan^{-1}\dfrac{0.05}{0.101} = 0.46$ rads

Hence displacement expression can be written:

$$x = Ae^{-\xi\omega_n t}\sin(\omega_d t + \phi) = 0.113e^{-2.094t}\sin(20.84t + 0.46)$$

The first peak occurs when the sine term = 1, i.e. when $20.84t + 0.46 = \dfrac{\pi}{2} \quad \rightarrow \quad t = 0.053$ s

and the displacement at this instant is $x = 0.113e^{-2.094\times 0.053} = 0.101$ m

Time at which amplitude drops to 1% of this value is given by:

$$\frac{0.101}{100} = 0.113e^{-2.094t} \quad \rightarrow \quad t = 2.25\text{ s}$$

7. An undamped SDOF system has mass 150 kg and natural frequency 1.8 Hz. It is loaded by a harmonic force $F = F_0\sin(2\pi ft)$ where $F_0 = 1.25$ kN and $f = 2.0$ Hz. a) Find the amplitude of the undamped steady state displacement response. b) Find the damping ratio that would be required to reduce the amplitude to 100 mm.

a) Stiffness of system $k = 4\pi^2 f_n^2 m = 19.2$ kN/m

Static displacement $x_0 = \frac{F_0}{k} = 0.065$ m , frequency ratio $\Omega = \frac{2.0}{1.8} = 1.11$

With no damping, Eq 2.30 gives: $x_{\max} = x_0 \frac{1}{\left|1-\Omega^2\right|} = 0.278$ m

b) If displacement is limited to 100 mm, this means the DAF $D$ is limited to 100/65 = 1.53.

For a damped system, Eq 2.30:

$$D = \frac{1}{\sqrt{(1-\Omega^2)^2 + (2\xi\Omega)^2}}$$

Rearranging:

$$\xi = \frac{1}{2\Omega}\sqrt{\frac{1}{D^2} - (1-\Omega^2)^2}$$

With $D = 1.53$, $\Omega = 1.11$, this gives $\xi = 0.274$ (i.e. 27.4%).

8. A SDOF system has mass 100 kg, natural frequency 1.0 Hz and a damping ratio of 0.05. It is loaded by a harmonic force $F = F_0\sin(2\pi ft)$ where $F_0 = 3.948$ kN. Consider cases where the forcing frequency $f$ takes the value a) 0.8 Hz, b) 1.0 Hz, c) 1.5 Hz. For each case, find an expression for the displacement response as a function of time and draw phasor diagrams showing the applied force together with the inertia, damping and stiffness forces.

System stiffness $k = m(2\pi f_n)^2 = 39.48$ kN/m , $c = \xi.2\sqrt{km} = 628$ Ns/m

Static displacement due to $F_0$ is $x_0 = \frac{F_0}{k} = 0.1$ m

a) $\Omega = 0.8$. Applying Eqs 2.29 - 2.31: $x = Dx_0 \sin(2\pi ft - \phi)$

where $D = \dfrac{1}{\sqrt{(1-\Omega^2)^2 + (2\xi\Omega)^2}} = \dfrac{1}{\sqrt{(1-0.8^2)^2 + (2\times 0.05\times 0.8)^2}} = 2.71$

$\phi = \tan^{-1}\dfrac{2\xi\Omega}{1-\Omega^2} = \tan^{-1}\dfrac{2\times 0.05\times 0.8}{1-0.8^2} = 0.219 \text{ rads} = 12.5°$

Giving $x = 0.271\sin(1.6\pi t - 0.219)$

Differentiating: $\dot{x} = 1.36\cos(1.6\pi t - 0.219) = 1.36\sin(1.6\pi t - 0.219 + \pi/2)$

$\ddot{x} = -6.85\sin(1.6\pi t - 0.219) = 6.85\sin(1.6\pi t - 0.219 + \pi)$

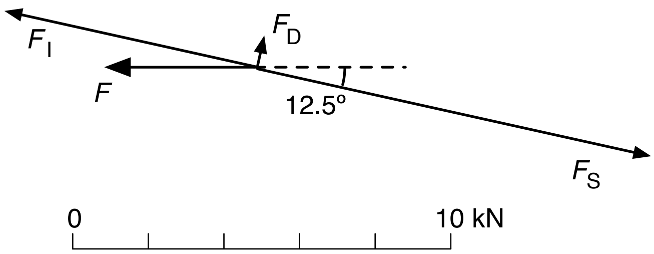
Hence:

Stiffness force: $F_S = 0.271\times 39.48 = 10.71 \text{ kN}$, $\phi_S = 0.219 \text{ rads} = 12.5°$

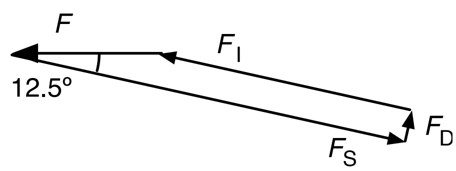
Damping force: $F_D = 1.36\times 628 = 0.86 \text{ kN}$, $\phi_D = 0.219 - \pi/2 = -1.35 \text{ rads} = -77.5°$

Inertia force: $F_I = 6.85\times 1000 = 6.85 \text{ kN}$, $\phi_I = 0.219 - \pi = -2.92 \text{ rads} = -167.5°$

Phasor diagram (plotted to scale):




Force polygon:




(Note on directions of vectors: the three internal force vectors must sum to give a resultant force vector equal and opposite to the applied external force $F$. With $F$ assumed acting to the left, the vector sum of the three internal forces therefore acts to the right. The displacement of the mass is at a phase angle of 12.5° to $F$, and the stiffness force opposes this displacement, therefore it makes a phase angle of 12.5° to $-F$.)

b) Following identical procedure but with Ω = 1.0:

$D = 10$, $\phi = \pi/2$ rads = 90°, giving $x = \sin(2\pi t - \pi/2)$

Differentiating: $\dot{x} = 2\pi\cos(2\pi t - \pi/2) = 6.28\sin(2\pi t)$

$$\ddot{x} = -4\pi^2\sin(2\pi t - \pi/2) = 39.48\sin(2\pi t + \pi/2)$$

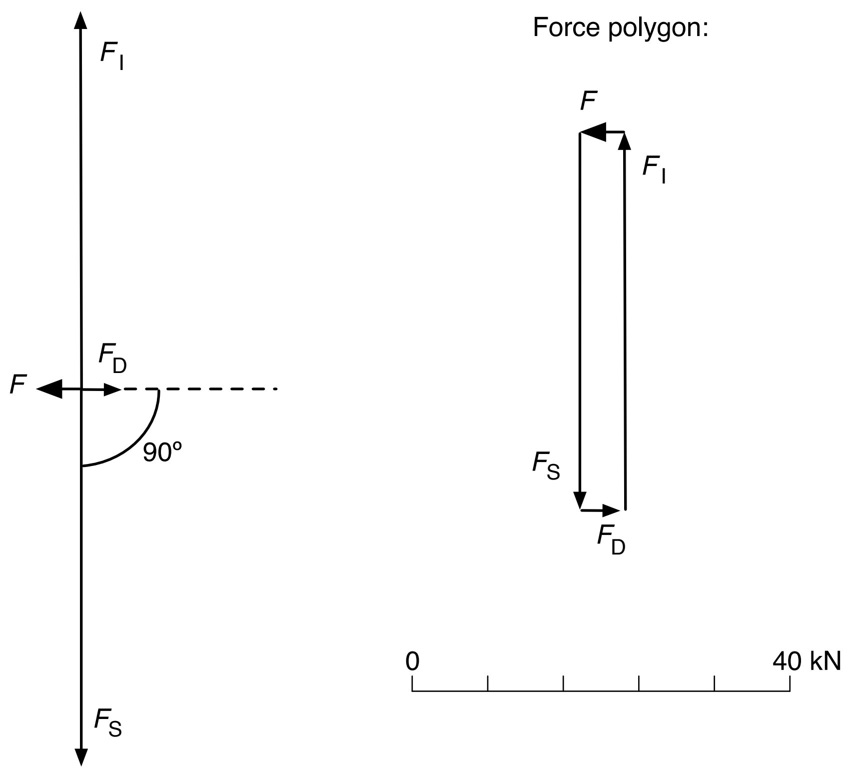
Hence:

Stiffness force: $F_S = 1\times 39.48 = 39.48$ kN, $\phi_S = \pi/2$ rads $= 90°$

Damping force: $F_D = 6.28\times 628 = 3.948$ kN, $\phi_D = 0$

Inertia force: $F_I = 39.48\times 1000 = 39.48$ kN, $\phi_I = -\pi/2$ rads $= -90°$

Phasor diagram – note different scale to part a):



c) Following identical procedure but with $\Omega = 1.5$:

$D = 0.79$, $\phi = 3.02$ rads = 173.2°, giving $x = 0.079\sin(3\pi t - 3.02)$

Differentiating: $\dot{x} = 0.75\cos(3\pi t - 3.02) = 0.75\sin(3\pi t - 3.02 + \pi/2)$

$$\ddot{x} = -7.06\sin(3\pi t - 3.02) = 7.06\sin(3\pi t - 3.02 + \pi)$$

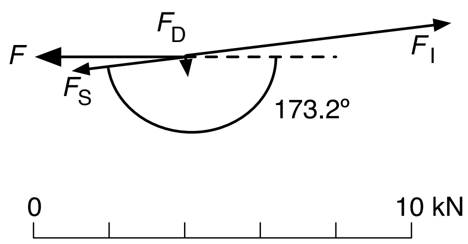
Hence:

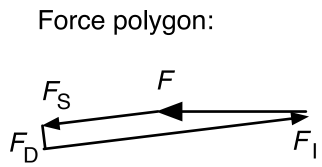
Stiffness force: $F_S = 0.079 \times 39.48 = 3.14$ kN, $\phi_S = 3.02 \text{ rads} = 173.2°$

Damping force: $F_D = 0.75 \times 628 = 0.47$ kN, $\phi_D = 3.02 - \pi/2 = 1.45 \text{ rads} = 83.2°$

Inertia force: $F_I = 7.06 \times 1000 = 7.06$ kN, $\phi_I = 3.02 - \pi = -0.12 \text{ rads} = -6.8°$

Phasor diagram – same scale as part a):



9. A SDOF system has mass 10 kg, natural frequency 2.0 Hz and damping ratio 0.05. It is loaded by a sequence of six equal impulses of magnitude 10 Ns, regularly spaced at 0.5 s intervals. Neglecting any distinction between damped and undamped natural frequency, find an expression for the variation of displacement with time, for times after the last impulse.

For a single impulse Eq 2.47 gives $x = I.h(t) = \dfrac{I}{m\omega_n} e^{-\xi\omega_n t} \sin\omega_n t$

So, for a train of six impulses spaced at τ: $x = \dfrac{I}{m\omega_n} \sum_{j=0}^{5} e^{-\xi\omega_n (t - j\tau)} \sin\omega_n (t - j\tau)$

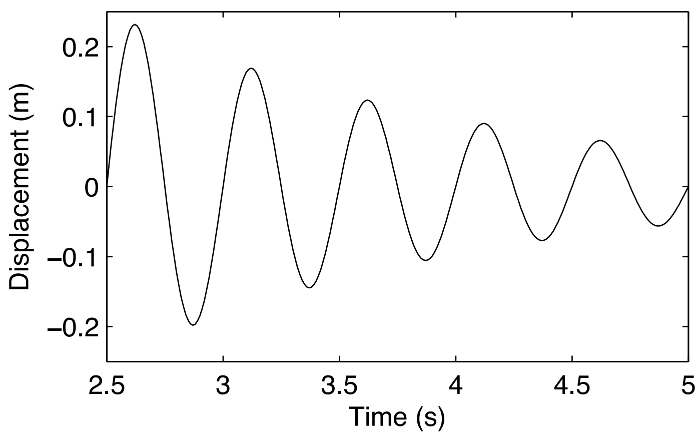
Noting that, for our problem, $\omega_n\tau = 2\pi$, this can be simplified to:

$$x = \frac{I}{m\omega_n} \sum_{j=0}^{5} e^{-\xi\omega_n t} e^{\xi\omega_n j\tau} \sin(\omega_n t - j.2\pi) = \frac{I}{m\omega_n} e^{-\xi\omega_n t} \sin\omega_n t \sum_{j=0}^{5} e^{2\pi\xi j}$$

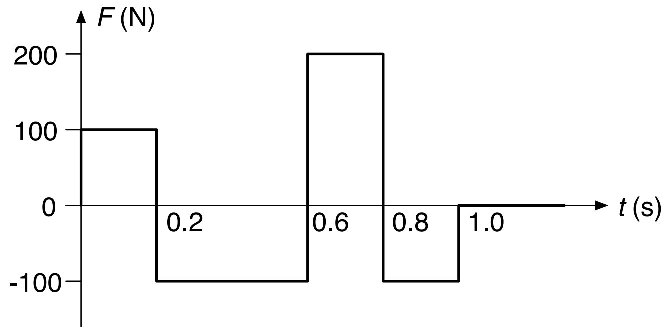
Substituting in numerical values and writing out the summation term by term:

$$x = \frac{10}{10\times 4\pi} e^{-0.05\times 4\pi t} \sin 4\pi t \left(1 + e^{0.05\times 2\pi} + e^{0.05\times 4\pi} + e^{0.05\times 6\pi} + e^{0.05\times 8\pi} + e^{0.05\times 10\pi}\right)$$
$$= \frac{1}{4\pi} e^{-0.2\pi t} \sin 4\pi t \times 15.13 = 1.204 e^{-0.2\pi t} \sin 4\pi t$$

Solution is valid for times after the last impulse, i.e. for $t > 2.5$ s, and is plotted below.



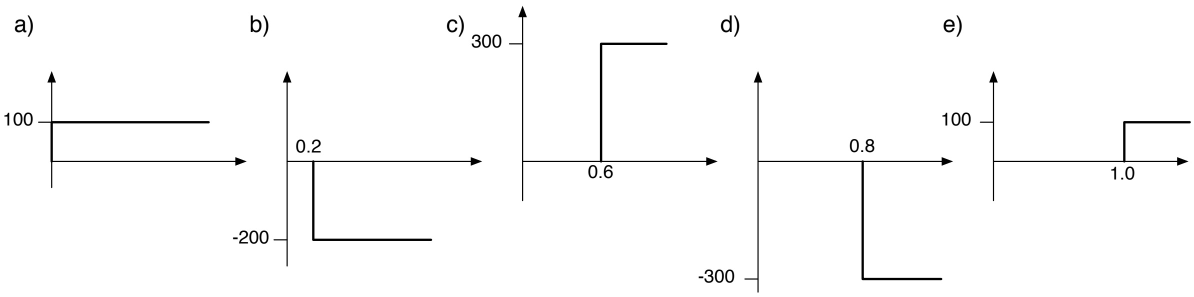
10. Find and plot the response of an undamped SDOF system of stifness 10 kN/m and natural period 0.4 s to the loading function shown below.




Duhamel's integral for a constant load $F$, undamped system, Eq 2.52:

$$x(t)=\frac{1}{m\omega_n}\int_0^t F\sin\omega_n(t-\tau)\mathrm{d}\tau=\frac{F}{m\omega_n^2}\left[\cos\omega_n(t-\tau)\right]_0^t=\frac{F}{k}\left[1-\cos\omega_n t\right]$$

Treat the stepped load pattern as the sum of a set of constant loads, all starting at different times:




a) $F$ = 100, starts at $t$ = 0: $\quad x(t)=\frac{100}{10{,}000}\left[1-\cos\frac{2\pi}{0.4}t\right]=0.01\left[1-\cos 5\pi t\right] \qquad t>0$

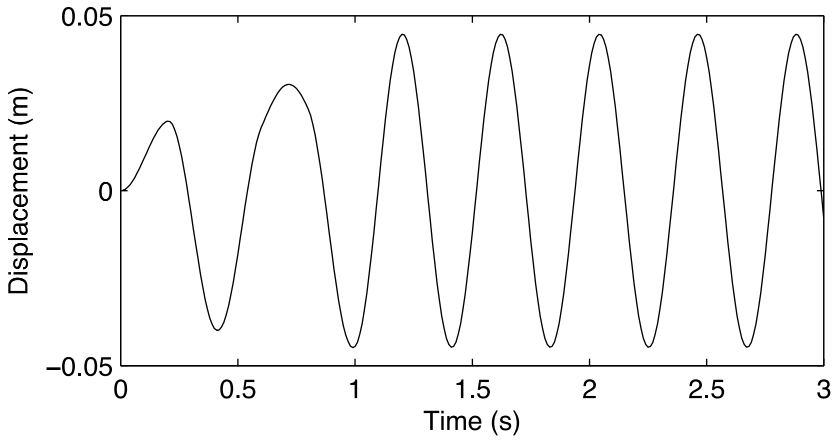
b) $F$ = –200, replace t by $t$ – 0.2: $\quad x(t)=-0.02\left[1-\cos 5\pi(t-0.2)\right] \qquad t>0.2$

c) $F$ = 300, replace t by $t$ – 0.6: $\quad x(t)=0.03\left[1-\cos 5\pi(t-0.6)\right] \qquad t>0.6$

d) $F$ = –300, replace t by $t$ – 0.8: $\quad x(t)=-0.03\left[1-\cos 5\pi(t-0.8)\right] \qquad t>0.8$

e) $F$ = 100, replace t by $t$ – 1: $\quad x(t)=0.01\left[1-\cos 5\pi(t-1)\right] \qquad t>1$

The full solution is found by summing a) – e), noting their respective start times, and is plotted below.




11. A SDOF structure has mass 1800 kg, 5% damping and natural period 0.5 s. Assume it is subjected to an earthquake defined by the EC8 response spectra shown in Fig. 2.22. Calculate the peak force experienced by the system and the peak displacement of the mass when the system is subjected to an earthquake with peak ground acceleration 0.3g if the structure is founded on: a) rock (ground type A), or b) stiff soil (ground type C).

    In a second structure, the mass is doubled and the stiffness halved. Calculate the new peak force and displacement in each case.

    For original system:

    a) $S_a = 2.0 \times 0.3 \times 9.81 = 5.89$ m/s$^2$

       Force = $mS_a = 1800 \times 5.89 = 10.6$ kN. Peak displacement = $5.89 \times 0.5^2/(4\pi^2) = 37$ mm

    b) $S_a = 2.875 \times 0.3 \times 9.81 = 8.46$ m/s$^2$

       Force = $mS_a = 1800 \times 8.46 = 15.2$ kN. Peak displacement = $8.46 \times 0.5^2/(4\pi^2) = 54$ mm

    For second structure, $m = 3600$ kg, period = 1.0 s (proportional to $\sqrt{(m/k)}$). Hence:

    a) $S_a = 1.0 \times 0.3 \times 9.81 = 2.94$ m/s$^2$

       Force = $mS_a = 3600 \times 2.94 = 10.6$ kN. Peak displacement = $2.94 \times 1^2/(4\pi^2) = 75$ mm

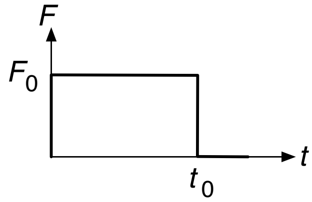
    b) $S_a = 1.7 \times 0.3 \times 9.81 = 5.00$ m/s$^2$

       Force = $mS_a = 3600 \times 5.0 = 18.0$ kN. Peak displacement = $5.0 \times 1^2/(4\pi^2) = 127$ mm

    So, for the rock case in a), doubling the mass and the period results in the same force and double the displacement. This is because both $T = 0.5$ and $T = 1.0$ s lie on the same part of the spectral curve, where $S_a$ is proportional to $1/T$, so that doubling the period halves the acceleration. For the stiff soil case in b) the relationship between the responses at the two periods is not so simple, because the first lies on the plateau of the spectrum and the second on the $1/T$ curve.

12. Find the Laplace transform of the load function plotted below.

Taking $F_0 = 10$ N, $t_0 = 0.6$ s, find the response to the load of a SDOF system of mass $m = 2$ kg, stiffness $k = 50$ N/m and 5% damping, with zero initial displacement and velocity.




$$\bar{F}(s) = \int_0^{\infty} F(t)\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^{t_0} F_0\mathrm{e}^{-st}\,\mathrm{d}t = \left[-\frac{F_0}{s}\mathrm{e}^{-st}\right]_0^{t_0} = \frac{F_0}{s}(1-\mathrm{e}^{-st_0})$$

Governing equation is: $m\ddot{x} + c\dot{x} + kx = F(t)$ . Taking Laplace transforms:

$$(ms^2 + cs + k)\bar{x}(s) = \frac{F_0}{s}(1-\mathrm{e}^{-st_0})$$

The damping coefficient is $c = \xi.2\sqrt{km} = 0.05 \times 2\sqrt{50 \times 2} = 1\ \text{Ns/m}$ . Therefore, substituting in numbers:

$$\bar{x}(s) = \frac{10}{s(2s^2 + s + 50)}(1-\mathrm{e}^{-0.6s})$$

After some manipulation this can be expressed as:

$$\begin{aligned}\bar{x}(s) &= \left[\frac{0.2}{s} - \frac{0.4s+0.2}{(2s^2+s+50)}\right](1-\mathrm{e}^{-0.6s}) = \left[\frac{2}{s} - \frac{0.2s+0.1}{(s^2+0.5s+25)}\right](1-\mathrm{e}^{-0.6s}) \\ &= \left[\frac{0.2}{s} - 0.2\frac{s+0.25}{(s+0.25)^2+5^2} - 0.01\frac{5}{(s+0.25)^2+5^2}\right](1-\mathrm{e}^{-0.6s})\end{aligned}$$

Note: strictly, the squared term in the denominator of the second and third terms in square brackets, shown as $5^2 = 25$, should be 24.9375. The difference, which corresponds to the difference between the damped and undamped natural frequency of the system, is neglected here. Finally, taking inverse Laplace transforms and using the shifting properties in Eq D.2 and D.3:

$$\begin{aligned}x(t) &= u(t)\left[0.2 - 0.2\mathrm{e}^{-0.25t}\cos 5t - 0.01\mathrm{e}^{-0.25t}\sin 5t\right] \\ &\quad -u(t-0.6)\left[0.2 - 0.2\mathrm{e}^{-0.25(t-0.6)}\cos 5(t-0.6) - 0.01\mathrm{e}^{-0.25(t-0.6)}\sin 5(t-0.6)\right]\end{aligned}$$

where $u(t)$ is the unit step function (=1 if argument is positive, 0 otherwise).

The resulting response is plotted below. The full solution is given by the solid line. The dotted line shows the response to a step load applied at time zero and sustained indefinitely (i.e. the first line of the solution above).

Note that the value 0.2 m is the static displacement produced by a constant force of 10 N on a spring of stiffness 50 N/m.