CHAPTER 2

Mathematical Preliminaries Solutions

2.1

$||A|| = \sqrt{(3-0)^2 + (4-(-1))^2 + ((-1)-1)^2} = 6.164$ units.

2.2

(a) **a**+**b**=**i**+5**j**+**k**

(b) **b**+**a**=**i**+5**j**+**k**

(c) 3**a**+3**b**=3**i**+15**j**+3**k**

(d) 3(**a**+**b**)=3**i**+15**j**+3**k**

(e) (**a**+**b**)+**c**=3**i**+8**j**-3**k**

(f) **a**+(**b**+**c**)=3**i**+8**j**-3**k**

(g) Observe that **a**+**b**=**b**+**a**, 3**a**+3**b**=3(**a**+**b**), and (**a**+**b**)+**c**=**a**+(**b**+**c**).

(h) Vector addition is Commutative, Associative and Distributive.

2.3

(a) **a**.**b**=-8

(b) **b**.**a**=-8

(c) Dot product of two vectors is commutative.

(d) $\theta = 113.4°$

2.4

(a) **a**.(**b**+**c**)=-1

(b) **a**.(**b**-**c**)=-49

(c) **a**.**b**+**a**.**c**=-1

(d) **a**.**b**-**a**.**c**=-49

(e) Observe that **a**.(**b**+**c**)=**a**.**b**+**a**.**c** and **a**.(**b**-**c**)=**a**.**b**-**a**.**c**.

2.5

(a) Work done = 0.

(b) The work done is zero because there is no displacement in the direction of the force.

2.6

(a) 90°

(b) 90°

(c) 90°

(d) **a**, **b**, and **c** are orthogonal.

2.7

$2x$-$5y$+14=0; Slope=$\frac{2}{5}$

2.8

$\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 1$

2.9

(a)
```
>> (A*B)'
ans =
     3     3
     3     4
```

(b)
```
>> B'*A'
ans =
     3     3
     3     4
```

Note that `(AB)'=B'A'`.

2.10

(a)
```
>> A'
ans =
     1     0     2
     0     3     1
     2     4     3
```

(b)
```
>> (A')'
ans =
     1     0     2
     0     3     4
     2     1     3
```

(c)
```
>> (A+B)'
ans =
     1     2     7
     2     6     2
     6     8     6
```

(d)
```
>> A'+B'
ans =
     1     2     7
     2     6     2
     6     8     6
```

(e) `(A')'=A`, `(A+B)'=A'+B'`

2.11

```
>> A'
ans =
     0    -2     1
     2     0     4
    -1    -4     0
```

`A'=-A`. Such matrices are called Skew Symmetric matrices.

2.12

(1) Lower Triangular Matrix:

```
>> A=[1 0 0;-2 3 0;5 4 1]
A =
     1     0     0
    -2     3     0
     5     4     1
>> det(A)
ans =
     3
```

It can be seen that the product of the diagonal elements $= 1 \times 3 \times 1 = 3$.

(2) Upper Triangular Matrix:

```
>> B=[-11 5 9;0 7 16;0 0 1]
B =
   -11     5     9
     0     7    16
     0     0     1
>> det(B)
ans =
   -77
```

It can be seen that the product of the diagonal elements $= -11 \times 7 \times 1 = -77$.

2.13

det(A)=-7

2.14

$I^{-1} = I$

2.15

$|A - \lambda I| = 0$
$(1-\lambda)(2-\lambda) - 2 \times 3 = 0$
$\lambda_1 = -1$
$\lambda_2 = 4$

```
>> [x,v]=eig([1 2;3 2])
x =
   -0.7071   -0.5547
    0.7071   -0.8321
v =
    -1     0
     0     4
```

2.16

$|A - \lambda I| = 0$
$(3-\lambda)(2-\lambda)(1-\lambda) + 2 + 2 - (3-\lambda) - 4(1-\lambda) - (2-\lambda) = 0$
$-\lambda^3 + 6\lambda^2 - 5\lambda + 1 = 0$

```
>> [x,v]=eig(A)
x =
   -0.3280   -0.5910    0.7370
    0.7370    0.3280    0.5910
   -0.5910    0.7370    0.3280
v =
    0.3080         0         0
         0    0.6431         0
         0         0    5.0489
```

The matrix is positive definite, because all of its eigen values are positive.

2.17

(a) $\frac{4}{7}$
(b) $\frac{1}{3}$
(c) $-\frac{1}{4}$

2.18

(a) Convex

(b) Convex for $x > 0$.

(c) Not Convex.

2.19

(a) $4x^3 + 3x^2 + 8x + 2$

(b) $2x + 1 - \frac{1}{x^2} - \frac{2}{x^3}$

(c) $log(x) + \frac{1}{ln10}$

(d) $\cos(x)$

(e) $\frac{-\cos(x)}{x^2} - \frac{\sin(x)}{x}$

(f) $\sec^2(x)$

2.20

(a) $3x^2 + 2xy^2 + 4x + 2$

(b) $2x + 1$

(c) $\log y$

(d) $-\frac{\cos y}{x^2}$

(e) $\frac{1}{\sqrt{2x+3y}}$

(f) $e^x(\tan(x-y) + \sec^2(x-y))$

2.21

(a) Ist order $f(x) = x$; IInd order $f(x) = x$. IIIrd order $f(x) = x - \frac{x^3}{3}$. If you consider more higher order terms, we get a better approximation of the function. See the plot in Fig. 2.1.

(b) 1

(c) 1

(d) 1

2.22

(a)

$$H = \begin{pmatrix} 2c_{11} & c_{12}+c_{21} & c_{13}+c_{31} & c_{14}+c_{41} \\ c_{12}+c_{21} & 2c_{22} & c_{23}+c_{32} & c_{24}+c_{42} \\ c_{13}+c_{31} & c_{23}+c_{32} & 2c_{33} & c_{34}+c_{43} \\ c_{14}+c_{41} & c_{24}+c_{42} & c_{34}+c_{43} & 2c_{44} \end{pmatrix} \tag{2.1}$$

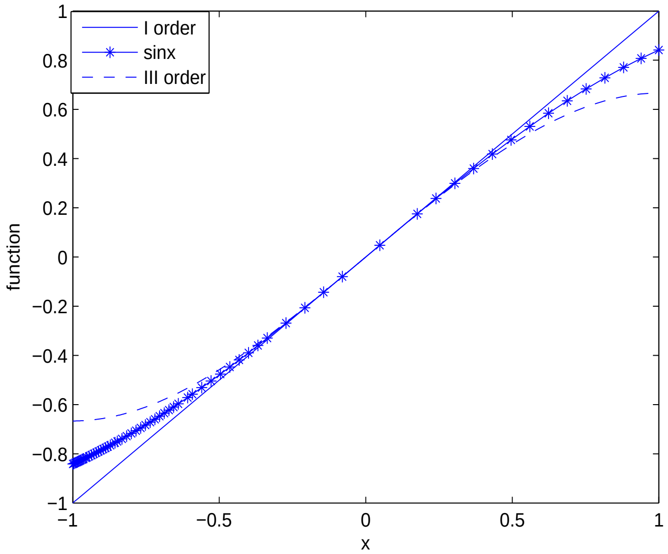

Figure 2.1: Plots for Prob 2.21

(b)

$$\frac{\partial f}{\partial x_1} = N(\sin x_1 + \cos x_2)^{N-1}\cos x_1 \tag{2.2}$$

$$\frac{\partial f}{\partial x_2} = N(\sin x_1 + \cos x_2)^{N-1}(-\sin x_2) \tag{2.3}$$

$$H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \tag{2.4}$$

where
$H_1 = N(N-1)(\sin x_1 + \cos x_2)^{N-2}(\cos x_1)^2 - N(\sin x_1 + \cos x_2)^{N-1}\sin x_1$,
$H_2 = -N(N-1)(\sin x_1 + \cos x_2)^{N-2}\sin x_2 \cos x_1$,
$H_3 = -N(N-1)(\sin x_1 + \cos x_2)^{N-2}\sin x_2 \cos x_1$,
$H_1 = N(N-1)(\sin x_1 + \cos x_2)^{N-2}(-\sin x_2)^2 - N(\sin x_1 + \cos x_2)^{N-1}\cos x_2$.

(c)

$$\frac{\partial f}{\partial x_1} = \ln x_2 + \frac{x_2}{x_1} \tag{2.5}$$

$$\frac{\partial f}{\partial x_2} = \frac{x_1}{x_2} + \ln x_1 \tag{2.6}$$

$$H = \begin{pmatrix} -\frac{x_2}{x_1^2} & \frac{1}{x_2} + \frac{1}{x_1} \\ \frac{1}{x_2} + \frac{1}{x_1} & -\frac{x_1}{x_2^2} \end{pmatrix} \tag{2.7}$$

2.23

(a)
```
clear;clc;
% lower and upper limits
lb = -5;
ub = 500;
% define matrix dimension
m=5; n=5;
% use rand(m, n) to generate matrix
% with random entries between 0 and 1
% and scale it to [-5, 500]
A = lb + (ub-lb).*rand(m,n);
e = eig(A);
% to ensure e does not contain complex number
while ~isreal(e)
    A = lb + (ub-lb).*rand(m,n);
    e = eig(A);
end
% number of pos., neg., and zero
pos = 0;
neg = 0;
zero = 0;
% browsing e
for i=1:1:5
    if e(i)> 0
        pos = pos+1;
    elseif e(i) < 0;
        neg = neg+1;
    else
        zero = zero+1;
    end
end

if pos == 5
    fprintf('The matrix is positive definite.\n');
elseif neg == 5
    fprintf('The matrix is negative definite.\n');
elseif neg == 0 && zero >=1
    fprintf('The matrix is positive semi-definite.\n');
elseif pos == 0 && zero >=1
    fprintf('The matrix is negative semi-definite.\n');
else
    fprintf('The matrix is indefinite.\n');
end
```

2.24

1.

$$\frac{\partial f}{\partial x_1} = 4x_1 + 4x_2 + 4 \tag{2.8}$$

$$\frac{\partial f}{\partial x_2} = -6x_2 + 4x_1 \tag{2.9}$$

$$\frac{\partial f}{\partial x_3} = 2x_3 + 4 \tag{2.10}$$

$$H = \begin{pmatrix} 4 & 4 & 0 \\ 4 & -6 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{2.11}$$

2. $f(x)$ has stationary points when the gradient is 0.

$$\frac{\partial f}{\partial x_1} = 4x_1 + 4x_2 + 4 = 0 \tag{2.12}$$

$$\frac{\partial f}{\partial x_2} = -6x_2 + 4x_1 = 0 \tag{2.13}$$

$$\frac{\partial f}{\partial x_3} = 2x_3 + 4 = 0 \tag{2.14}$$

$$x_1 = \frac{-3}{5}, x_2 = \frac{-2}{5}, x_3 = -2 \tag{2.15}$$

Point $(x_1, x_2, x_3) = (\frac{-3}{5}, \frac{-2}{5}, -2)$ is a stationary point.

3. The eigenvalues of the Hessian are -7.4031, 2, and 5.4031. This means that the hessian matrix is indefinite and the stationary point defined above can be classified as a saddle point.

2.25

1.

$$\frac{\partial f}{\partial x_1} = 5x_1 + 2x_2 + 4 \tag{2.16}$$

$$\frac{\partial f}{\partial x_2} = 5x_2 + 2x_1 \tag{2.17}$$

$$H = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \tag{2.18}$$

2. $f(x)$ has stationary points when the gradient is 0.

$$\frac{\partial f}{\partial x_1} = 5x_1 + 2x_2 + 4 = 0 \tag{2.19}$$

$$\frac{\partial f}{\partial x_2} = 5x_2 + 2x_1 = 0 \tag{2.20}$$

$$x_1 = \frac{-40}{42}, x_2 = \frac{8}{21} \tag{2.21}$$

Point $(x_1, x_2) = (\frac{-40}{42}, \frac{8}{21})$ is a stationary point.

2.26

(a) Necessary conditions for x^* to be a local minimum:

$$\frac{d(\sin x)}{dx}|_{x=x*} = 0 \Rightarrow \cos(x)|_{x=x*} = 0 \tag{2.22}$$

$$\Rightarrow x = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2 \ldots) \tag{2.23}$$

$$\frac{d^2(\sin x)}{dx^2}|_{x=x*} \geq 0 \Rightarrow -\sin(x)|_{x=x^*} \geq 0 \tag{2.24}$$

$$\Rightarrow x = \frac{3\pi}{2} + 2n\pi \quad (n = 0, \pm 1, \pm 2 \ldots) \tag{2.25}$$

Necessary conditions for x^* to be a local maximum:

$$\frac{d(\sin x)}{dx}|_{x=x*} = 0 \Rightarrow \cos(x)|_{x=x*} = 0 \tag{2.26}$$

$$\Rightarrow x = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2 \ldots) \tag{2.27}$$

$$\frac{d^2(\sin x)}{dx^2}|_{x=x*} \leq 0 \Rightarrow -\sin(x)|_{x=x^*} \leq 0 \tag{2.28}$$

$$\Rightarrow x = \frac{\pi}{2} + 2n\pi \quad (n = 0, \pm 1, \pm 2 \ldots) \tag{2.29}$$

(b) Stationary points: $x_1^* = \frac{\pi}{2}$, $x_1^* = \frac{3\pi}{2}$.

(c) $x_1^* = \frac{\pi}{2}$ is a global maximum in the given interval, and $x_2^* = \frac{3\pi}{2}$ is a global minimum in the given interval.

(d) See Fig. 2.2.

2.27

(a) $f'(x) = (e^{-ax^2})(-a)(2x) = -2axe^{-ax^2}$
$x = 0$ is a stationary point, for both $a > 0$ and $a < 0$. If $a = 0$, the function $f(x)$ will have a constant value of 1.

(b) $f''(x) = -2a[x(-2axe^{-ax^2}) + e^{-ax^2}] = 4a^2x^2e^{-ax^2} - 2ae^{-ax^2}$
For $a > 0$, $f''(x) = -2a < 0$. $x = 0$ is maximum.
For $a < 0$, $f''(x) = -2a > 0$. $x = 0$ is minimum.
For $a = 0$, $f(x)$ has a constant value.

(c) See Fig. 2.3. The global minimum of $f(x)$ is at $x = -\infty$ and $x = +\infty$, and the function value at the minimum is $f(x) = 0$.

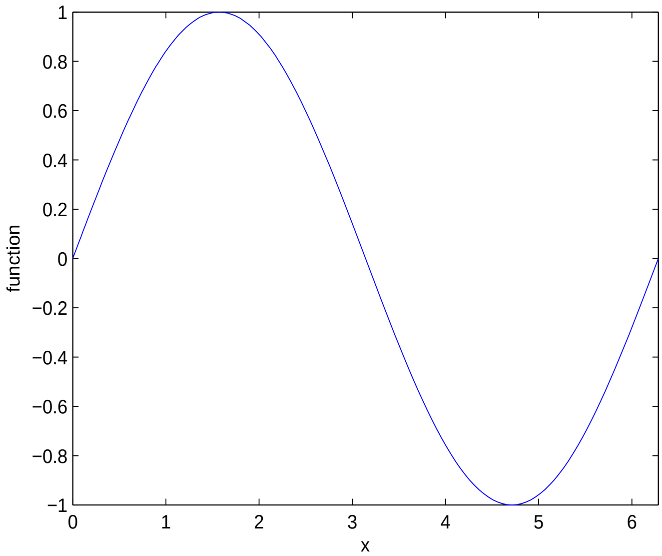

Figure 2.2: Plot for Prob 2.26

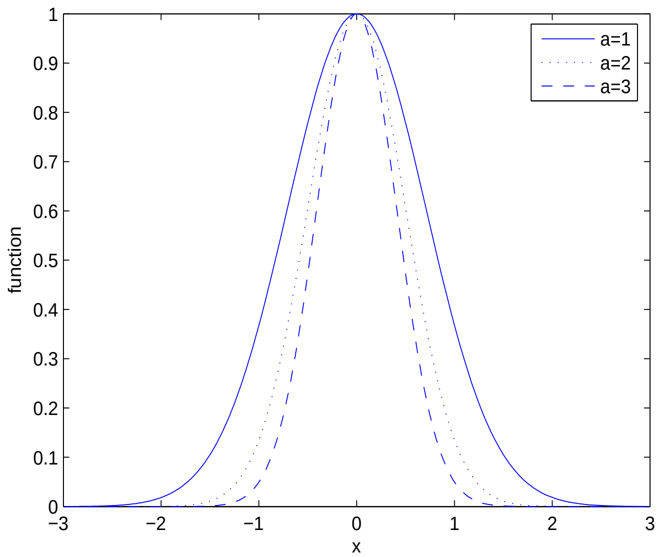

Figure 2.3: Plots for Prob 2.27