# 2 Chapter Two

1. Using Fourier transform basic definition, prove the Parseval's energy theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

**Solution:** The proof readily follows from the basic Fourier integral definition. We have:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t) dt = \int_{-\infty}^{\infty} x(t)[\int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df]^* dt$$

After rearranging the integral, we will have:

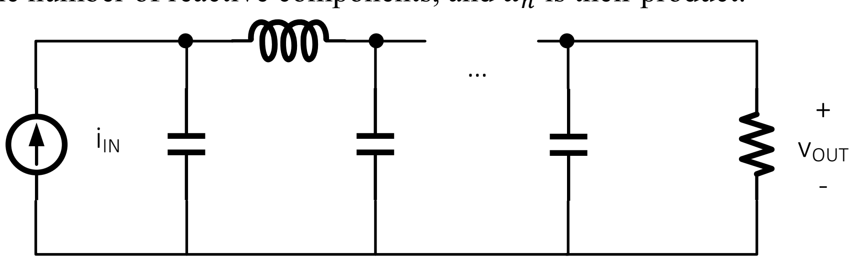
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} X^*(f) \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt\right] df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Note that the term inside the bracket: $\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt$ is $X(f)$.

2. In the ladder structure shown below, show that the transfer function is of the form:

$$\frac{v_{OUT}(s)}{i_{IN}(s)} = \frac{1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Where n is the number of reactive components, and $a_n$ is their product.




**Solution:** Considering how the current is divided within the ladder,
To find $a_n$ intuitively, as we discussed in the chapter, at high frequencies, with the capacitors being short, and the inductors open, we can monitor the current dividing in any given branch as follows: For an arbitrary part of the ladder comprising of the shunt capacitance $C_i$, and the series inductance $L_i$, the total impedance at the right side of the capacitance is mostly dominated by the inductance, and is roughly $L_i s$. Consequently, the current entering the branch is divided by:

$$\approx \frac{\frac{1}{C_i s}}{\frac{1}{C_i s} + L_{ni} s} \approx \frac{1}{L_i C_i s^2}$$

When all the branches considered, the output current, which determines the output voltage appearing on the resistor, is of the form:

$$\prod_{i=1}^{n/2} \frac{1}{L_i C_i s^2} = \frac{1}{[\prod L_i C_i] s^n}$$

assuming $n$ is even (we have equal number of capacitances and inductances). The case of $n$ being odd is very similar.

Since at high frequencies only the term $\frac{1}{a_n s^n}$ of the transfer function is dominant, we conclude:

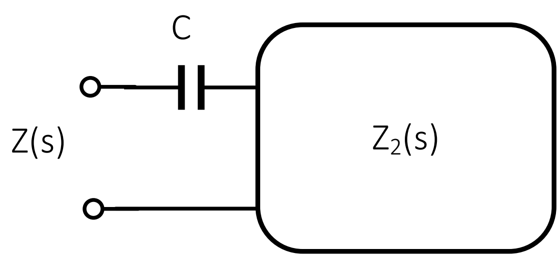
$$a_n = \prod_{1}^{n} L_i C_i$$

which is the proof.

3. The input impedance of the circuit shown below is expressed as:

$$Z(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s}$$

Where $|n - m| \leq 1$.




Show that the value of the series input capacitor is:

$$C = \frac{\left.\frac{\partial}{\partial s} D(s)\right|_{s=0}}{N(0)}$$

**Solution:** We can rewrite $Z(s)$ as:

$$Z(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{s(b_m s^{m-1} + b_{m-1} s^{m-1} + \cdots + b_1)}$$

Using partial fraction expansion (note $Z(s)$ has a pool at $s = 0$):

$$Z(s) = \frac{K}{s} + \frac{a'_{n-1} s^{n-1} + a'_{n-2} s^{n-2} + \cdots + a'_1}{b_m s^{m-1} + b_{m-1} s^{m-1} + \cdots + b_1}$$

Clearly:

$$K = sZ(s)|_{s=0} = \frac{a_0}{b_1} = \frac{N(0)}{\left.\frac{\partial}{\partial s} D(s)\right|_{s=0}}$$

It is evident that $Z(s)$ consists of a capacitance of $\frac{1}{K}$ in series with another one-port. Hence:

$$C = \frac{1}{K} = \frac{\left.\frac{\partial}{\partial s} D(s)\right|_{s=0}}{N(0)}$$

The condition $|n - m| \leq 1$ arises from the fact that for $Z(s)$ to be realizable by a lumped RLCM network, it needs to be a *positive real function*, that is:

$$Re[Z(s)] \geq 0$$

when $Re[s] \geq 0$. The proof is given by Brune, and is a sufficient and necessary condition that must be satisfied when synthesizing the circuits. It follows that in a positive real function, the difference between degrees of the numerator and denominator cannot exceed one. For more details, see Dimopoulos[2] or Temes[3].

4. A one-port has the input impedance:

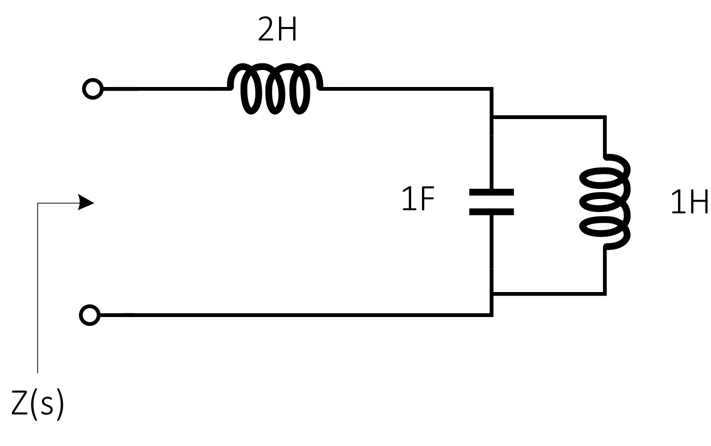
$$Z(s) = \frac{2s^3 + 2s + 1}{s^2 + 1}$$

Using a similar approach as problem 3, find the value of the series input inductance, and synthesize the rest of the circuit.

**Solution:** We may write:

$$Z(s) = \frac{2s(s^2 + 1) + 1}{s^2 + 1} = 2s + \frac{1}{s^2 + 1}$$

Accordingly, the one-port may be synthesized though a 2H inductor, in series with a parallel LC circuit with 1H inductance and 1F capacitance (to realize the $\frac{1}{s^2+1}$ portion).




The synthesized circuit is depicted above.

[2] H. Dimopoulos, Analog filters, Theory, design, and synthesis, Springer 2011.
[3] G. Temes, J. Laparta, Introduction to circuit synthesis and design, McGraw Hill, 1977.

5. Prove that the Butterworth function: $|H(f)| = \frac{1}{\sqrt{1+(\frac{f}{B})^{2n}}}$ is maximally flat, that is, its first n derivatives are equal to zero at $f = 0$.

   **Solution**: All we need to show is that the first n derivatives of the transfer function is equal to zero. For simplicity, we assume $B = 1$ as it does not impact the derivatives. Forming:

$$G(f) = \frac{1}{\sqrt{1+f^{2n}}} = (1+f^{2n})^{-1/2}$$

   We have:

$$G'(f) = -nf^{2n-1}(1+f^{2n})^{-3/2}$$

   Which is zero for $f = 0$.
   The second derivative is:

$$G''(f) = -n(2n-1)f^{2n-2}(1+f^{2n})^{-\frac{3}{2}} + 3n^2 f^{4n-2}(1+f^{2n})^{-5/2}$$

   Which is also zero for $f = 0$. The rest of the derivatives may be done very easily as well.
   For instance, for $n = 1$, we can see that

$$G'(f) = -f(1+f^2)^{-3/2}$$

   And clearly the second derivative is not zero anymore.

6. Show that all the poles of the normalized Butterworth filter (B = 1) lie on unity circle in the s-plane. Discuss the pole locations for even and odd values of n. **Hint:** Prove: $H(s)H(-s) = \left.\frac{1}{1+(-s^2)^n}\right|_{s=j\omega}$. Show that the roots of $H(s)$ are: $s_k = e^{j(\frac{2k+1}{2n}\pi+\frac{\pi}{2})}$, where $k = 0, 1, \cdots, n-1$.

   **Solution:** Since $|H(f)| = \frac{1}{\sqrt{1+(\frac{f}{B})^{2n}}}$, it follows:

$$H(s)H(-s) = \left.\frac{1}{1+(-s^2)^n}\right|_{s=j\omega}$$

   To find the roots, we shall set:

$$1 + (-s^2)^n = 0$$

   It follows:

$$(-s^2)^n = -1 = e^{j(2k+1)\pi}$$

   where $k = 0, 1, \cdots, n-1$. Consequently:

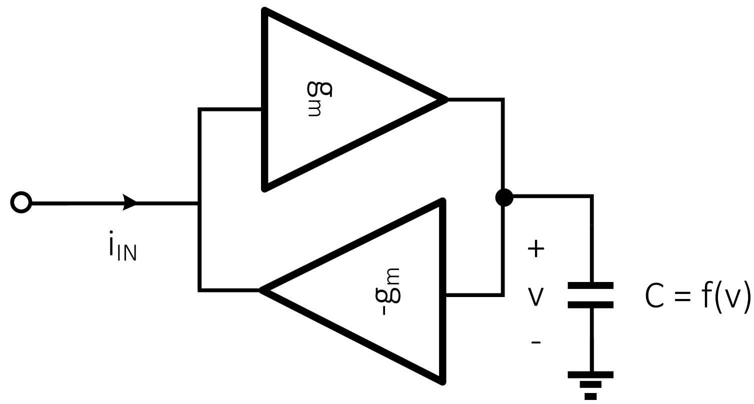
$$-s^2 = e^{\frac{j(2k+1)\pi}{n}}$$

   Thus:

$$s^2 = e^{\frac{j(2k+1)\pi}{n}} e^{j\pi}$$

   Which leads to: $s = e^{j(\frac{(2k+1)\pi}{2n}+\frac{\pi}{2})}$. Clearly, all the roots are on the unity circuit as $|s| = 1$.

7. For the following active gyrator, the capacitor $C$ is voltage-dependent given by $C = f(v)$, prove that the effective inductor is current-dependent with value given by $L = \frac{1}{gm^2} f(\frac{1}{gm} i_{IN})$.




**Solution:** Assuming an input voltage of $v_{IN}$, we have:

$$g_m v_{IN} = C\frac{dv}{dt} = f(v)\frac{dv}{dt}$$

Furthermore:

$$i_{IN} + (-g_m v) = 0$$

Thus:

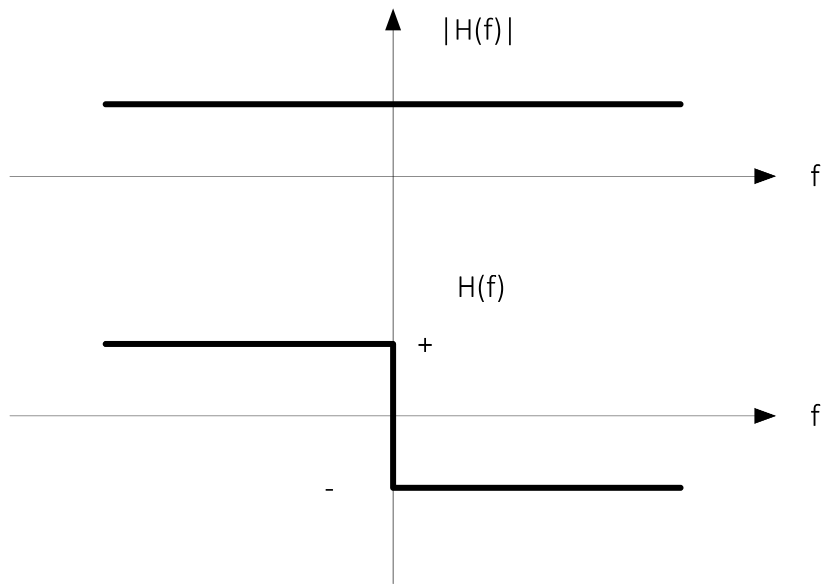
$$v_{IN} = \frac{1}{g_m} f(\frac{i_{IN}}{g_m})\frac{d}{dt}(\frac{i_{IN}}{g_m})$$

Or:

$$v_{IN} = [\frac{1}{{g_m}^2} f\left(\frac{i_{IN}}{g_m}\right)]\frac{di_{IN}}{dt}$$

which indicates a nonlinear inductance at the input, whose value is: $\frac{1}{{g_m}^2} f(\frac{i_{IN}}{g_m})$.

8. A transfer function $H(f)$ is plotted below, which has a unity magnitude and phase of $-\alpha$ and $+\alpha$ for positive and negative frequencies, respectively. Prove that the impulse response is given by the following expression: $h(t) = \frac{sin\alpha}{\pi t} + cos\alpha \times \delta(t)$.




**Solution:** We show first that:

$$H(f) = cos\alpha + sin\alpha \times \begin{cases} -j & f > 0 \\ +j & f < 0 \end{cases}$$

Evidently:

$$|H(f)| = \sqrt{cos\alpha^2 + (\mp sin\alpha)^2} = 1$$

for all $f$.
And:

$$\angle H(f) = \begin{cases} \tan^{-1} \dfrac{-sin\alpha}{cos\alpha} = -\alpha & f > 0 \\ \tan^{-1} \dfrac{+sin\alpha}{cos\alpha} = +\alpha & f < 0 \end{cases}$$

We showed in the Hilbert transform section that the impulse response of a Hilbert function is:

$$h_Q(t) = \frac{1}{\pi t}$$

Thus, for the spectrum described in the figure above, the corresponding impulse response is:

$$h(t) = cos\alpha\delta(t) + \frac{sin\alpha}{\pi t}$$

which concludes the proof.

9. Find the input impedance of a polyphase filter loaded by an arbitrary load $Z_L$.

**Solution:** Consider a sequence of $(+V_{IN}, -jV_{IN}, -V_{IN}, +jV_{IN})$ applied to the input. Given the symmetry, it produces an output sequence of $(+V_{OUT}, -jV_{OUT}, -V_{OUT}, +jV_{OUT})$. Writing KCL for the top RC branch input, we have:

$$I_{IN} = G(V_{IN} - V_{OUT}) + jC\omega(V_{IN} + jV_{OUT})$$

Where $G = \frac{1}{R}$. A KCL at the top output node yields,

$$Y_L V_{OUT} + jC\omega(V_{IN} - jV_{OUT}) = G(V_{IN} - V_{OUT})$$

This latter equation directly leads to the transfer function of a loaded polyphase filter:

$$\frac{V_{OUT}}{V_{IN}} = \frac{G - C\omega}{G + jC\omega + Y_L}$$

It reveals that the null location is not affected if the filter is loaded. However, the passband loss is increased in general, as expected. Replacing $V_{OUT}$ given the transfer function just calculated, the input current is:

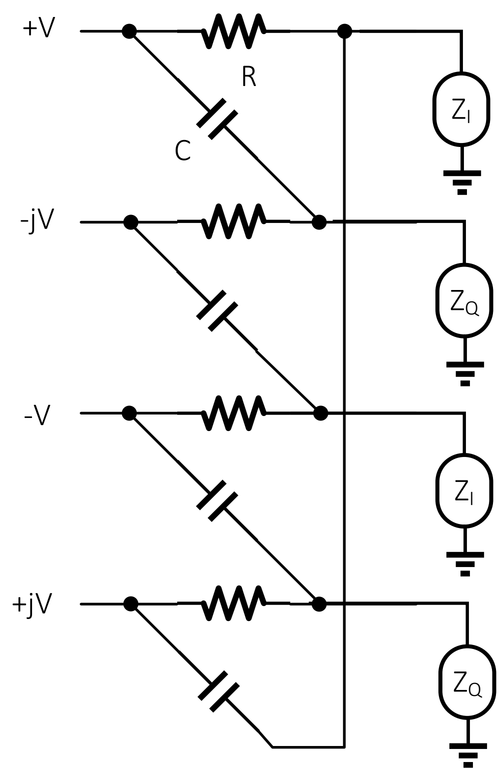
$$I_{IN} = GV_{IN}\left(1 - \frac{G - C\omega}{G + jC\omega + Y_L}\right) + jC\omega V_{IN}(1 + j\frac{G - C\omega}{G + jC\omega + Y_L})$$

This leads to:

$$Y_{IN} = G + jC\omega - \frac{(G + C\omega)(G - C\omega)}{G + jC\omega + Y_L}$$

Interestingly, at $\omega = \frac{\pm 1}{RC}$, always a parallel RC is seen. Accordingly, if a polyphase filter is loaded by an identical stage, at the passband for the first stage, $Y_L = G + jC\omega$, and the pass band gain is exactly reduced by 2×.

10. Consider the following 1-stage polyphase filter, where all the resistors and capacitors are identical, loaded by non-identical loads. Calculate the transfer function. What is the passband loss, and image rejection if the loads are identical? Repeat for the case the I and Q loads are different as shown.




**Solution:** The input sequence $(+V, -jV, -V, +jV)$, creates an output sequence of $(+V_1, +V_2, -V_1, -V_2)$. Given the asymmetry, we expect $V_2 \neq -jV_1$. However, we still expect differential voltages on I and Q sides. By writing a KCL at the top branch output, we obtain:

$$G(V - V_1) + jC\omega(jV - V_1) = Y_I V_1$$

Thus,

$$\frac{V_1}{V} = \frac{G - C\omega}{G + jC\omega + Y_I}$$

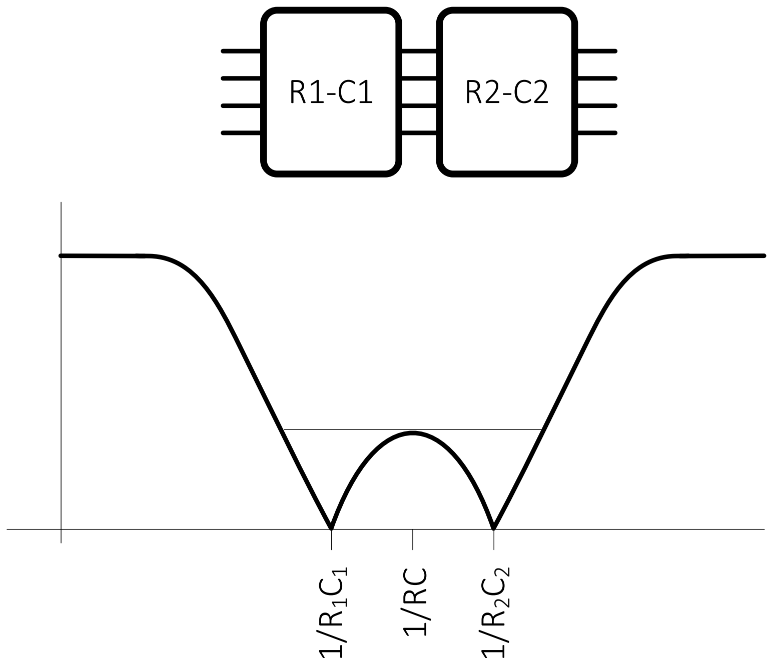
Similarly, we obtain:

$$\frac{V_2}{V} = -j\frac{G - C\omega}{G + jC\omega + Y_Q}$$

Unless the loads are identical, the two outputs are not necessarily $90^{\mathrm{o}}$ out of phase. However, the null location remains the same, as it is independent of the load impedance.
An expression for the transfer function in the case of matched load has been already derived in the previous problem

11. Design a 2-stage passive poly-phase filter to produce quadrature outputs from a differential clock signal at 1.6GHz. Assume the resistors and capacitors have each a process variation of 10%. Design for the best possible quadrature accuracy given the variation of the components.

**Solution:** Assume the stages are tuned to $\frac{1}{R_1C_1}$ and $\frac{1}{R_2C_2}$ as shown below. When driven by a complex 4-phase sequence, we expect the frequency response depicted in the figure. If this response satisfies the requirements, we expect the filter to produce the desired quadrature signal when driven by a differential input only.




Since RC product can vary by as much as 20%, or ±10%, the two nulls must be chosen such that on extreme case the nominal frequency (1.6GHz) is covered. Hence,

$$\frac{1}{R_2C_2} > \frac{1.6G}{0.9}$$
$$\frac{1}{R_1C_1} < \frac{1.6G}{1.1}$$

For simplicity, suppose:

$$R_1 = R_2 = R$$

Then, we must have:

$$C_1 = 1.1C$$
$$C_2 = 0.9C$$

Clearly, $\frac{1}{RC} \approx 1.6GHz$. The value of $R$ is determined based noise concerns, the acceptable loading, and how small the size of the capacitors may be. Once $R$ is known, the capacitances are obtained accordingly.
To cover the process variation, the nulls are not exactly at 1.6GHz, and thus, despite perfect matching, the IQ accuracy is finite, and set by the hump depth (figure above). To find that, we notice that the first stage is loaded by $R_2||C_2$. Thus,

$$\frac{V_{OUT1}}{V_{IN}} = \frac{1 - R_1C_1\omega}{1 + jR_1C_1\omega + R_1Y_{L2}} = \frac{1 - R_1C_1\omega}{1 + jR_1C_1\omega + R_1(\frac{1}{R_2} + jC_2\omega)}$$

Setting $\omega = \frac{1}{RC}$, we have:

$$\frac{V_{OUT1}}{V_{IN}} = \frac{1 - 1.1}{1 + j1.1 + 1 + j0.9} = \frac{-0.1}{2 + j2}$$

The second stage is not loaded, and hence:

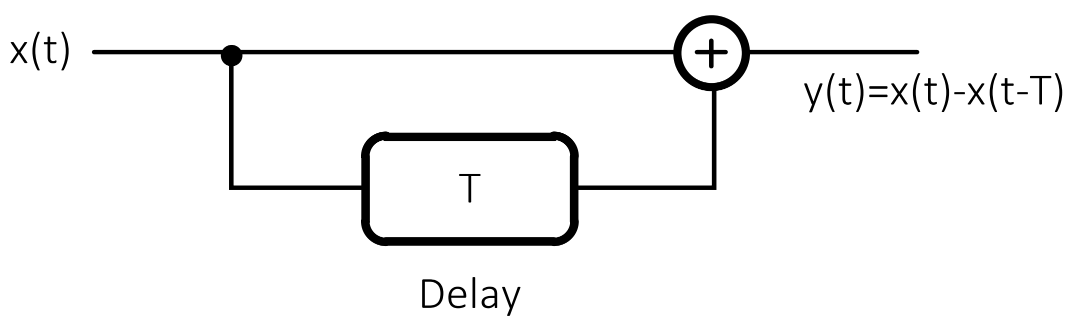
$$\frac{V_{OUT2}}{V_{OUT1}} = \frac{1 - R_2C_2\omega}{1 + jR_2C_2\omega} = \frac{1 - 0.9}{1 + j0.9} = \frac{0.1}{1 + j0.9}$$

Thus, the total rejection at 1.6GHz is:

$$\left|\frac{-0.1}{2 + j2}\right| \times \left|\frac{0.1}{1 + j0.9}\right| = 0.0026$$

Which is equivalent to the amount of quadrature accuracy obtainable under nominal case.

12. Consider the following comb filter, where the input $x(t)$ is a stationary random signal. Prove that the auto-correlation of the input and output are related according to the following equation: $R_y(\tau) = 2R_x(\tau) - R_x(\tau - T) - R_x(\tau + T)$.




**Solution:** We have:

$$R_y(\tau) = E[y(t + \tau)y^*(t)] = E[(x(t + \tau) - x(t - T + \tau))(x^*(t) - x^*(t - T))]$$

After expansion:

$$R_y(\tau) = E[x(t + \tau)x^*(t)] - E[x(t - T + \tau)x^*(t)] - E[x(t + \tau)x^*(t - T)] + E[x(t - T + \tau)x^*(t - T)] = R_x(\tau) - R_x(\tau - T) - R_x(\tau + T) + R_x(\tau)$$

Thus: $R_y(\tau) = 2R_x(\tau) - R_x(\tau - T) - R_x(\tau + T)$.

13. For the process: $x(t) = \boldsymbol{r}cos(\omega t + \boldsymbol{\phi})$, we assume that random variables **r** and **ϕ** are independent, and **ϕ** is uniform in the interval (-π , π). Find the mean and auto-correlation.

**Solution:** Given the independence:

$$E[x(t)] = \int_{-\infty}^{\infty}\int_{0}^{2\pi} f_R(r)\frac{1}{2\pi}rcos(\omega t + \phi)\, drd\phi = E[r]\int_{0}^{2\pi}\frac{1}{2\pi}cos(\omega t + \phi)d\phi$$

The integral $\int_{-\infty}^{\infty}\frac{1}{2\pi}cos(\omega t+\phi)d\phi$ is zero given the cosine term averaging out. Thus:

$$E[x(t)] = 0$$

Similarly,

$$R_x(t_1,t_1) = E[r^2\cos(\omega t_1+\phi)\cos(\omega t_2+\phi)]$$

Given the independence,

$$R_x(t_1,t_1) = E[r^2]E[\cos(\omega t_1+\phi)\cos(\omega t_2+\phi)]$$

As for the $E[\cos(\omega t_1+\phi)\cos(\omega t_2+\phi)]$ component, upon expansion, the terms at sum frequency averages out, and thus:

$$E[\cos(\omega t_1+\phi)\cos(\omega t_2+\phi)] = \frac{1}{2}\int_0^{2\pi}\frac{1}{2\pi}cos\omega(t_1-t_2)d\phi = \frac{1}{2}cos\omega(t_1-t_2)$$

Thus:

$$R_x(t_1,t_1) = E[r^2]cos\omega(t_1-t_2)$$

Note that $r$ is a random variable (and not a random process).

14. For the process: $x(t) = acos(\boldsymbol{\omega t}+\boldsymbol{\phi})$, the random variable **ω** has the probability density function of f(ω), and the random variable **ϕ** is uniform in the interval (-π , π), and independent of **ω**. Find the mean and autocorrelation of x(t).

**Solution:** We have:

$$E[x(t)] = \int_{-\infty}^{\infty}\int_0^{2\pi} f(\omega)\frac{1}{2\pi}acos(\omega t+\phi)\,d\phi d\omega$$

After expansion:

$$E[x(t)] = a\int_{-\infty}^{\infty}\int_0^{2\pi}\frac{f(\omega)}{2\pi}[cos\omega t cos\phi - sin\omega t sin\phi]\,d\phi d\omega$$

Given the independence,

$$E[x(t)] = a\left[\int_{-\infty}^{\infty} f(\omega)\,cos\omega t d\omega\right]\left[\int_0^{2\pi}\frac{cos\phi}{2\pi}d\phi\right] - a\left[\int_{-\infty}^{\infty} f(\omega)\,sin\omega t d\omega\right]\left[\int_0^{2\pi}\frac{sin\phi}{2\pi}d\phi\right]$$

Which clearly yields:

$$E[x(t)] = 0$$

Similarly,

$$R_x(t_1,t_1) = a^2E[cos(\omega t_1+\phi)cos(\omega t_2+\phi)]$$

After expansion:

$$R_x(t_1,t_1) = \frac{a^2}{2}E[cos(\omega(t_1+t_2)+2\phi)+cos\omega(t_1-t_2)]$$

The same argument as we used for the mean, reveals that the first terms is zero, and thus:

$$R_x(t_1,t_1) = \frac{a^2}{2}E[cos\omega(t_1-t_2)]$$

Which indicates that the process is stationary in a wide-sense.

15. Prove the equation: $E[L\{x(t)\}] = L\{E[x(t)]\}$, where $L$ denotes the linear operation imposed by convolution integral.

**Solution:** As $L$ denotes the linear operations, we can write:

$$E[L\{x(t)\}] = E\left[\int_{-\infty}^{\infty} x(\theta)h(t-\theta)d\,\theta\right] = \int_{-\infty}^{\infty} E[x(\theta)]\,h(t-\theta)d\theta = L\{E[x(t)]\}$$

Which concludes the proof.

16. Suppose x(t) is a stationary process with zero mean, and autocorrelation $R_x(\tau)$, and random variable s is defined as: $s = \int_{-T}^{T} x(t)dt$. Find the variance of s (${\sigma_s}^2$) in terms of $R_x(\tau)$.

**Answer:** ${\sigma_s}^2 = \int_{-2T}^{2T}(2T - |\tau|)R_x(\tau)d\tau$.

**Solution:** Clearly,

$$E[s] = E[\int_{-T}^{T} x(t)dt] = \int_{-T}^{T} E[x(t)]dt = 0$$

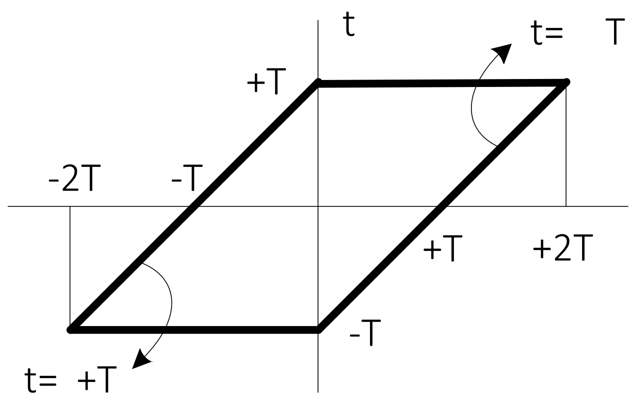
Thus:

$${\sigma_s}^2 = E[s^2] = E\left[\int_{-T}^{T}\int_{-T}^{T} x(t_1)x(t_2)dt_1dt_2\right] = \int_{-T}^{T}\int_{-T}^{T} E[x(t_1)x(t_2)]dt_1dt_2$$

Consequently:

$${\sigma_s}^2 = \int_{-T}^{T}\int_{-T}^{T} R_x(t_1 - t_2)dt_1dt_2$$

The integral may be simplified considering the figure below.




We shall make the following two variable changes:

$$t_1 - t_2 = \tau$$
$$t_1 = t$$

Thus: $t = \tau + t_2$. The figure above plots $t_1$ and $t_2$ on $t - \tau$ plane. Since $t_1$ and $t_2$ vary between $-T$ and $T$, $\tau$ varies from $-2T$ to $2T$, and $t$ follows $\tau$ on $\tau + T$ line on the left half of the plane, and $\tau - T$ line on the right half. Accordingly, the integral may be expressed as:

$$\sigma_s{}^2 = \int_{-2T}^{0} R_x(\tau)(\int_{-T}^{\tau+T} dt)d\tau + \int_{0}^{2T} R_x(\tau)(\int_{\tau-T}^{T} dt)d\tau$$

This leads to:

$$\sigma_s{}^2 = \int_{-2T}^{0} R_x(\tau)(2T + \tau)d\tau + \int_{0}^{2T} R_x(\tau)(2T - \tau)d\tau$$

Or:

$$\sigma_s{}^2 = \int_{-2T}^{2T} R_x(\tau)(2T - |\tau|)d\tau$$

If $x(t)$ is white, $R_x(\tau)$ will be an impulse, and $\sigma_s{}^2 = 2T$.

17. For the pulse amplitude modulated process: $v(t) = \sum_{n=-\infty}^{\infty} a_n h(t - nT)$, we assume $a_n$ is a stationary sequence, with autocorrelation: $R_a(n) = E[a_{n+m}a_m]$, and spectral density: $S_a(f) = \sum_{n=-\infty}^{\infty} R_a(n)\, e^{-j2\pi f n}$. We form the impulse train $w(t) = \sum_{n=-\infty}^{\infty} a_n \delta(t - nT)$. Show that for the impulse train process the autocorrelation of the shifted process $w_s(t) = w(t - \theta)$ is: $R_{w_s}(\tau) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R_a(n)\delta(\tau - nT)$. Show that the spectral density for the shifted process $v_s(t) = v(t - \theta)$ is : $S_{v_s}(f) = \frac{1}{T} S_a(f)|H(f)|^2$. **Hint:** $v(t)$ is the output of a linear system with input $w(t)$. Thus: $v(t) = h * w(t)$, and $v_s(t) = h * w_s(t)$.

**Solution:** We form:

$$R_w(t,\tau) = E\left[\sum_n a_n \delta(t + \tau - nT) \sum_m a_m{}^* \delta(t - mT)\right]$$

Which may be rearranged as:

$$R_w(t,\tau) = \sum_n \sum_m E[a_n a_m{}^*]\delta(t + \tau - nT)\delta(t - mT)$$

Thus, by definition:

$$R_w(t,\tau) = \sum_n \sum_m R_a(n - m)\delta(t + \tau - nT)\delta(t - mT)$$

The shifted process was proven to be of the form:

$$R_{w_s}(\tau) = \frac{1}{T}\int_T R_w(t,\tau)\, dt$$

Hence:

$$R_{w_s}(\tau) = \frac{1}{T}\int_T \sum_n \sum_m R_a(n-m)\delta(t+\tau-nT)\delta(t-mT)\, dt$$

Rearranging the integral:

$$R_{w_s}(\tau) = \frac{1}{T}\sum_n \sum_m R_a(n-m)\int_T \delta(t+\tau-nT)\delta(t-mT)dt = \frac{1}{T}\sum_n R_a(n)\delta(\tau-nT)$$

Hence:

$$S_{w_s}(f) = \frac{1}{T}\sum_n R_a(n)e^{-j2\pi fn} = \frac{1}{T}S_a(f)$$

As far as $v(t)$ is concerned, it is evidently produced by passing $w(t)$ through an LTI system whose impulse response is: $h(t)$. That is because:

$$v(t) = \sum_n a_n h(t-nT) = h * [\sum_n a_n\delta(t-nT)] = h * w(t)$$

Clearly,

$$v_s(t) = v(t-\theta) = h * w_s(t)$$

Thus:

$$S_{v_s}(f) = S_{w_s}(f)|H(f)|^2 = \frac{1}{T}S_a(f)|H(f)|^2$$

Which concludes the proof.

18. In the previous problem, suppose $h(t)$ is a pulse with width T, and is $a_n$ is white noise taking the values ±1 with equal probability. The resulting process is called *binary transmission*. It is a cyclostationary process taking the values of ±1 in every interval T. Show that: $S_{v_s}(\omega) = \frac{4sin^2(\omega T/2)}{\omega^2 T}$.

**Solution:** If $h(t)$ is a pulse with width T, its Fourier transform is:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt = \int_0^T e^{-j\omega t}dt = 2e^{-j\omega T/2}\frac{sin\omega T/2}{\omega}$$

Thus, from the previous problem:

$$S_{v_s}(\omega) = \frac{1}{T}S_a(\omega)|H(\omega)|^2 = \frac{4sin^2(\omega T/2)}{\omega^2 T}$$

Where $S_a(\omega) = 1$, given it is white.

19. Assume that $x$ is a random variable with mean $\mu$ and standard deviation $\sigma$. (a) Prove that the following inequality is held for any arbitrary values of $a$ and $b$ ($b$ is positive): $p\{|x-a| \geq b\} \leq \frac{E[(x-a)^2]}{b^2}$. (b) Use the above inequality to prove the following extension of Chebyshev's inequality: $p\{k_1 < x < k_2\} \geq \frac{4\{(\mu-k_1)(k_2-\mu)-\sigma^2\}}{(k_2-k_1)^2}$. Chebyshev's inequality is a special case of the above inequality where $k_1 = \mu - k\sigma$ and $k_2 = \mu + k\sigma$.

**Solution:** For $|x-a| \geq b$, we must have $x \geq a+b$ or $x \leq a-b$. Thus, based on the definition, we have:

$$p\{|x-a| \geq b\} = \int_{-\infty}^{a-b} f_X(x)dx + \int_{a+b}^{+\infty} f_X(x)dx$$

Since over these two intervals $\frac{(x-a)^2}{b^2} > 1$, we can write:

$$p\{|x-a| \geq b\} \leq \int_{-\infty}^{a-b} \frac{(x-a)^2}{b^2} f_X(x)dx + \int_{a+b}^{+\infty} \frac{(x-a)^2}{b^2} f_X(x)\, dx$$

$$\leq \int_{-\infty}^{+\infty} \frac{(x-a)^2}{b^2} f_x(x)dx = \frac{E[(x-a)^2]}{b^2}$$

For the second part, we have:

$$E[(x-a)^2] = E[(x-\mu+\mu-a)^2]$$

After expansion:

$$E[(x-a)^2] = E[(x-\mu)^2] + 2(\mu-a)E[x-\mu] + (\mu-a)^2$$

Since $E[x] = \mu$, the equation simplifies to:

$$E[(x-a)^2] = \sigma^2 + (\mu-a)^2$$

If we replace $a = \frac{k_1+k_2}{2}$ and $b = \frac{k_2-k_1}{2}$ in inequality of part (a), and knowing the fact that $p\{k_1 < x < k_2\} = 1 - p\{|x-a| \geq b\}$, we will have:

$$p\{k_1 < x < k_2\} \geq 1 - \frac{E[(x-a)^2]}{b^2} = 1 - \frac{\sigma^2 + (\mu-a)^2}{b^2}$$

Hence:

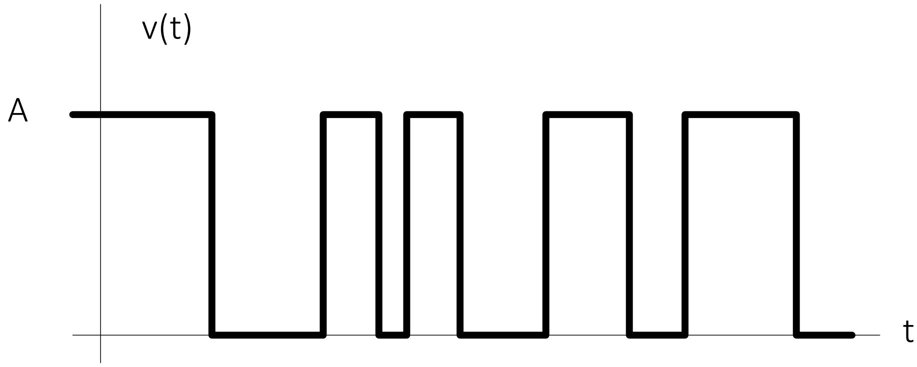
$$p\{k_1 < x < k_2\} \geq \frac{b^2 - \sigma^2 - (\mu-a)^2}{b^2}$$

Or:

$$p\{k_1 < x < k_2\} \geq \frac{(\mu-a+b)(b+a-\mu) - \sigma^2}{b^2}$$

Plugging $a = \frac{k_1+k_2}{2}$ and $b = \frac{k_2-k_1}{2}$ into the equation above, leads to the proof of the inequality in part (b).

20. A random telegraph signal assumes only two values, 0 and A, which happen with equal probabilities. The signal makes independent random shifts between these two values. The number of shifts per unit time is governed by the Poisson distribution with μ being the average shift rate ($p\{x = k, over\ time\ \tau\} = \frac{\mu^k}{k!} e^{-\mu\tau}$). (a) Prove that the auto-correlation function of the signal is found to be: $R_v(\tau) = \frac{A^2}{4}(1 + e^{-2\mu|\tau|})$. (b) Use the above auto-correlation to find the power spectral density.




**Solution:** The autocorrelation is:

$$R_v(\tau) = E[v(t+\tau)v(t)]$$

From the definition of conditional probability, we may rewrite:

$$R_v(\tau) = p\{v(t) = A\}E[v(t)v(t+\tau)|v(t) = A] + p\{v(t) = 0\}E[v(t)v(t+\tau)|v(t) = 0]$$

Noticing that:

$$E[v(t)v(t+\tau)|v(t) = 0] = 0$$

and that there is 50% probability for $v(t)$ to assume a value of $A$, we will have:

$$R_v(\tau) = \frac{1}{2}E[v(t)v(t+\tau)|v(t) = A]$$

When $v(t) = A$ at $t$, the next transition would lead $v$ to be 0 and the one after leads $v$ back to $A$. Thus, $v(t)v(t+\tau)$ will be equal to $A^2$ only if *even* number of transitions take place between $t$ and $t+\tau$, and zero otherwise. Thus:

$$R_v(\tau) = \frac{1}{2}E\{v(t)v(t+\tau)|v(t) = A\} = \frac{1}{2}A^2 p\{even\ number\ of\ transitions\}$$

Given the probability density function of the random telegraph signal is Poisson, we have:

$$R_v(\tau) = \frac{1}{2}A^2\left\{\frac{\mu^0}{0!}e^{-\mu|\tau|} + \frac{\mu^2}{2!}e^{-\mu|\tau|} + \frac{\mu^4}{4!}e^{-\mu|\tau|} + ..\right\}$$

Using the exponential function Taylor series expansion, we can write:

$$e^x + e^{-x} = 2(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)$$

Which leads to:

$$R_v(\tau) = \frac{1}{4}A^2\left(e^{+\mu|\tau|} + e^{-\mu|\tau|}\right)e^{-\mu|\tau|} = \frac{A^2}{4}(1 + e^{-2\mu|\tau|})$$

Finally, from the Fourier transform definition, we can readily show:

$$S_v(f) = \frac{A^2}{4}\delta(f) + \frac{A^2}{4\mu(1 + (\pi f/\mu)^2)}$$

which is spectral density of the random telegraph signal.

21. Show that the FM signal: $v_C(t) = A_C\cos\,(\omega_C t + 2\pi f_\Delta \int x(\tau)d\tau)$, satisfies the following equation, known as *FM differential equation*:

$$v_C(t) - \frac{v_C{}'(t)\omega_i{}'(t)}{\omega_i(t)^3} + \frac{v_C{}''(t)}{\omega_i(t)^2} = 0$$

Where $\omega_i(t)$ is the instantaneous frequency.

**Solution:** We can express the FM signal as:

$$v_C(t) = A_C cos\,(\int \omega_i(\tau)\, d\tau)$$

where $\omega_i(t) = \omega_C + 2\pi f_\Delta x(t)$ is the instantaneous frequency. Taking the derivative of $v_C(t)$:

$$v_C' = -\omega_i A_C sin\,(\int \omega_i(\tau)\, d\tau)$$

And the second derivative is:

$$v_C'' = -\omega_i' A_C sin\left(\int \omega_i(\tau)\, d\tau\right) - \omega_i{}^2 A_C cos\left(\int \omega_i(\tau)\, d\tau\right) = \omega_i' \frac{v_C'}{\omega_i} - \omega_i{}^2 v_C$$

Rearranging, and dividing by $\omega_i{}^2$ leads to the FM differential equation.

22. Show that the FM signal also satisfies the following integro-differential equation:

$$v_C(t) + \int_0^t \omega_i(\theta)\left(\int_0^\theta \omega_i(\tau) v_C(\tau) d\tau\right) d\theta = 0$$

Accordingly, devise an FM modulator by a circuit comprising of multipliers and integrators (known as *analog computer*) that satisfies the equation.

**Solution:** Taking a derivative from the equation above yields:

$$v_C' + \omega_i \int_0^\theta \omega_i(\tau) v_C(\tau) d\tau = 0$$

And the second derivative gives:

$$v_C'' + \omega_i{}' \int_0^\theta \omega_i(\tau) v_C(\tau) d\tau + \omega_i{}^2 v_C = 0$$

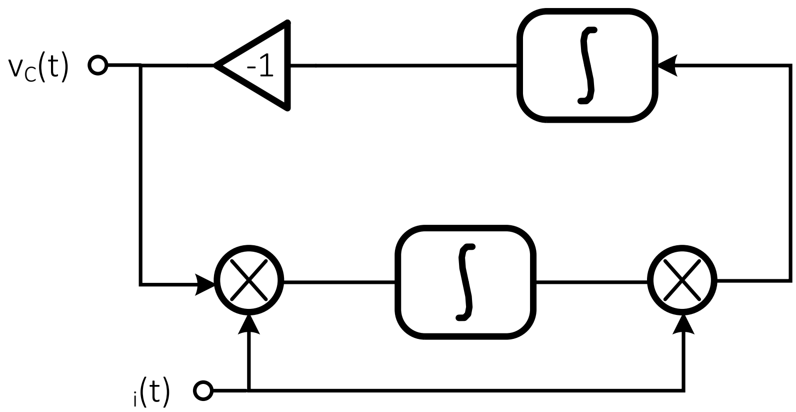
On the other hand, from the previous equation:

$$\int_0^\theta \omega_i(\tau) v_C(\tau) d\tau = -\frac{v_C'}{\omega_i}$$

Thus, we can write:

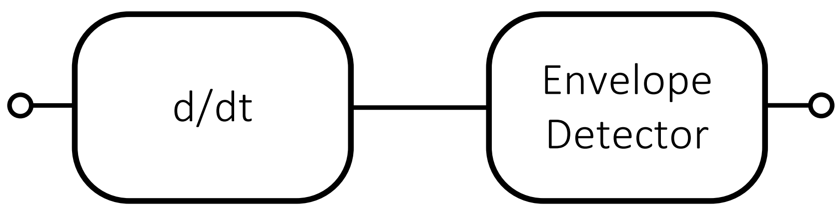
$$v_C'' - \omega_i{}' \frac{v_C'}{\omega_i} + \omega_i{}^2 v_C = 0$$

Which is same as the FM differential equation obtained in the previous problem.
A system comprising integrators, and multipliers to realize the FM integro-differential equation is depicted below.

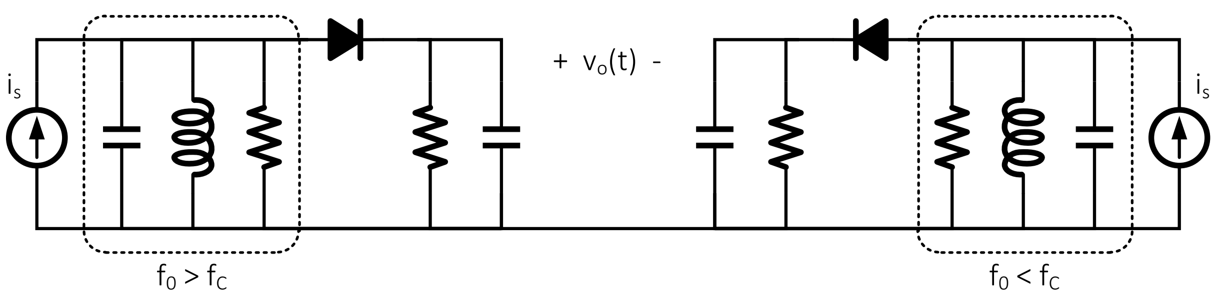



From the equation, the signal must be multiplied by the instantaneous frequency ($\omega_i$), integrated, multiplied by $\omega_i$ and integrated again, and subtracted from itself. Thus, the output of the system shown above is a sinusoid whose instantaneous frequency (the input of the system) is set by $\omega_i(t)$.

23. Show that the following circuit may be employed as an FM demodulator. Propose a proper circuitry to perform the differentiation suitable for high frequencies.




**Hint:** Use the circuit below known as a *balanced slope demodulator*.



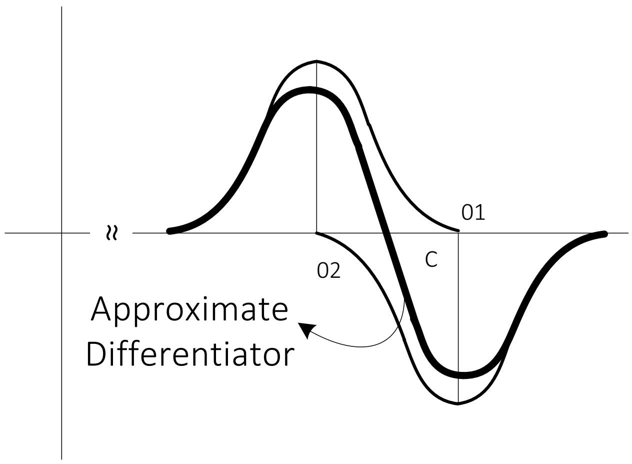

**Solution:** Given the FM signal:

$$v_C(t) = A_C cos\,(\int \omega_i(\tau)\,d\tau)$$

once differentiated, we will have:

$$v_C' = A_C \omega_i(t) sin\,(\int \omega_i(\tau)\, d\tau)$$

Clearly, the envelope of the differentiated signal above is proportional to the instantaneous frequency. Thus, all needed is to pass $v_C'$ thought an envelope detector, and extract $\omega_i(t)$. This is then essentially an FM demodulator.
Realizing a good differentiator at high frequency may not be a trivial task. A suitable circuit to accomplish this is shown above, and whose frequency response is depicted below.




The circuit, known as a *balanced slope detector*, comprises of two LC tanks, one slightly high tuned, where as the other is slightly low tuned with respect to the carrier frequency. The FM signal is fed to each tank, performing the differentiation, after which is passed to an envelope detector as desired. From the frequency response plotted above, once the two LC tanks outputs are subtracted, around the carrier frequency ($\omega_C$), the composite response may be approximated by a straight line, hence representing a differentiator (whose frequency response is ideally $K\omega$).
To understand better how the slope detector works, suppose the carrier frequency is $\omega_C$, and that the LC tanks are tuned to $\omega_{01}$ and $\omega_{02}$ respectively. Furthermore, we shall assume the two tanks have equal resistor ($R$) and capacitor ($C$), and only the inductors are different to tune them as desired. If the tank Q is high enough (say ten or more), the composite response may be expressed as:

$$|Z_T(j\omega)| \approx \frac{R}{\sqrt{1+\left(\frac{\omega-\omega_{01}}{\alpha}\right)^2}} - \frac{R}{\sqrt{1+(\frac{\omega-\omega_{02}}{\alpha})^2}}$$

where $\omega_{01}$ and $\omega_{02}$ are the tank center frequencies, and $\alpha = \frac{1}{2RC} = \frac{BW}{2}$. In addition, we shall set:

$$\omega_{01} = \omega_C + \Delta\omega$$
$$\omega_{02} = \omega_C - \Delta\omega$$

Which is to say the two tanks are symmetrically high and low tuned around carrier frequency ($\omega_C$) by an arbitrary amount of $\Delta\omega$, whose optimum value, we shall determine shortly.
To gain more insight, let us use Taylor series to expand $|Z_T(j\omega)|$:

$$|Z_T(j\omega)| = |Z_T(j\omega)|'(\omega-\omega_C) + |Z_T(j\omega)|'''(\omega-\omega_C)^3 + |Z_T(j\omega)|'''''(\omega-\omega_C)^5 + \cdots$$

Note that the balanced operation results in even-order derivatives not to appear.
One can verify that by choosing:

$$\Delta\omega = \sqrt{\frac{3}{2}}\alpha$$

the 3rd-order derivative is eliminated, and thus:

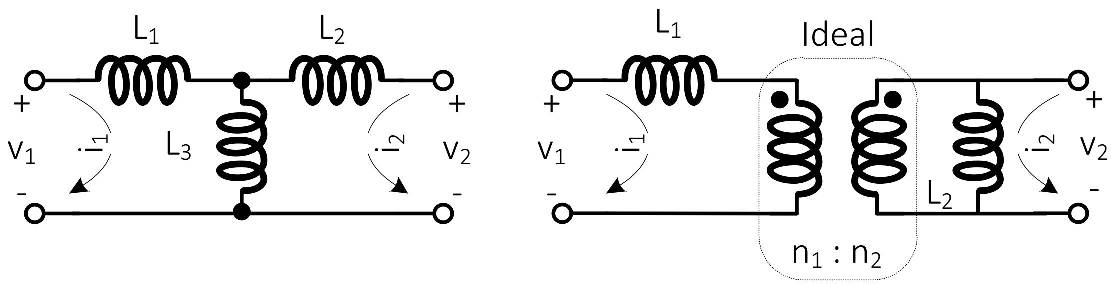
$$|Z_T(j\omega)| = \frac{4R}{5}\sqrt{\frac{3}{5}}\left[\frac{\omega-\omega_C}{\alpha} - \frac{64}{625}\left(\frac{\omega-\omega_C}{\alpha}\right)^5 + \cdots\right]$$

For small frequency deviations with respect to bandwidth, the second terms may be ignored, and thus a linear response is obtained as desired. To be precise, for $\left|\frac{\omega-\omega_C}{\alpha}\right| < 0.5$, the second terms is less than 0.5% and shall be comfortably ignored.
The slope detector may be realized using only one LC tank based on the same principle.
However, the particular circuit proposed above has all the nice properties of the balanced circuits in general. More details of the circuit operation may be found in Clarke and Hess[4].

[4] K. Clarke, and D. Hess, Communications circuits: analysis and design.

# 3 Chapter Three

1. Find the L matrix for the following circuits:




**Solution:** The voltages and currents are labeled as shown in the figure. For the circuit on the left, we can write:

$$v_1 = L_1 \frac{d}{dt} i_1 + L_3 \frac{d}{dt}(i_1 + i_2) = (L_1 + L_3)\frac{d}{dt} i_1 + L_3 \frac{d}{dt} i_2$$
$$v_2 = L_2 \frac{d}{dt} i_2 + L_3 \frac{d}{dt}(i_1 + i_2) = L_3 \frac{d}{dt} i_1 + (L_2 + L_3)\frac{d}{dt} i_2$$

Thus:

$$L = \begin{bmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{bmatrix}$$

For the circuit on right, we have:

$$v_1 = L_1 \frac{d}{dt} i_1 + \frac{n_1}{n_2} v_2$$
$$v_2 = L_2 \frac{d}{dt}\left(i_2 + \frac{n_1}{n_2} i_1\right) = \left(\frac{n_1}{n_2} L_2\right)\frac{d}{dt} i_1 + L_2 \frac{d}{dt} i_2$$

Expressing $v_2$ based on the currents in the $v_1$ equation yields:

$$v_1 = L_1 \frac{d}{dt} i_1 + \frac{n_1}{n_2}\left[L_2 \frac{d}{dt}\left(i_2 + \frac{n_1}{n_2} i_1\right)\right] = \left[L_1 + (\frac{n_1}{n_2})^2 L_2\right]\frac{d}{dt} i_1 + (\frac{n_1}{n_2} L_2)\frac{d}{dt} i_2$$

Thus:

$$L = \begin{bmatrix} L_1 + (\frac{n_1}{n_2})^2 L_2 & \frac{n_1}{n_2} L_2 \\ \frac{n_1}{n_2} L_2 & L_2 \end{bmatrix}$$

Notice that for both circuits, $L_{12} = L_{21}$.

2. Find the transfer function and input impedance of the double-tuned circuit shown below.