CHAPTER 2

First-Order and Simple Higher-Order Differential Equations

Problem 2.1

Solve $\cos^2 y\,\mathrm{d}x + (1+\mathrm{e}^{-x})\sin y\,\mathrm{d}y = 0.$

Case 1. $\cos y = 0 \implies y = (2n+1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$

Case 2. $\cos y \neq 0$

$$\int \frac{1}{1+\mathrm{e}^{-x}}\mathrm{d}x = -\int \frac{\sin y}{\cos^2 y}\mathrm{d}y + C \implies \int \frac{\mathrm{e}^x}{\mathrm{e}^x+1}\mathrm{d}x = -\int \frac{\sin y}{\cos^2 y}\mathrm{d}y + C$$

$$\int \frac{1}{\mathrm{e}^x+1}\mathrm{d}(\mathrm{e}^x+1) = \int \frac{1}{\cos^2 y}\mathrm{d}(\cos y) + C \implies \ln(\mathrm{e}^x+1) = -\frac{1}{\cos y} + C$$

Problem 2.2

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^3\mathrm{e}^{x^2}}{y\ln y}.$

$$\int y\ln y\,\mathrm{d}y = \int x^3\mathrm{e}^{x^2}\,\mathrm{d}x + C$$

$$I_1 = \int y\ln y\,\mathrm{d}y = \tfrac{1}{2}\int \ln y\,\mathrm{d}(y^2) = \tfrac{1}{2}\left(y^2\ln y - \int y^2\cdot\frac{1}{y}\mathrm{d}y\right) = \tfrac{1}{2}\left(y^2\ln y - \tfrac{1}{2}y^2\right)$$

$$I_2 = \int x^3\mathrm{e}^{x^2}\,\mathrm{d}x = \int x\cdot x^2\mathrm{e}^{x^2}\,\mathrm{d}x = \tfrac{1}{2}\int x^2\mathrm{e}^{x^2}\,\mathrm{d}(x^2) = \tfrac{1}{2}\int u\mathrm{e}^u\,\mathrm{d}u, \quad u = x^2$$

$$= \tfrac{1}{2}\int u\,\mathrm{d}(\mathrm{e}^u) = \tfrac{1}{2}\left(u\mathrm{e}^u - \int \mathrm{e}^u\,\mathrm{d}u\right) = \tfrac{1}{2}(u\mathrm{e}^u - \mathrm{e}^u)$$

$$\therefore\ \tfrac{1}{2}y^2\left(\ln y - \tfrac{1}{2}\right) = \tfrac{1}{2}\mathrm{e}^{x^2}(x^2-1) + C \implies y^2\left(\ln y - \tfrac{1}{2}\right) = \mathrm{e}^{x^2}(x^2-1) + C$$

Problem 2.3

Solve $x\cos^2 y\,\mathrm{d}x + \mathrm{e}^x\tan y\,\mathrm{d}y = 0.$

$$\int x\mathrm{e}^{-x}\,\mathrm{d}x = \int -\frac{\sin y}{\cos^3 y}\mathrm{d}y + C, \quad \cos y \neq 0$$

$$\because \int x\,\mathrm{e}^{-x}\,\mathrm{d}x = -\int x\,\mathrm{d}(\mathrm{e}^{-x}) = -\Big[x\,\mathrm{e}^{-x} - \int \mathrm{e}^{-x}\,\mathrm{d}y\Big] = -x\,\mathrm{e}^{-x} - \mathrm{e}^{-x}$$

$$\int -\frac{\sin y}{\cos^3 y}\,\mathrm{d}y = \int \frac{1}{\cos^3 y}\,\mathrm{d}(\cos y) = -\frac{1}{2\cos^2 y}$$

$$\therefore\ -\mathrm{e}^{-x}(x+1) = -\frac{1}{2\cos^2 y} + C \implies \mathrm{e}^{-x}(x+1) = \frac{1}{2\cos^2 y} + C$$

Problem 2.4

Solve $\quad x\,(y^2+1)\,\mathrm{d}x + (2y+1)\,\mathrm{e}^{-x}\,\mathrm{d}y = 0.$

$$\int x\,\mathrm{e}^{x}\,\mathrm{d}x = \int -\frac{2y+1}{y^2+1}\,\mathrm{d}y + C$$

$$\because \int x\,\mathrm{e}^{x}\,\mathrm{d}x = \int x\,\mathrm{d}(\mathrm{e}^{x}) = x\,\mathrm{e}^{x} - \int \mathrm{e}^{x}\,\mathrm{d}x = x\,\mathrm{e}^{x} - \mathrm{e}^{x}$$

$$\int \frac{2y+1}{y^2+1}\,\mathrm{d}y = \int \frac{2y}{y^2+1}\,\mathrm{d}y + \int \frac{1}{y^2+1}\,\mathrm{d}y = \ln(y^2+1) + \tan^{-1}y$$

$$\therefore\ (x-1)\,\mathrm{e}^{x} = -\Big[\ln(y^2+1) + \tan^{-1}y\Big] + C$$

$$(x-1)\,\mathrm{e}^{x} + \ln(y^2+1) + \tan^{-1}y = C.$$

Problem 2.5

Solve $\quad x y^3\,\mathrm{d}x + \mathrm{e}^{x^2}\,\mathrm{d}y = 0.$

$$x y^3 \mathrm{d}x = -\mathrm{e}^{x^2}\mathrm{d}y$$ ✍ Variable separable

Case 1. $y = 0$ is a solution.

Case 2. $y \neq 0$: $\quad \displaystyle\int x\,\mathrm{e}^{-x^2}\mathrm{d}x = \int -\frac{1}{y^3}\mathrm{d}y + C$

$$\tfrac{1}{2}\int \mathrm{e}^{-x^2}\mathrm{d}(x^2) = -\int y^{-3}\mathrm{d}y + C \implies -\tfrac{1}{2}\mathrm{e}^{-x^2} = -\frac{y^{-2}}{-2} + C$$

$$\therefore\ \mathrm{e}^{-x^2} + \frac{1}{y^2} = C$$

Problem 2.6

Solve $\quad x\cos^2 y\,\mathrm{d}x + \tan y\,\mathrm{d}y = 0$

$$x\cos^2 y\,\mathrm{d}x = -\tan y\,\mathrm{d}y$$ ✍ Variable separable

$$\int x\,\mathrm{d}x = -\int \frac{\tan y}{\cos^2 y}\,\mathrm{d}y + C \implies \tfrac{1}{2}x^2 = -\int \tan y\,\mathrm{d}(\tan y) + C$$

$$\therefore\ \tfrac{1}{2}x^2 = -\tfrac{1}{2}\tan^2 y + C \implies x^2 + \tan^2 y = C$$

Problem 2.7

Solve $\quad x y^3\,\mathrm{d}x + (y+1)\,\mathrm{e}^{-x}\,\mathrm{d}y = 0$

Case 1. $y=0$ is a solution.

Case 2. $y\neq 0$:

$$x\mathrm{e}^x\,\mathrm{d}x = -\frac{y+1}{y^3}\,\mathrm{d}y + C \implies \int x\mathrm{e}^x = -\int (y^{-2}+y^{-3})\,\mathrm{d}y + C$$

$$\therefore\ \mathrm{e}^x(x-1) = -\left(\frac{1}{-2+1}y^{-2+1} + \frac{1}{-3+1}y^{-3+1}\right) + C$$

$$\therefore\ \mathrm{e}^x(x-1) - \frac{1}{y} - \frac{1}{2y^2} = C \qquad \text{✍ General solution}$$

Problem 2.8

Solve $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{x}{y} + 2 = 0$

The DE is homogeneous. Let $\dfrac{y}{x} = u \implies y = xu \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = u + x\dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore\ u + x\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{u} + 2 = 0$$

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = -u - \frac{1}{u} - 2 = -\frac{u^2+1+2u}{u} = -\frac{(u+1)^2}{u} \qquad \text{✍ Variable separable}$$

Case 1. $u+1=0 \implies y=-x$ is a solution.

Case 2. $u+1\neq 0 \implies \displaystyle\int \frac{u}{(u+1)^2}\,\mathrm{d}u = \int -\frac{1}{x}\,\mathrm{d}x + C$

$$\int \frac{u}{(u+1)^2}\,\mathrm{d}u = \int \frac{u+1-1}{(u+1)^2}\,\mathrm{d}u = \int \frac{1}{u+1}\,\mathrm{d}u - \int \frac{1}{(u+1)^2}\,\mathrm{d}u$$

$$= \ln|u+1| + \frac{1}{u+1}$$

$$\therefore\ \ln|u+1| + \frac{1}{u+1} = -\ln|x| + C$$

$$\ln\left|\frac{y}{x}+1\right| + \frac{1}{\frac{y}{x}+1} + \ln|x| = C \implies \ln|x+y| + \frac{x}{x+y} = C$$

Problem 2.9

Solve $\quad x\,\mathrm{d}y - y\,\mathrm{d}x = x\cot\!\left(\dfrac{y}{x}\right)\mathrm{d}x$

Divide by $x\,\mathrm{d}x$: $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{y}{x} = \cot\!\left(\dfrac{y}{x}\right) \qquad$ ✍ Homogeneous DE

Let $u = \dfrac{y}{x} \implies y = xu \implies \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

$$\therefore\ u + x\frac{du}{dx} - u = \cot u$$

Case 1. $\cos u = 0 \implies \cos\dfrac{y}{x} = 0$ is a solution.

Case 2. $\cos u \neq 0$.

$$\int \tan u \, du = \int \frac{1}{x}\, dx + C \implies -\ln\left|\cos u\right| = \ln\left|x\right| + C$$

$$\frac{1}{\cos u} = Cx \implies \cos\left(\frac{y}{x}\right) = \frac{C}{x}$$

✍ Case 1 is included in Case 2 for $C = 0$.

Problem 2.10

Solve $\left[x\cos^2\left(\dfrac{y}{x}\right) - y\right] dx + x\, dy = 0$

$$\frac{dy}{dx} = -\frac{x\cos^2\left(\frac{y}{x}\right) - y}{x} \implies \frac{dy}{dx} = -\cos^2\left(\frac{y}{x}\right) + \frac{y}{x}$$ ✍ Homogeneous

Let $u = \dfrac{y}{x} \implies y = xu \implies \dfrac{dy}{dx} = u + x\dfrac{dy}{dx}$

$$\therefore\ u + x\frac{dy}{dx} = -\cos^2 u + u$$ ✍ Variable separable

Case 1. $\cos u = 0 \implies \cos\dfrac{y}{x} = 0$ is a solution.

Case 2. $\cos u \neq 0$.

$$\int \frac{du}{\cos^2 u} = -\int \frac{dx}{x} + C \implies \tan u = -\ln\left|x\right| + C \implies \ln\left|x\right| + \tan\left(\frac{y}{x}\right) = C$$

Problem 2.11

Solve $x\, dy = y(1 + \ln y - \ln x)\, dx$

$$\frac{dy}{dx} = \frac{y}{x}\left(1 + \ln\frac{y}{x}\right)$$ ✍ Homogeneous

Let $u = \dfrac{y}{x} \implies y = xu \implies \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

$$\therefore\ u + x\frac{du}{dx} = u(1 + \ln u) \implies x\frac{du}{dx} = u\ln u$$ ✍ Variable separable

$$\int \frac{1}{u \ln u}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x + C \implies \int \frac{1}{\ln u}\,\mathrm{d}(\ln u) = \ln x + C$$

$$\ln\left|\ln u\right| = \ln\left|Cx\right| \implies \ln u = Cx \implies u = \mathrm{e}^{Cx} \implies \frac{y}{x} = \mathrm{e}^{Cx}$$

Problem 2.12

Solve $\quad xy\,\mathrm{d}x + (x^2+y^2)\,\mathrm{d}y = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{xy}{x^2+y^2} = -\frac{\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2} \qquad \text{✍ Homogeneous DE}$$

Let $\dfrac{y}{x} = u \implies y = xu \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = u + x\dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore\ u + x\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{u}{1+u^2} \implies x\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{u\,(2+u^2)}{1+u^2} \qquad \text{✍ Variable separable}$$

Case 1. $\quad u=0 \implies y=0$ is a solution.

Case 2. $\quad u\neq 0$.

$$\int \frac{1+u^2}{u\,(2+u^2)}\,\mathrm{d}u = -\int \frac{\mathrm{d}x}{x} + C$$

$$\because\ I = \int \frac{1+u^2}{u\,(2+u^2)}\,\mathrm{d}u = \int \frac{1+u^2}{u^2\,(2+u^2)}\,u\,\mathrm{d}u = \tfrac{1}{2}\int \frac{1+u^2}{u^2(2+u^2)}\,\mathrm{d}(u^2) \qquad \text{✍ } v=u^2$$

$$= \tfrac{1}{2}\int \frac{1+v}{v\,(2+v)}\mathrm{d}v = \tfrac{1}{4}\int\left(\frac{1}{v}+\frac{1}{2+v}\right)\mathrm{d}v = \tfrac{1}{4}\big[\ln v + \ln(2+v)\big]$$

$$\therefore\ \tfrac{1}{4}\big[\ln u^2 + \ln(2+u^2)\big] = -\ln|x| + \tfrac{1}{4}\ln|C|$$

$$\ln\big[u^2(2+u^2)\big] = \ln\frac{C}{x^4} \implies u^2(2+u^2) = \frac{C}{x^4}$$

$$x^4\left\{\left(\frac{y}{x}\right)^2\left[2+\left(\frac{y}{x}\right)^2\right]\right\} = C \implies y^2(2x^2+y^2) = C$$

Alternatively, it is obvious that $y=0$ is a solution; for $y\neq 0$,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{x^2+y^2}{xy} = -\frac{\left(\frac{x}{y}\right)^2+1}{\frac{x}{y}} \qquad \text{✍ Homogeneous}$$

Let $\dfrac{x}{y} = u \implies x = yu \implies \dfrac{\mathrm{d}x}{\mathrm{d}y} = u + y\dfrac{\mathrm{d}u}{\mathrm{d}y}$

$$u + y\frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{u^2+1}{u} \implies y\frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{2u^2+1}{u} \qquad \text{✍ Variable separable}$$

$$\int \frac{u}{2u^2+1}\,\mathrm{d}u = -\int \frac{\mathrm{d}y}{y} + C$$

$$\because \int \frac{u}{2u^2+1}\,\mathrm{d}u = \frac{1}{4}\int \frac{\mathrm{d}(2u^2+1)}{2u^2+1} = \frac{1}{4}\ln(2u^2+1)$$

$$\therefore \frac{1}{4}\ln(2u^2+1) = -\ln|y| + \frac{1}{4}\ln|C| \implies \ln(2u^2+1) = \ln\frac{C}{y^4}$$

$$(2u^2+1)\,y^4 = C \implies \left[2\left(\frac{x}{y}\right)^2+1\right]y^4 = C \implies y^2(2x^2+y^2) = C$$

Problem 2.13

Solve $\left[1+\exp\left(-\frac{y}{x}\right)\right]\mathrm{d}y + \left(1-\frac{y}{x}\right)\mathrm{d}x = 0$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{y}{x}-1}{1+\exp\left(-\frac{y}{x}\right)}$$ ✍ Homogeneous

Let $\frac{y}{x} = u \implies y = xu \implies \frac{\mathrm{d}y}{\mathrm{d}x} = u + x\,\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore u + x\,\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u-1}{1+\mathrm{e}^{-u}} \implies x\,\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1+u\mathrm{e}^{-u}}{1+\mathrm{e}^{-u}}$$ ✍ Variable separable

$$\int \frac{1+\mathrm{e}^{-u}}{1+u\mathrm{e}^{-u}}\,\mathrm{d}u = -\int \frac{\mathrm{d}x}{x} + C \implies \int \frac{\mathrm{e}^u+1}{\mathrm{e}^u+u}\,\mathrm{d}u = -\ln|x| + \ln|C|$$

$$\therefore \ln|\mathrm{e}^u+u| = \ln\left|\frac{C}{x}\right| \implies \mathrm{e}^u+u = \frac{C}{x}$$

$$\exp\left(\frac{y}{x}\right) + \frac{y}{x} = \frac{C}{x} \quad \text{or} \quad x\exp\left(\frac{y}{x}\right) + y = C$$

Problem 2.14

Solve $(x^2 - xy + y^2)\,\mathrm{d}x - xy\,\mathrm{d}y = 0$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2-xy+y^2}{xy} = \frac{1-\frac{y}{x}+\left(\frac{y}{x}\right)^2}{\frac{y}{x}}$$ ✍ Homogeneous DE

Let $u = \frac{y}{x} \implies y = xu \implies \frac{\mathrm{d}y}{\mathrm{d}x} = u + x\,\frac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore u + x\,\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1-u+u^2}{u} \implies x\,\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1-u}{u}$$

Case 1. $u = 1 \implies y = x$ is a solution.

Case 2. $u \neq 1$.

$$\frac{u}{1-u}\,\mathrm{d}u = \frac{\mathrm{d}x}{x} \implies \int\Big(-1-\frac{1}{u-1}\Big)\,\mathrm{d}u = \int\frac{\mathrm{d}x}{x} + C$$

$$\therefore\ -u-\ln|u-1| = \ln|x| + \ln|C| \implies -\ln\big(\mathrm{e}^{u}|u-1|\big) = \ln|Cx|$$

$$\therefore\ \mathrm{e}^{u}(u-1) = \frac{C}{x} \implies x\Big(\frac{y}{x}-1\Big)\,\mathrm{e}^{\frac{y}{x}} = C \implies (y-x)\,\mathrm{e}^{\frac{y}{x}} = C$$

✍ Case 1 is included in Case 2 for $C = 0$.

Problem 2.15

Solve $(3+2x+4y)\,y' = 1+x+2y$

$$y' = \frac{1+x+2y}{3+2(x+2y)}$$ ✍ Special transformation

Let $u = x+2y \implies \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1+2\,\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$\therefore\ \frac{1}{2}\Big(\frac{\mathrm{d}u}{\mathrm{d}x}-1\Big) = \frac{1+u}{3+2u} \implies \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{5+4u}{3+2u}$$ ✍ Variable separable

Case 1. $5+4u = 0 \implies 4x+8y+5 = 0$ is a solution.

Case 2. $5+4u \neq 0 \implies \displaystyle\int\frac{2u+3}{4u+5}\,\mathrm{d}u = \int\mathrm{d}x + C$

$$\begin{array}{r|lll} & & \frac{1}{2} & \\ 4u+5 & 2u & + & 3 \\ & 2u & + & \frac{5}{2} \quad (- \\ \hline & & & \frac{1}{2} \end{array}$$

$$\int\Big(\frac{1}{2}+\frac{\frac{1}{2}}{4u+5}\Big)\,\mathrm{d}u = x + C \implies \frac{u}{2}+\frac{1}{8}\ln|4u+5| = x + C$$

$$\therefore\ \frac{x+2y}{2}+\frac{1}{8}\ln|4x+8y+5| = x+C \implies y-\frac{x}{2}+\frac{1}{8}\ln|4x+8y+5| = C$$

Problem 2.16

Solve $y' = \dfrac{2x+y-1}{x-y-2}$

Point of intersection of lines

$$\begin{cases} 2x+y-1=0 \\ x-y-2=0 \end{cases} \implies x=1,\quad y=-1.$$

Let $x = X+1, \quad y = Y-1 \implies y' = \dfrac{\mathrm{d}Y}{\mathrm{d}X}$

$$\therefore \quad \frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{2\,(X+1)+(Y-1)-1}{(X+1)-(Y-1)-2} = \frac{2X+Y}{X-Y} = \frac{2+\frac{Y}{X}}{1-\frac{Y}{X}}$$

Let $u = \dfrac{Y}{X} \implies Y = Xu \implies \dfrac{\mathrm{d}Y}{\mathrm{d}X} = u + X\,\dfrac{\mathrm{d}u}{\mathrm{d}X}$

$$u + X\,\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{2+u}{1-u}$$

$$X\,\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{2+u}{1-u} - u = \frac{2+u-u+u^2}{1-u} = \frac{u^2+2}{1-u}$$ ✍ Variable Separable

$$\int \frac{1-u}{u^2+2}\,\mathrm{d}u = \int \frac{\mathrm{d}X}{X} + C \qquad (\because u^2+2 \neq 0)$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} - \frac{1}{2}\ln(u^2+2) = \ln|x| + C$$

$$\sqrt{2}\tan^{-1}\frac{y+1}{\sqrt{2}\,(x-1)} = \ln\Big[\Big(\frac{y+1}{x-1}\Big)^2 + 2\Big] + 2\ln|x-1| + C$$

$$\therefore \quad \sqrt{2}\,\tan^{-1}\frac{y+1}{\sqrt{2}\,(x-1)} = \ln\big[(y+1)^2 + 2\,(x-1)^2\big] + C$$

Problem 2.17

Solve $\qquad (y+2)\,\mathrm{d}x = (2x+y-4)\,\mathrm{d}y$

Point of intersection of lines

$$\begin{cases} y+2=0 \\ 2x+y-4=0 \end{cases} \implies x=3, \quad y=-2$$

Let $x = X+3, \quad y = Y-2 \implies y' = \dfrac{\mathrm{d}Y}{\mathrm{d}X}$

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{(Y-2)+2}{2(X+3)+(Y-2)-4} = \frac{Y}{2X+Y} = \frac{\frac{Y}{X}}{2+\frac{Y}{X}}$$ ✍ Homogeneous DE

Let $u = \dfrac{Y}{X} \implies Y = Xu \implies \dfrac{\mathrm{d}Y}{\mathrm{d}X} = u + X\,\dfrac{\mathrm{d}u}{\mathrm{d}X}$

$$\therefore \quad u + x\,\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{u}{2+u} - u = \frac{u-2u-u^2}{2+u} = -\frac{u\,(u+1)}{u+2}$$

Case 1. $u = 0 \implies y+2=0$

Case 2. $u = -1 \implies \dfrac{y+2}{x-3} = -1 \implies y = 1-x$

Case 3. $u \neq 0, \quad u \neq -1$: $$\int \frac{u+2}{u(u+1)}\,\mathrm{d}u = -\int \frac{1}{X}\,\mathrm{d}X + C$$

Partial fractions: $$\frac{u+2}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$A = \frac{u+2}{u+1}\bigg|_{u=0} = 2, \qquad B = \frac{u+2}{u}\bigg|_{u=-1} = -1$$

$$\therefore \quad \int \left(\frac{2}{u} - \frac{1}{u+1}\right)\mathrm{d}u = -\int \frac{1}{X}\,\mathrm{d}x + C$$

$$2\ln|u| - \ln|u+1| = -\ln|x| + \ln|C| \implies \ln\left|\frac{u^2}{u+1}\right| = \ln\left|\frac{C}{X}\right|$$

$$\therefore \quad \frac{u^2}{u+1} = \frac{C}{X} \implies \frac{\left(\frac{y+2}{x-3}\right)^2}{\frac{y+2}{x-3}+1} = \frac{C}{x-3} \implies (y+2)^2 = C\,(x+y-1)$$

Since $y+2=0$ is included in the general solution with $C=0$, the solutions are:

$$(y+2)^2 = C\,(x+y-1); \qquad y = 1-x$$

Problem 2.18

Solve $\quad y' = \sin^2(x-y)$

Let $x-y=u \implies 1 - \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore \quad 1 - \frac{\mathrm{d}u}{\mathrm{d}x} = \sin^2 u \implies 1 - \sin^2 u = \frac{\mathrm{d}u}{\mathrm{d}x} \implies \cos^2 u = \frac{\mathrm{d}u}{\mathrm{d}x}$$ ✍ Separable

Case 1. $\cos u = 0 \implies x - y = u = \frac{1}{2}\pi \pm k\pi, \quad k = 0, 1, 2, \ldots$

Case 2. $\cos u \neq 0 \implies \displaystyle\int \mathrm{d}x = \int \frac{1}{\cos^2 u}\,\mathrm{d}u + C$

$$\therefore \quad x = \tan u + C \implies x = \tan(x-y) + C.$$

Problem 2.19

Solve $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = (x+1)^2 + (4y+1)^2 + 8xy + 1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + 2x + 8xy + (4y+1)^2 + 2$$

$$= x^2 + 2\cdot x\cdot(4y+1) + (4y+1)^2 + 2 = (x+4y+1)^2 + 2$$

Let $x + 4y + 1 = u \implies 1 + 4\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore \quad \frac{1}{4}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - 1\right) = u^2 + 2 \implies \frac{\mathrm{d}u}{\mathrm{d}x} = 4u^2 + 9 = 4\left[u^2 + \left(\tfrac{3}{2}\right)^2\right]$$

$$\int \frac{du}{u^2+\left(\frac{3}{2}\right)^2} = \int 4\,dx + C \implies \frac{1}{\frac{3}{2}}\tan^{-1}\frac{u}{\frac{3}{2}} = 4x + C$$

$$\tan^{-1}\frac{2u}{3} = 6x + C \implies \tfrac{2}{3}(x+4y+1) = \tan(6x+C).$$

Problem 2.20

Solve $\quad (3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$

$$M = 3x^2 + 6xy^2, \quad N = 6x^2y + 4y^3$$

$$\frac{\partial M}{\partial y} = 12xy, \qquad \frac{\partial N}{\partial x} = 12xy \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ ✍ Exact DE

$$\begin{array}{ccccccc} 3x^2\,dx & + & 6xy^2\,dx & + & 6x^2y\,dy & + & 4y^3\,dy = 0 \\ {\scriptstyle\int dx}\Big\downarrow & & \searrow{\scriptstyle\int dx} & & \nearrow{\scriptstyle\frac{\partial}{\partial y}} & & {\scriptstyle\int dy}\Big\downarrow \\ \boxed{x^3} & & & \boxed{3x^2y^2} & & & \boxed{y^4} \end{array}$$

General solution: $\quad x^3 + 3x^2y^2 + y^4 = C$

Problem 2.21

Solve $\quad (2x^3 - xy^2 - 2y + 3)\,dx - (x^2y + 2x)\,dy = 0$

$$M = 2x^3 - xy^2 - 2y + 3. \qquad N = -x^2y - 2x$$

$$\frac{\partial M}{\partial y} = -2xy - 2, \qquad \frac{\partial N}{\partial x} = -2xy - 2 \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ ✍ Exact DE

$$\begin{array}{ccccccccccc} 2x^3\,dx & - & xy^2\,dx & - & x^2y\,dy & - & 2y\,dx & - & 2x\,dy & + & 3\,dx = 0 \\ {\scriptstyle\int dx}\Big\downarrow & & \searrow{\scriptstyle\int dx} & & \nearrow{\scriptstyle\frac{\partial}{\partial y}} & & \searrow{\scriptstyle\int dx} & & \nearrow{\scriptstyle\frac{\partial}{\partial y}} & & {\scriptstyle\int dx}\Big\downarrow \\ \boxed{\frac{1}{2}x^4} & & & \boxed{-\frac{1}{2}x^2y^2} & & & & \boxed{-2xy} & & & \boxed{3x} \end{array}$$

General solution: $\quad \frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x = C$

Problem 2.22

Solve $\quad (xy^2 + x - 2y + 3)\,dx + x^2y\,dy = 2(x+y)\,dy$

$$(xy^2 + x - 2y + 3)\,dx + (x^2y - 2x - 2y)\,dy = 0$$

$$M = xy^2 + x - 2y + 3, \qquad N = x^2y - 2x - 2y$$

$$\frac{\partial M}{\partial y} = 2xy - 2, \qquad \frac{\partial N}{\partial x} = 2xy - 2 \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ ✍ Exact DE

$$xy^2\,\mathrm{d}x + x^2y\,\mathrm{d}y + (x+3)\,\mathrm{d}x - 2y\,\mathrm{d}x - 2x\,\mathrm{d}y - 2y\,\mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{\tfrac{1}{2}x^2y^2} \quad \frac{\partial}{\partial y} \qquad \int \mathrm{d}x \quad \boxed{\tfrac{1}{2}x^2 + 3x} \qquad \int \mathrm{d}x \quad \boxed{-2xy} \quad \frac{\partial}{\partial y} \qquad \int \mathrm{d}y \quad \boxed{-y^2}$$

General solution: $\frac{1}{2}x^2y^2 + \frac{1}{2}x^2 + 3x - 2xy - y^2 = C$

Problem 2.23

Solve $\quad 3y(x^2-1)\,\mathrm{d}x + (x^3 + 8y - 3x)\,\mathrm{d}y = 0, \quad$ when $x = 0,\ y = 1$

$M = 3y(x^2-1), \qquad N = x^3 + 8y - 3x$

$$\frac{\partial M}{\partial y} = 3(x^2-1), \quad \frac{\partial N}{\partial x} = 3x^2 - 3 = 3(x^2-1) \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ ✍ Exact DE

$$3x^2y\,\mathrm{d}x + x^3\,\mathrm{d}y - 3y\,\mathrm{d}x - 3x\,\mathrm{d}y + 8y\,\mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{x^3y} \quad \frac{\partial}{\partial y} \qquad \int \mathrm{d}x \quad \boxed{-3xy} \quad \frac{\partial}{\partial y} \qquad \int \mathrm{d}y \quad \boxed{4y^2}$$

General solution: $\quad x^3y - 3xy + 4y^2 = C$

$x=0,\ y=1 \implies C=4 \implies x^3y - 3xy + 4y^2 = 4$

Problem 2.24

Solve $\quad (x^2 + \ln y)\,\mathrm{d}x + \dfrac{x}{y}\,\mathrm{d}y = 0$

$$M = x^2 + \ln y, \quad N = \frac{x}{y} \implies \frac{\partial M}{\partial y} = \frac{1}{y} = \frac{\partial N}{\partial x}$$ ✍ Exact DE

$$x^2\,\mathrm{d}x + \ln y\,\mathrm{d}x + \frac{x}{y}\,\mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{\tfrac{1}{3}x^3} \qquad \int \mathrm{d}x \quad \boxed{x\ln y} \quad \frac{\partial}{\partial y}$$

General solution: $\quad \frac{1}{3}x^3 + x\ln y = C$

Problem 2.25

Solve $\quad 2x(3x + y - ye^{-x^2})\,\mathrm{d}x + (x^2 + 3y^2 + e^{-x^2})\,\mathrm{d}y = 0$

$M = 2x(3x + y - ye^{-x^2}), \qquad N = x^2 + 3y^2 + e^{-x^2}$

$$\frac{\partial M}{\partial y} = 2x(1 - e^{-x^2}), \quad \frac{\partial N}{\partial x} = 2x + e^{-x^2}(-2x) \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ ✍ Exact DE

$$6x^2\,\mathrm{d}x + 2xy\,\mathrm{d}x \;+\; x^2\,\mathrm{d}y + \mathrm{e}^{-x^2}\,\mathrm{d}y \;-\; 2xy\,\mathrm{e}^{-x^2}\,\mathrm{d}x \;+\; 3y^2\,\mathrm{d}y = 0$$

$$6x^2\,\mathrm{d}x \xrightarrow{\int \mathrm{d}x} \boxed{2x^3}, \qquad 2xy\,\mathrm{d}x \xrightarrow{\int \mathrm{d}x} \boxed{x^2y} \xrightarrow{\frac{\partial}{\partial y}} x^2\,\mathrm{d}y, \qquad \mathrm{e}^{-x^2}\,\mathrm{d}y \xrightarrow{\int \mathrm{d}y} \boxed{y\,\mathrm{e}^{-x^2}} \xrightarrow{\frac{\partial}{\partial x}} -2xy\,\mathrm{e}^{-x^2}\,\mathrm{d}x, \qquad 3y^2\,\mathrm{d}y \xrightarrow{\int \mathrm{d}y} \boxed{y^3}$$

General solution: $\quad 2x^3 + x^2y + y\,\mathrm{e}^{-x^2} + y^3 = C$

Problem 2.26

Solve $\qquad (3 + y + 2y^2\sin^2 x)\,\mathrm{d}x + (x + 2xy - y\sin 2x)\,\mathrm{d}y = 0$

$$M = 3 + y + 2y^2\sin^2 x = 3 + y + y^2 - y^2\cos 2x, \qquad N = x + 2xy - y\sin 2x$$

$$\frac{\partial M}{\partial y} = 1 + 2y - 2y\cos 2x, \qquad \frac{\partial N}{\partial x} = 1 + 2y - 2y\cos 2x \implies \text{Exact DE}$$

$$3\,\mathrm{d}x \;+\; y\,\mathrm{d}x \;+\; x\,\mathrm{d}y \;+\; y^2\mathrm{d}x \;+\; 2xy\,\mathrm{d}y \;-\; y^2\cos 2x\,\mathrm{d}x \;-\; y\sin 2x\,\mathrm{d}y = 0$$

$$3\,\mathrm{d}x \xrightarrow{\int \mathrm{d}x} \boxed{3x}, \qquad y\,\mathrm{d}x \xrightarrow{\int \mathrm{d}x} \boxed{xy} \xrightarrow{\frac{\partial}{\partial y}} x\,\mathrm{d}y, \qquad y^2\mathrm{d}x \xrightarrow{\int \mathrm{d}x} \boxed{xy^2} \xrightarrow{\frac{\partial}{\partial y}} 2xy\,\mathrm{d}y$$

$$-y^2\cos 2x\,\mathrm{d}x \xrightarrow{\int \mathrm{d}x} \boxed{-\tfrac{1}{2}y^2\sin 2x} \xrightarrow{\frac{\partial}{\partial y}} -y\sin 2x\,\mathrm{d}y$$

General solution: $\quad 3x + xy + xy^2 - \tfrac{1}{2}y^2\sin 2x = C$

Problem 2.27

Solve $\qquad (2xy + y^2)\,\mathrm{d}x + (x^2 + 2xy + y^2)\,\mathrm{d}y = 0$

$$M = 2xy + y^2, \qquad N = x^2 + 2xy + y^2$$

$$\frac{\partial M}{\partial y} = 2x + 2y, \qquad \frac{\partial N}{\partial x} = 2x + 2y \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{Exact DE}$$

$$2xy\,\mathrm{d}x \;+\; x^2\mathrm{d}y \;+\; y^2\,\mathrm{d}x \;+\; 2xy\,\mathrm{d}y \;+\; y^2\,\mathrm{d}y = 0$$

$$2xy\,\mathrm{d}x \xrightarrow{\int \mathrm{d}x} \boxed{x^2y} \xrightarrow{\frac{\partial}{\partial y}} x^2\mathrm{d}y, \qquad y^2\,\mathrm{d}x \xrightarrow{\int \mathrm{d}x} \boxed{y^2x} \xrightarrow{\frac{\partial}{\partial y}} 2xy\,\mathrm{d}y, \qquad y^2\,\mathrm{d}y \xrightarrow{\int \mathrm{d}y} \boxed{\tfrac{1}{3}y^3}$$

General solution: $\quad x^2y + xy^2 + \tfrac{1}{3}y^3 = C$

Problem 2.28

Solve $\quad (x^2 - \sin^2 y)\,\mathrm{d}x + x \sin 2y\,\mathrm{d}y = 0$

$$M = x^2 - \sin^2 y, \quad N = x \sin 2y$$

$$\frac{\partial M}{\partial y} = -2 \sin y \, \cos y = -\sin 2y, \qquad \frac{\partial N}{\partial x} = \sin 2y$$ ✍ Not exact DE

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x\,\sin 2y}\,(-\sin 2y - \sin 2y) = -\frac{2}{x}$$

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mathrm{d}x\right] = \exp\left(-\int \frac{2}{x}\,\mathrm{d}x\right) = \mathrm{e}^{-2\ln x} = \frac{1}{x^2}$$

Multiply the DE by $\mu(x) = \dfrac{1}{x^2}$: $\quad \left(1 - \dfrac{\sin^2 y}{x^2}\right)\mathrm{d}x + \dfrac{\sin 2y}{x}\,\mathrm{d}y = 0$

$$1\,\mathrm{d}x - \frac{\sin^2 y}{x^2}\,\mathrm{d}x + \frac{2 \sin y \cos y}{x}\,\mathrm{d}y = 0$$

$$\int \mathrm{d}x: \ \boxed{x} \qquad \int \mathrm{d}x: \ \boxed{\frac{\sin^2 y}{x}} \qquad \frac{\partial}{\partial y}$$

General solution: $\quad x + \dfrac{\sin^2 y}{x} = C$

Problem 2.29

Solve $\quad y\,(2x - y + 2)\,\mathrm{d}x + 2\,(x - y)\,\mathrm{d}y = 0$

$$M = 2xy - y^2 + 2y, \qquad N = 2x - 2y)$$

$$\frac{\partial M}{\partial y} = 2x - 2y + 2, \qquad \frac{\partial N}{\partial x} = 2 \implies \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ ✍ Not exact DE

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2\,(x-y)}\,\big[(2x-2y+2)-2\big] = 1$$

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mathrm{d}x\right] = \exp\left(\int \mathrm{d}x\right) = \mathrm{e}^x$$

Multiply the DE by $\mu(x) = \mathrm{e}^x$: $\quad (2xy - y^2 + 2y)\,\mathrm{e}^x\,\mathrm{d}x + (2x - 2y)\,\mathrm{e}^x\,\mathrm{d}y = 0$

$$2x\mathrm{e}^x\,\mathrm{d}y + (2y\mathrm{e}^x + 2xy\mathrm{e}^x)\,\mathrm{d}x - y^2\mathrm{e}^x\,\mathrm{d}x - 2y\mathrm{e}^x\,\mathrm{d}y = 0$$

$$\int \mathrm{d}y: \ \boxed{2xy\mathrm{e}^x} \quad \frac{\partial}{\partial x} \qquad\qquad \int \mathrm{d}x: \ \boxed{-y^2\mathrm{e}^x} \quad \frac{\partial}{\partial y}$$

General solution: $\quad 2xy\mathrm{e}^x - y^2\mathrm{e}^x = C \implies y\mathrm{e}^x(2x - y) = C$

Problem 2.30

Solve $\quad (4xy + 3y^2 - x)\,\mathrm{d}x + x(x+2y)\,\mathrm{d}y = 0$

$$M = 4xy + 3y^2 - x, \quad N = x^2 + 2xy)$$

$$\frac{\partial M}{\partial y} = 4x + 6y, \quad \frac{\partial N}{\partial x} = 2x + 2y \implies \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ ✍ Not exact DE

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(4x+6y)-(2x+2y)}{x\,(x+2y)} = \frac{2}{x}$$ ✍ A function of x only

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mathrm{d}x\right] = \exp\left(\int \frac{2}{x}\,\mathrm{d}x\right) = \mathrm{e}^{2\ln x} = x^2$$

Muiltiply the DE by $\mu(x) = x^2$: $\quad (4x^3y + 3x^2y^2 - x^3)\,\mathrm{d}x + (x^4 + 2x^3y)\,\mathrm{d}y = 0$

$$4x^3y\,\mathrm{d}x \; + \; x^4\,\mathrm{d}y \; + \; 3x^2y^2\,\mathrm{d}x \; + \; 2x^3y\,\mathrm{d}y \; - \; x^3\,\mathrm{d}x = 0$$

$$\int \mathrm{d}x \quad \boxed{x^4y} \quad \frac{\partial}{\partial y} \qquad \int \mathrm{d}x \quad \boxed{x^3y^2} \quad \frac{\partial}{\partial y} \qquad \int \mathrm{d}x \quad \boxed{-\tfrac{1}{4}x^4}$$

General solution: $\quad x^4y + x^3y^2 - \frac{1}{4}x^4 = C$

Problem 2.31

Solve $\quad y\,\mathrm{d}x + x(y^2 + \ln x)\,\mathrm{d}y = 0$

$$M = y, \quad N = x(y^2 + \ln x)$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = (y^2 + \ln x) + x \cdot \frac{1}{x} = y^2 + \ln x + 1$$ ✍ Not exact DE

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1 - (y^2 + \ln x + 1)}{x\,(y^2 + \ln x)} = -\frac{1}{x}$$ ✍ A function of x only

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mathrm{d}x\right] = \exp\left(\int -\frac{1}{x}\,\mathrm{d}x\right) = \mathrm{e}^{-\ln x} = \frac{1}{x}$$

Multiply the DE by $\mu(x) = \dfrac{1}{x}$: $\quad \dfrac{y}{x}\,\mathrm{d}x + (y^2 + \ln x)\,\mathrm{d}y = 0$

$$\frac{y}{x}\,\mathrm{d}x \; + \; \ln x\,\mathrm{d}y \; + \; y^2\,\mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{y\,\ln x} \quad \frac{\partial}{\partial y} \qquad \int \mathrm{d}y \quad \boxed{\tfrac{1}{3}y^3}$$

General solution: $\quad y\ln x + \frac{1}{3}y^3 = C$

Problem 2.32

Solve $\quad (x^2+2x+y)\,\mathrm{d}x + (3x^2y - x)\,\mathrm{d}y = 0$

$$M = x^2+2x+y, \qquad N = 3x^2y - x$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 6xy - 1 \implies \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{✍ Not exact DE}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1-(6xy-1)}{x\,(3xy-1)} = -\frac{2}{x} \quad \text{✍ A function of } x \text{ only}$$

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mathrm{d}x\right] = \exp\left(\int -\frac{2}{x}\,\mathrm{d}x\right) = \mathrm{e}^{-2\ln|x|} = \frac{1}{x^2}$$

Multiply the DE by $\mu(x) = \dfrac{1}{x^2}$: $\quad \left(1+\dfrac{2}{x}+\dfrac{y}{x^2}\right)\mathrm{d}x + \left(3y - \dfrac{1}{x}\right)\mathrm{d}y = 0$

$$\left(1+\frac{2}{x}\right)\mathrm{d}x \;-\; \frac{1}{x}\,\mathrm{d}y \;+\; \frac{y}{x^2} \;+\; 3y\,\mathrm{d}y = 0$$

$$\int\mathrm{d}x \downarrow \boxed{x+2\ln|x|} \qquad \int\mathrm{d}y \searrow \boxed{-\frac{y}{x}} \nearrow \frac{\partial}{\partial x} \qquad \downarrow \boxed{\frac{3}{2}y^2}$$

General solution: $\quad x + 2\ln|x| - \dfrac{y}{x} + \frac{3}{2}y^2 = C$

Problem 2.33

Solve $\quad y^2\,\mathrm{d}x + (xy + y^2 - 1)\,\mathrm{d}y = 0$

$$M = y^2, \quad N = xy + y^2 - 1 \implies \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = y \quad \text{✍ Not exact DE}$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y^2}\,(y-2y) = -\frac{1}{y} \quad \text{✍ A function of } y \text{ only}$$

$$\mu(y) = \exp\left[\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}y\right] = \exp\left(-\int \frac{1}{y}\,\mathrm{d}y\right) = \mathrm{e}^{-\ln|y|} = \frac{1}{y}$$

Multiplying the DE by $\mu(y) = \dfrac{1}{y}$: $\quad y\,\mathrm{d}x + \left(x+y-\dfrac{1}{y}\right)\mathrm{d}y = 0$

$$y\,\mathrm{d}x \;+\; x\,\mathrm{d}y \;+\; \left(y - \frac{1}{y}\right)\mathrm{d}y = 0$$

$$\int\mathrm{d}x \searrow \boxed{xy} \nearrow \frac{\partial}{\partial y} \qquad \int\mathrm{d}y \downarrow \boxed{\frac{1}{2}y^2 - \ln|y|}$$

General solution: $\quad xy + \frac{1}{2}y^2 - \ln|y| = C$

Problem 2.34

Solve $\quad 3(x^2+y^2)\,dx + x(x^2+3y^2+6y)\,dy = 0$

$$M = 3x^2+3y^2, \qquad N = x^3+3xy^2+6xy$$

$$\frac{\partial M}{\partial y} = 6y, \qquad \frac{\partial N}{\partial x} = 3x^2+3y^2+6y$$ ✍ Not exact DE

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{(3x^2+3y^2+6y)-6y}{3(x^2+y^2)} = 1$$ ✍ A function of y only

$$\mu(y) = \exp\left[\int \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dy\right] = \exp\left(\int 1\cdot dy\right) = e^y$$

Multiply the DE by e^y: $\quad (3x^2e^y+3y^2e^y)\,dx + (x^3e^y+3xy^2e^y+6xye^y)\,dy = 0$

$$x^3e^y\,dy \;+\; 3x^2e^y\,dx + 3y^2e^y\,dx \;+\; (6xye^y+3xy^2e^y)\,dy = 0$$

$$\int dy \quad \boxed{x^3e^y} \quad \frac{\partial}{\partial x} \qquad\qquad \int dx \quad \boxed{3xy^2e^y} \quad \frac{\partial}{\partial y}$$

General solution: $\quad x^3e^y + 3xy^2e^y = C$

Problem 2.35

Solve $\quad 2y(x+y+2)\,dx + (y^2-x^2-4x-1)\,dy = 0$

$$M = 2xy+2y^2+4y, \quad N = y^2-x^2-4x-1$$

$$\frac{\partial M}{\partial y} = 2x+4y+4, \qquad \frac{\partial N}{\partial x} = -2x-4$$ ✍ Not exact DE

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{(-2x-4)-(2x+4y+4)}{2y(x+y+2)} = -\frac{2}{y}$$ ✍ A function of y only

$$\mu(y) = \exp\left[\int \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dy\right] = \exp\left(\int -\frac{2}{y}\right) = e^{-2\ln y} = \frac{1}{y^2}$$

Multiply the DE by $\mu(y)=\dfrac{1}{y^2}$: $\quad \left(\dfrac{2x}{y}+2+\dfrac{4}{y}\right)dx+\left(1-\dfrac{x^2}{y^2}-\dfrac{4x}{y^2}-\dfrac{1}{y^2}\right)dy=0$

$$\frac{2x}{y}\,dx - \frac{x^2}{y^2}\,dy + \frac{4}{y}\,dx - \frac{4x}{y^2}\,dy + 2\,dx + 1\,dy - \frac{1}{y^2}\,dy = 0$$

$$\int dx \quad \boxed{\frac{x^2}{y}} \quad \frac{\partial}{\partial y} \qquad \int dx \quad \boxed{\frac{4x}{y}} \quad \frac{\partial}{\partial y} \qquad \int dx \;\; \boxed{2x} \qquad \int dy \;\; \boxed{y} \qquad \int dy \;\; \boxed{\frac{1}{y}}$$

General solution: $\quad \dfrac{x^2}{y} + \dfrac{4x}{y} + 2x + y + \dfrac{1}{y} = C$

Problem 2.36

Solve $\quad (2+y^2+2x)\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$

$$M = 2+y^2+2x,\quad N = 2y \implies \frac{\partial M}{\partial y} = 2y,\quad \frac{\partial N}{\partial x} = 0$$ ✍ Not exact DE

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2y}\,(2y-0) = 1$$ ✍ a function of x only

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial M}{\partial x}\right)\mathrm{d}x\right] = \exp\left(\int \mathrm{d}x\right) = \mathrm{e}^x$$

Multiply the DE by $\mu(x) = \mathrm{e}^x$: $\quad (2\,\mathrm{e}^x + y^2\,\mathrm{e}^x + 2x\,\mathrm{e}^x)\,\mathrm{d}x + 2y\,\mathrm{e}^x\,\mathrm{d}y = 0$

$$(2\mathrm{e}^x + 2x\mathrm{e}^x)\,\mathrm{d}x + y^2\,\mathrm{e}^x\,\mathrm{d}x + 2y\,\mathrm{e}^x\,\mathrm{d}y = 0$$

$$\int\mathrm{d}x \downarrow \ \boxed{2x\mathrm{e}^x} \qquad\qquad \int\mathrm{d}x \searrow \ \boxed{y^2\,\mathrm{e}^x} \ \nearrow \frac{\partial}{\partial y}$$

General solution: $\quad 2x\,\mathrm{e}^x + y^2\,\mathrm{e}^x = C$

Problem 2.37

Solve $\quad (2xy^2 - y)\,\mathrm{d}x + (y^2+x+y)\,\mathrm{d}y = 0$

$$M = 2xy^2 - y,\quad N = y^2+x+y \implies \frac{\partial M}{\partial y} = 4xy-1,\quad \frac{\partial N}{\partial x} = 1$$ ✍ Not exact

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1-(4xy-1)}{2xy^2-y} = -\frac{2}{y}$$ ✍ A function of y only

$$\mu(y) = \exp\left[\int \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mathrm{d}y\right] = \exp\left(\int -\frac{2}{y}\,\mathrm{d}y\right) = \mathrm{e}^{-2\ln y} = \frac{1}{y^2}$$

Multiply the DE by $\mu(y) = \dfrac{1}{y^2}$: $\quad \left(2x - \dfrac{1}{y}\right)\mathrm{d}x + \left(1 + \dfrac{x}{y^2} + \dfrac{1}{y}\right)\mathrm{d}y = 0$

$$2x\,\mathrm{d}x - \frac{1}{y}\,\mathrm{d}x + \frac{x}{y^2}\,\mathrm{d}y + \left(1+\frac{1}{y}\right)\mathrm{d}y = 0$$

$$\int\mathrm{d}x \downarrow \ \boxed{x^2} \qquad \int\mathrm{d}x \searrow \ \boxed{-\frac{x}{y}} \ \nearrow \frac{\partial}{\partial y} \qquad \int\mathrm{d}y \downarrow \ \boxed{y+\ln|y|}$$

General solution: $\quad x^2 - \dfrac{x}{y} + y + \ln|y| = C$

Problem 2.38

Solve $\quad y\,(x+y)\,\mathrm{d}x + (x+2y-1)\,\mathrm{d}y = 0$

$$M = xy + y^2, \quad N = x + 2y - 1$$

$$\frac{\partial M}{\partial y} = x + 2y, \quad \frac{\partial N}{\partial x} = 1 \implies \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ ✍ Not exact DE

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(x+2y)-1}{x+2y-1} = 1$$ ✍ A function of x only

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mathrm{d}x\right] = \exp\left(\int 1\cdot \mathrm{d}x\right) = \mathrm{e}^x$$

Multiplying the DE by $\mu(x) = \mathrm{e}^x$: $\quad y\,\mathrm{e}^x(x+y)\,\mathrm{d}x + \mathrm{e}^x(x+2y-1)\,\mathrm{d}y = 0$

$$y^2\,\mathrm{e}^x\,\mathrm{d}x \quad + \quad 2y\,\mathrm{e}^x\,\mathrm{d}y \; + \; (x-1)\,\mathrm{e}^x\,\mathrm{d}y \quad + \quad xy\,\mathrm{e}^x\,\mathrm{d}x \; = 0$$

$$\int \mathrm{d}x \quad \boxed{y^2\,\mathrm{e}^x} \quad \frac{\partial}{\partial y} \qquad\qquad \int \mathrm{d}y \quad \boxed{(x-1)\,\mathrm{e}^x\,y} \quad \frac{\partial}{\partial x}$$

General solution: $\quad y^2\,\mathrm{e}^x + (x-1)\,\mathrm{e}^x y = C$

Problem 2.39

Solve $\quad 2x\,(x^2 - \sin y + 1)\,\mathrm{d}x + (x^2+1)\cos y\,\mathrm{d}y = 0$

$$M = 2x\,(x^2+1) - 2x\sin y, \quad N = (x^2+1)\cos y$$

$$\frac{\partial M}{\partial y} = -2x\cos y, \quad \frac{\partial N}{\partial x} = 2x\cos y \implies \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ ✍ Not exact DE

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(x^2+1)\cos y}\,(-2x\cos y - 2x\cos y) = -\frac{4x}{x^2+1}$$

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mathrm{d}x\right] = \exp\left[-\int \frac{4x}{x^2+1}\,\mathrm{d}x\right]$$

$$= \exp\left[-2\int \frac{1}{x^2+1}\,\mathrm{d}(x^2+1)\right] = \exp\left[-2\ln(x^2+1)\right] = \frac{1}{(x^2+1)^2}$$

Multiplying the DE by $\mu(x)$:

$$\frac{2x}{x^2+1}\,\mathrm{d}x + \frac{\cos y}{x^2+1}\,\mathrm{d}y \quad - \quad \frac{2x\sin y}{(x^2+1)^2}\,\mathrm{d}x \; = 0$$

$$\int \mathrm{d}x \quad \boxed{\ln(x^2+1)} \qquad\qquad \int \mathrm{d}y \quad \boxed{\frac{\sin y}{x^2+1}} \quad \frac{\partial}{\partial x}$$

General solution: $\quad \ln(x^2+1) + \dfrac{\sin y}{x^2+1} = C$

Problem 2.40

Consider a homogeneous differential equation of the form

$$M(u)\,\mathrm{d}x + N(u)\,\mathrm{d}y = 0, \qquad u = \frac{y}{x}.$$

If $Mx+Ny=0$, i.e., $M(u)+N(u)u=0$, show that $\dfrac{1}{xM}$ is an integrating factor.

Multiply $\dfrac{1}{Mx}$ to the differential equation

$$\underbrace{\frac{1}{x}}_{\bar{M}}\,\mathrm{d}x + \underbrace{\frac{N}{Mx}}_{\bar{N}}\,\mathrm{d}y = 0 \qquad (*)$$

$$\frac{\partial \bar{M}}{\partial y} = \frac{\partial}{\partial y}\Big(\frac{1}{x}\Big) = 0, \qquad \frac{\partial \bar{N}}{\partial x} = \frac{\partial}{\partial x}\Big(\frac{N}{Mx}\Big) = \frac{\partial}{\partial x}\Big(\frac{N}{M}\Big)\cdot\frac{1}{x} + \frac{N}{M}\cdot\Big(-\frac{1}{x^2}\Big)$$

$$\because\ \frac{\partial}{\partial x}\Big(\frac{N}{M}\Big) = \frac{\partial}{\partial u}\Big(\frac{N}{M}\Big)\frac{\partial u}{\partial x} = \frac{N'M-NM'}{M^2}\Big(-\frac{y}{x^2}\Big)$$ ✍ Chain rule

$$\because\ M+Nu=0 \implies M=-Nu$$

$$M' = -N'u - N$$ ✍ Differentiate with respect to u

$$\therefore\ \frac{\partial \bar{N}}{\partial x} = \frac{N'M-NM'}{M^2}\Big(-\frac{y}{x^2}\Big)\cdot\frac{1}{x} + \frac{N}{M}\Big(-\frac{1}{x^2}\Big)$$

$$= -\frac{1}{x^2M}\Big(\frac{N'M-NM'}{M}u+N\Big) = -\frac{1}{x^2M}\Big(\frac{N'Mu-NM'u+MN}{M}\Big)$$

$$= -\frac{1}{x^2M}\,\frac{N'(-Nu)u-N(-N'u-N)u+N(-Nu)}{M} = 0$$

$$\therefore\ \frac{\partial \bar{M}}{\partial y} = \frac{\partial \bar{N}}{\partial x} = 0 \implies \text{DE } (*) \text{ is exact or } \frac{1}{Mx} \text{ is an integrating factor.}$$

Problem 2.41

Solve $\quad (x^2+y+y^2)\,\mathrm{d}x - x\,\mathrm{d}y = 0$

$$(x^2+y^2)\,\mathrm{d}x + y\,\mathrm{d}x - x\,\mathrm{d}y = 0$$

$$\mathrm{d}x - \frac{-y\,\mathrm{d}x+x\,\mathrm{d}y}{x^2+y^2} = 0$$

$$\mathrm{d}x - \mathrm{d}\Big(\tan^{-1}\frac{y}{x}\Big) = 0 \implies x - \tan^{-1}\frac{y}{x} = C$$

Problem 2.42

Solve $\quad \big(x-\sqrt{x^2+y^2}\big)\,\mathrm{d}x + \big(y-\sqrt{x^2+y^2}\big)\,\mathrm{d}y = 0$

$$(x\,\mathrm{d}x+y\,\mathrm{d}y)-\sqrt{x^2+y^2}\,(\mathrm{d}x+\mathrm{d}y)=0$$

$$\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{\sqrt{x^2+y^2}}-(\mathrm{d}x+\mathrm{d}y)=0$$

$$\mathrm{d}\left(\sqrt{x^2+y^2}\right)-(\mathrm{d}x+\mathrm{d}y)=0 \implies \sqrt{x^2+y^2}-x-y=C$$

Problem 2.43

Solve $\quad y\sqrt{1+y^2}\,\mathrm{d}x+\left(x\sqrt{1+y^2}-y\right)\mathrm{d}y=0$

$$\sqrt{1+y^2}\,(y\,\mathrm{d}x+x\,\mathrm{d}y)-y\,\mathrm{d}y=0$$

$$(y\,\mathrm{d}x+x\,\mathrm{d}y)-\frac{y\,\mathrm{d}y}{\sqrt{1+y^2}}=0$$

$$\mathrm{d}\,(xy)-\mathrm{d}\left(\sqrt{1+y^2}\right)=0 \implies xy-\sqrt{1+y^2}=C$$

Problem 2.44

Solve $\quad y^2\,\mathrm{d}x-(xy+x^3)\,\mathrm{d}y=0$

$$y\,(y\,\mathrm{d}x-x\,\mathrm{d}y)-x^3\,\mathrm{d}y=0 \implies \frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{x^2}-\frac{x}{y}\,\mathrm{d}y=0$$

$$-\mathrm{d}\left(\frac{y}{x}\right)-\frac{x}{y}\,\mathrm{d}y=0 \implies \frac{y}{x}\mathrm{d}\left(\frac{y}{x}\right)+\mathrm{d}y=0$$

$$\mathrm{d}\left[\frac{1}{2}\left(\frac{y}{x}\right)^2\right]+\mathrm{d}y=0 \implies \frac{1}{2}\left(\frac{y}{x}\right)^2+y=C$$

Problem 2.45

Solve $\quad y\,\mathrm{d}x-x\,\mathrm{d}y-2x^3\tan\frac{y}{x}\,\mathrm{d}x=0$

$$\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{x^2}=2x\tan\frac{y}{x}\,\mathrm{d}x \implies -\mathrm{d}\left(\frac{y}{x}\right)=2x\tan\frac{y}{x}\,\mathrm{d}x$$

$$-\cot\frac{y}{x}\,\mathrm{d}\left(\frac{y}{x}\right)=2x\,\mathrm{d}x \implies -\mathrm{d}\left(\ln\left|\sin\frac{y}{x}\right|\right)=\mathrm{d}(x^2)$$

$$\therefore\ \ln\left|\sin\frac{y}{x}\right|=-x^2+C \implies \sin\frac{y}{x}=C\,\mathrm{e}^{-x^2}$$

Problem 2.46

Solve $\quad (2x^2y^2+y)\,\mathrm{d}x+(x^3y-x)\,\mathrm{d}y=0$

$$xy\left(2xy\,\mathrm{d}x+x^2\,\mathrm{d}y\right)+(y\,\mathrm{d}x-x\,\mathrm{d}y)=0$$

$$\frac{xy}{y^2}\,\mathrm{d}(x^2y)+\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{y^2}=0 \implies \frac{x}{y}\,\mathrm{d}(x^2y)+\mathrm{d}\left(\frac{x}{y}\right)=0$$

$$\mathrm{d}(x^2y) + \frac{y}{x}\,\mathrm{d}\Big(\frac{x}{y}\Big) = 0 \implies \mathrm{d}(x^2y) + \mathrm{d}\Big(\ln\Big|\frac{x}{y}\Big|\Big) = 0$$

$$\therefore\ x^2y + \ln\Big|\frac{x}{y}\Big| = C$$

Problem 2.47

Solve $\quad y^2\,\mathrm{d}x + \big[xy + \tan(xy)\big]\mathrm{d}y = 0$

$$y\,(y\,\mathrm{d}x + x\,\mathrm{d}y) + \tan(xy)\,\mathrm{d}y = 0 \implies y\,\mathrm{d}(xy) + \tan(xy)\,\mathrm{d}y = 0$$

$$\cot(xy)\,\mathrm{d}(xy) + \frac{1}{y}\,\mathrm{d}y = 0 \implies \mathrm{d}\big[\ln\big|\sin(xy)\big|\big] + \mathrm{d}\big[\ln|y|\big] = 0$$

$$\ln\big|y\sin(xy)\big| = C \implies y\sin(xy) = C$$

Problem 2.48

Solve $\quad (2x^2y^4 - y)\,\mathrm{d}x + (4x^3y^3 - x)\,\mathrm{d}y = 0$

$$(2x^2y^4\,\mathrm{d}x + 4x^3y^3\,\mathrm{d}y) - (y\mathrm{d}x + x\mathrm{d}y) = 0$$

$$2x^2y^2\,(y^2\mathrm{d}x + 2xy\,\mathrm{d}y) - \mathrm{d}(xy) = 0 \implies 2x^2y^2\,\mathrm{d}(xy^2) - \mathrm{d}(xy) = 0$$

$$2\,\mathrm{d}(xy^2) - \frac{1}{(xy)^2}\,\mathrm{d}(xy) = 0 \implies 2\,\mathrm{d}(xy^2) + \mathrm{d}\Big(\frac{1}{xy}\Big) = 0$$

$$\therefore\ 2xy^2 + \frac{1}{xy} = C$$

Problem 2.49

Solve $\quad (x^2y^3 + y)\,\mathrm{d}x + (x^3y^2 - x)\,\mathrm{d}y = 0$

$$x^2y^2(y\,\mathrm{d}x + x\,\mathrm{d}y) + (y\,\mathrm{d}x - x\,\mathrm{d}y) = 0 \implies x^2y^2\mathrm{d}(xy) + (y\,\mathrm{d}x - x\,\mathrm{d}y) = 0$$

$$x^2\mathrm{d}(xy) + \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{y^2} = 0 \implies x^2\mathrm{d}(xy) + \mathrm{d}\Big(\frac{x}{y}\Big) = 0$$

$$\mu_1 = \frac{1}{x^2},\ u_1 = xy;\ \ \mu_2 = 1,\ u_2 = \frac{x}{y} \implies \frac{1}{x^2}\cdot xy = 1\cdot\Big(\frac{x}{y}\Big)^{-1} = \frac{y}{x} = \mu$$

Multiply the differential equation by integrating factor $\mu = \dfrac{y}{x}$:

$$\frac{y}{x}\cdot x^2\mathrm{d}(xy) + \frac{y}{x}\cdot\mathrm{d}\Big(\frac{x}{y}\Big) = 0 \implies (xy)\mathrm{d}(xy) + \frac{y}{x}\cdot\mathrm{d}\Big(\frac{x}{y}\Big) = 0$$

$$\tfrac{1}{2}\mathrm{d}\big[(xy)^2\big] + \mathrm{d}\Big(\ln\Big|\frac{x}{y}\Big|\Big) = 0 \implies \tfrac{1}{2}x^2y^2 + \ln\Big|\frac{x}{y}\Big| = C$$

Problem 2.50

Solve $y(y^2+1)\,\mathrm{d}x + x(y^2-x+1)\,\mathrm{d}y = 0$

$$(y^3\,\mathrm{d}x + xy^2\,\mathrm{d}y) + (y\,\mathrm{d}x + x\,\mathrm{d}y) - x^2\mathrm{d}y = 0$$

$$y^2(y\,\mathrm{d}x + x\,\mathrm{d}y) + (y\,\mathrm{d}x + x\,\mathrm{d}y) - x^2\mathrm{d}y = 0$$

$$\underbrace{(y^2+1)\,\mathrm{d}(xy)}_{\text{Group 1}} - \underbrace{x^2\mathrm{d}y}_{\text{Group 2}} = 0$$

Group 1: $\mu_1 = \dfrac{1}{y^2+1}, \qquad u_1 = xy$

Group 2: $\mu_2 = \dfrac{1}{x^2}, \qquad u_2 = y$

$$\mu = \frac{1}{y^2+1}\cdot\frac{1}{x^2y^2} = \frac{1}{x^2}\cdot\frac{1}{y^2(1+y^2)} = \frac{1}{x^2y^2(1+y^2)}$$

Multiply the differential equation by integrating factor μ:

$$\frac{1}{x^2y^2}\mathrm{d}(xy) - \frac{1}{y^2(1+y^2)}\mathrm{d}y = 0 \implies \frac{1}{(xy)^2}\mathrm{d}(xy) - \left(\frac{1}{y^2} - \frac{1}{1+y^2}\right)\mathrm{d}y = 0$$

$$-\mathrm{d}\left(\frac{1}{xy}\right) + \mathrm{d}\left(\frac{1}{y}\right) + \mathrm{d}\left(\tan^{-1}y\right) = 0 \implies \frac{1}{xy} - \frac{1}{y} - \tan^{-1}y = C.$$

Problem 2.51

Solve $y^2\,\mathrm{d}x + (\mathrm{e}^x - y)\,\mathrm{d}y = 0$

$$\underbrace{(y^2\,\mathrm{d}x - y\,\mathrm{d}y)}_{\text{Group 1}} + \underbrace{\mathrm{e}^x\,\mathrm{d}y}_{\text{Group 2}} = 0$$

Group 1: $M_1 = y^2, \quad N_1 = -y \implies \dfrac{\partial M_1}{\partial y} = 2y, \quad \dfrac{\partial N_1}{\partial x} = 0$

$$\frac{1}{M_1}\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) = \frac{1}{y^2}(0-2y) = -\frac{2}{y}$$ ✍ A function of y only

$$\mu_1(y) = \exp\left(\int -\frac{2}{y}\,\mathrm{d}y\right) = \mathrm{e}^{-2\ln|y|} = \frac{1}{y^2}$$

Multiply Group 1 by integrating factor μ_1:

$$\mathrm{d}x - \frac{1}{y}\mathrm{d}y = \mathrm{d}x - \mathrm{d}(\ln|y|) = \mathrm{d}[\,\ln(\mathrm{e}^x)] - \mathrm{d}[\,\ln|y|] = \mathrm{d}\left(\ln\left|\frac{\mathrm{e}^x}{y}\right|\right)$$

$$\therefore\ \mu_1 = \frac{1}{y^2}, \quad u_1 = \ln\left|\frac{\mathrm{e}^x}{y}\right|$$

Group 2: $\mu_2 = \dfrac{1}{\mathrm{e}^x}, \quad u_2 = y$

$$\mu = \mu_1\, g_1(u_1) = \mu_2\, g_2(u_2) = \frac{1}{y^2}\,\mathrm{e}^{-\ln\left|\frac{\mathrm{e}^x}{y}\right|} = \frac{1}{\mathrm{e}^x}\cdot\frac{1}{y}$$

Multiply the differential equation by $\mu = \dfrac{1}{\mathrm{e}^x y}$: $\quad \dfrac{y}{\mathrm{e}^x}\,\mathrm{d}x - \dfrac{1}{\mathrm{e}^x}\,\mathrm{d}y + \dfrac{1}{y}\,\mathrm{d}y = 0$

$$y\mathrm{e}^{-x}\,\mathrm{d}x \quad - \quad \mathrm{e}^{-x}\,\mathrm{d}y \quad + \quad \frac{1}{y}\,\mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{-\,y\mathrm{e}^{-x}} \quad \frac{\partial}{\partial y} \qquad\qquad \int \mathrm{d}y \quad \boxed{\ln|y|}$$

General solution: $\quad -\,y\mathrm{e}^{-x} + \ln|y| = C$

Problem 2.52

Solve $\quad (x^2y^2 - 2y)\,\mathrm{d}x + (x^3y - x)\,\mathrm{d}y = 0$

Method 1: $\quad \underbrace{(x^2y^2\,\mathrm{d}x + x^3y\,\mathrm{d}y)}_{\text{Group 1}} - \underbrace{(2y\,\mathrm{d}x + x\,\mathrm{d}y)}_{\text{Group 2}} = 0$

Group 1: $\quad x^2y^2\,\mathrm{d}x + x^3y\,\mathrm{d}y = x^2y\,(y\,\mathrm{d}x + x\,\mathrm{d}y) = x^2y\,\mathrm{d}(xy)$

Group 2: $\quad M_2 = 2y, \quad N_2 = x \implies \dfrac{\partial M_2}{\partial y} = 2, \quad \dfrac{\partial N_2}{\partial x} = 1$

$$\frac{1}{N_2}\left(\frac{\partial M_2}{\partial y} - \frac{\partial N_2}{\partial x}\right) = \frac{1}{x}(2-1) = \frac{1}{x}$$ ✍ A function of x only

$$\mu_2(x) = \exp\left(\int \frac{1}{x}\,\mathrm{d}x\right) = \mathrm{e}^{\ln x} = x$$

Multiply Group 2 by integrating factor μ_2:

$$x\cdot(2y\,\mathrm{d}x + x\,\mathrm{d}y) = 2xy\,\mathrm{d}x \quad + \quad x^2\,\mathrm{d}y = \mathrm{d}(x^2y).$$

$$\int \mathrm{d}x \quad \boxed{x^2y} \quad \frac{\partial}{\partial y}$$

$$\mu_1 = \frac{1}{x^2y}, \quad u_1 = xy; \qquad \mu_2 = x, \quad u_2 = x^2y:$$

$$\mu_1\, g_1(u_1) = \mu_2\, g_2(u_2) = \frac{1}{x^2y}\,(xy)^\alpha = x\,(x^2y)^\beta \implies x^{\alpha-2}y^{\alpha-1} = x^{2\beta+1}y^\beta$$

$$\left.\begin{aligned} \alpha - 2 &= 2\beta + 1 \\ \alpha - 1 &= \beta \end{aligned}\right\} \implies \left\{\begin{aligned} \alpha &= -1 \\ \beta &= -2 \end{aligned}\right.$$

$$\therefore\ \mu = x^{\alpha-2}y^{\alpha-1} = x^{-1-2}y^{-1-1} = \frac{1}{x^3y^2}$$

Multiply the differential equation by μ:

$$\begin{array}{ccccccccc} \frac{1}{x}\,\mathrm{d}x & + & \frac{1}{y}\,\mathrm{d}y & - & \frac{2}{x^3y}\,\mathrm{d}x & & - & \frac{1}{x^2y^2}\,\mathrm{d}y & = 0\\ {\scriptstyle\int \mathrm{d}x}\Big\downarrow & & {\scriptstyle\int \mathrm{d}y}\Big\downarrow & & {\scriptstyle\int \mathrm{d}x}\searrow & & \nearrow{\scriptstyle\frac{\partial}{\partial y}} & & \\ \boxed{\ln|x|} & & \boxed{\ln|y|} & & & \boxed{\frac{1}{x^2y}} & & & \end{array}$$

General solution: $\ln|x| + \ln|y| + \dfrac{1}{x^2y} = C \implies \ln|xy| + \dfrac{1}{x^2y} = C$

Method 2: $x^2y(y\,\mathrm{d}x + x\,\mathrm{d}y) - (2y\,\mathrm{d}x + x\,\mathrm{d}y) = 0$

Multiply the differential equation by x: $x^3y\,(y\,\mathrm{d}x + x\,\mathrm{d}y) - (2xy\,\mathrm{d}x + x^2\,\mathrm{d}y) = 0$

$$\underbrace{x^3y\,\mathrm{d}(xy)}_{\text{Group 1}} - \underbrace{\mathrm{d}(x^2y)}_{\text{Group 2}} = 0$$

Group 1: $\mu_1 = \dfrac{1}{x^3y}, \quad u_1 = xy$

Group 2: $\mu_2 = 1, \quad u_2 = x^2y$

$$\mu_1 g_1(u_1) = \mu_2 g_2(u_2) \implies \frac{1}{x^3y}(xy)^{\alpha} = 1\cdot(x^2y)^{\beta} \implies x^{\alpha-3}y^{\alpha-1} = x^{2\beta}y^{\beta}$$

$$\left.\begin{array}{l} \alpha - 3 = 2\beta \\ \alpha - 1 = \beta \end{array}\right\} \implies \left\{\begin{array}{l} \beta = -2 \\ \alpha = -1 \end{array}\right. \implies \mu = x^{-4}y^{-2}$$

Multiply the differential equation by μ: $\dfrac{1}{xy}\,\mathrm{d}(xy) - \dfrac{1}{x^4y^2}\,\mathrm{d}(x^2y) = 0$

$$\frac{1}{xy}\mathrm{d}(xy) - \frac{1}{(x^2y)^2}\mathrm{d}(x^2y) = 0 \implies \mathrm{d}\big[\ln|xy|\big] + \mathrm{d}\Big(\frac{1}{x^2y}\Big) = 0$$

General solution: $\ln|xy| + \dfrac{1}{x^2y} = C$

Problem 2.53

Solve $(2x^3y + y^3)\,\mathrm{d}x - (x^4 + 2xy^2)\,\mathrm{d}y = 0$

$$\underbrace{2x^3y\,\mathrm{d}x - x^4\,\mathrm{d}y}_{\text{Group 1}} + \underbrace{y^3\mathrm{d}x - 2xy^2\,\mathrm{d}y}_{\text{Group 2}} = 0$$

Group 1: $M_1 = 2x^3y, \quad N_1 = -x^4 \implies \dfrac{\partial M_1}{\partial y} = 2x^3, \quad \dfrac{\partial N_1}{\partial x} = -4x^3$

$$\frac{1}{N_1}\Big(\frac{\partial M_1}{\partial y} - \frac{\partial N_1}{\partial x}\Big) = -\frac{1}{x^4}(2x^3 + 4x^3) = -\frac{6}{x}$$ ✍ A function of x only

$$\mu_1 = \exp\Big(\int -\frac{6}{x}\mathrm{d}x\Big) = \mathrm{e}^{-6\ln x} = \frac{1}{x^6}$$

Multiply Group 1 by $\mu_1(x)$:

$$2\frac{y}{x^3}\,\mathrm{d}x \quad - \quad \frac{1}{x^2}\,\mathrm{d}y \implies u_1 = \frac{y}{x^2}$$

$$\boxed{-\frac{y}{x^2}} \quad \frac{\partial}{\partial y}$$

Group 2: $M_2 = y^3, \quad N_2 = -2xy^2 \implies \dfrac{\partial M_2}{\partial y} = 3y^2, \quad \dfrac{\partial N_2}{\partial x} = -2y^2$

$$\frac{1}{N_2}\Big(\frac{\partial M_2}{\partial y} - \frac{\partial N_2}{\partial x}\Big) = \frac{1}{-2xy^2}\,(3y^2+2y^2) = -\frac{5}{2x} \quad \text{A function of } x \text{ only}$$

$$\mu_2(x) = \exp\Big(\int -\frac{5}{2x}\mathrm{d}x\Big) = \mathrm{e}^{-\frac{5}{2}\ln x} = x^{-\frac{5}{2}}$$

Multiply Group 2 by $\mu_2(x)$:

$$-2x^{-\frac{3}{2}}y^2\,\mathrm{d}y \quad + \quad x^{-\frac{5}{2}}y^3\,\mathrm{d}x \implies u_2 = x^{-\frac{3}{2}}y^3$$

$$\int \mathrm{d}y \quad \boxed{-\frac{2}{3}x^{-\frac{3}{2}}y^3} \quad \frac{\partial}{\partial x}$$

$$\therefore\ \mu_1 = \frac{1}{x^6},\quad u_1 = \frac{y}{x^2};\qquad \mu_2 = x^{-\frac{5}{2}},\quad u_2 = x^{-\frac{3}{2}}y^3$$

$$\mu_1 g_1(u_1) = \mu_2 g_2(u_2) \implies x^{-6}(x^{-2}y)^{\alpha} = x^{-\frac{5}{2}}(x^{-\frac{3}{2}}y^3)^{\beta}$$

$$x^{-6-2\alpha}y^{\alpha} = x^{-\frac{5}{2}-\frac{3}{2}\beta}y^{3\beta}$$

$$\left.\begin{aligned} -6-2\alpha &= -\frac{5}{2}-\frac{3}{2}\beta \\ \alpha &= 3\beta \end{aligned}\right\} \implies \left\{\begin{aligned} \alpha &= -\frac{7}{3} \\ \beta &= -\frac{7}{9} \end{aligned}\right.$$

$$\therefore\ \mu = x^{-6-2\alpha}y^{\alpha} = x^{-6+\frac{14}{3}}y^{-\frac{7}{3}} = x^{-\frac{4}{3}}y^{-\frac{7}{3}}$$

Multiply the differential equation by μ:

$$2x^{\frac{5}{3}}y^{-\frac{4}{3}}\mathrm{d}x \quad - \quad x^{\frac{8}{3}}y^{-\frac{7}{3}}\mathrm{d}y \ + \ x^{-\frac{4}{3}}y^{\frac{2}{3}}\mathrm{d}x \quad - \quad 2x^{-\frac{1}{3}}y^{-\frac{1}{3}}\mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{\frac{3}{4}x^{\frac{8}{3}}y^{-\frac{4}{3}}} \quad \frac{\partial}{\partial y} \qquad\qquad \int \mathrm{d}x \quad \boxed{-3x^{-\frac{1}{3}}y^{\frac{2}{3}}} \quad \frac{\partial}{\partial y}$$

General solution: $\dfrac{3}{4}x^{\frac{8}{3}}y^{-\frac{4}{3}} - 3x^{-\frac{1}{3}}y^{\frac{2}{3}} = C \implies x^{\frac{8}{3}}y^{-\frac{4}{3}} - 4x^{-\frac{1}{3}}y^{\frac{2}{3}} = C$

Problem 2.54

Solve $\quad (1+y\cos x)\,\mathrm{d}x - \sin x\,\mathrm{d}y = 0$

Divide the differential equation by $\sin x\,\mathrm{d}x$: $\quad \dfrac{1+y\cos x}{\sin x} - \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$\therefore\ y' - \frac{\cos x}{\sin x}y = \frac{1}{\sin x}, \quad P(x) = -\frac{\cos x}{\sin x}, \quad Q(x) = \frac{1}{\sin x}$$

$$\int P(x)\,\mathrm{d}x = -\int \frac{\cos x}{\sin x}\,\mathrm{d}x = -\ln\big|\sin x\big|, \quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{-\ln|\sin x|} = \frac{1}{\sin x}$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x = \int \frac{1}{\sin x}\cdot\frac{1}{\sin x}\,\mathrm{d}x = -\cot x$$

$$\therefore\ \ y = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x + C\Big] = \sin x\,(-\cot x + C)$$

$$= -\cos x + C\sin x$$

Problem 2.55

Solve $\quad (\sin^2 y + x\cot y)\,y' = 1$

$$\frac{\mathrm{d}x}{\mathrm{d}y} - \cot y\cdot x = \sin^2 y, \quad P(y) = -\cot y, \quad Q(y) = \sin^2 y$$ ✍ Linear first-order

$$P(y) = -\cot y, \qquad Q(y) = \sin^2 y$$

$$\int P(y)\,\mathrm{d}y = -\int \cot y\,\mathrm{d}y = -\ln\big|\sin y\big|, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{-\ln|\sin y|} = \frac{1}{\sin y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y = \int \sin^2 y\cdot\frac{1}{\sin y}\,\mathrm{d}y = \int \sin y\,\mathrm{d}y = -\cos y$$

$$\therefore\ x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y + C\Big] = \sin y\,(-\cos y + C)$$

General solution: $\quad x = \sin y\,(C - \cos y)$

Problem 2.56

Solve $\quad \mathrm{d}x - (y - 2xy)\,\mathrm{d}y = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + 2y\cdot x = y, \quad P(y) = 2y, \quad Q(y) = y$$ ✍ Linear first-order

$$\int P(y)\,\mathrm{d}y = \int 2y\,\mathrm{d}y = y^2, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{y^2}$$

$$x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y + C\Big]\mathrm{e}^{-y^2}\Big(\int y\,\mathrm{e}^{y^2}\mathrm{d}y + C\Big)$$

$$= \mathrm{e}^{-y^2}\Big[\int \mathrm{e}^{y^2}\cdot\tfrac{1}{2}\mathrm{d}(y^2) + C\Big] = \mathrm{e}^{-y^2}\big(\tfrac{1}{2}\mathrm{e}^{y^2} + C\big) = \tfrac{1}{2} + C\mathrm{e}^{-y^2}$$

Problem 2.57

Solve $\quad \mathrm{d}x - (1 + 2x\tan y)\,\mathrm{d}y = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}y} - 2\tan y \cdot x = 1, \quad P(y) = -2\tan y, \quad Q(y) = 1$$

✍ Linear first-order

$$\int P(y)\,\mathrm{d}y = -2\int \tan y\,\mathrm{d}y = 2\ln\left|\cos y\right|, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{2\ln|\cos y|} = \cos^2 y$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int 1\cdot\cos^2 y\,\mathrm{d}y = \int \frac{1+\cos 2y}{2}\,\mathrm{d}y = \frac{y}{2} + \frac{\sin 2y}{4}$$

$$x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = \frac{1}{\cos^2 y}\Big(\frac{y}{2} + \frac{\sin 2y}{4} + C\Big)$$

$$\therefore\ \ 2x\cos^2 y = y + \sin y\cos y + C$$

Problem 2.58

Solve $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x}\Big(y^3 + \dfrac{x}{y}\Big) = 1$

$$\frac{\mathrm{d}x}{\mathrm{d}y} = y^3 + \frac{x}{y} \implies \frac{\mathrm{d}x}{\mathrm{d}y} - \frac{1}{y}\cdot x = y^3, \quad P(y) = -\frac{1}{y}, \quad Q(y) = y^3$$

$$\int P(y)\,\mathrm{d}y = \int -\frac{1}{y}\,\mathrm{d}y = -\ln\left|y\right|, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{-\ln|y|} = \frac{1}{y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int y^3\cdot\frac{1}{y}\,\mathrm{d}y = \int y^2\,\mathrm{d}y = \tfrac{1}{3}y^3$$

$$x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = y\big(\tfrac{1}{3}y^3 + C\big)$$

$$\therefore\ \ x - \tfrac{1}{3}y^4 - Cy = 0$$

Problem 2.59

Solve $\quad \mathrm{d}x + (x - y^2)\,\mathrm{d}y = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + x = y^2, \quad P(y) = 1, \quad Q(y) = y^2$$

✍ Linear first-order

$$\int P(y)\,\mathrm{d}y = \int 1\cdot\mathrm{d}y = y, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int y^2\,\mathrm{e}^{y}\,\mathrm{d}y = (y^2 - 2y + 2)\,\mathrm{e}^{y}$$

$$x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = \mathrm{e}^{-y}\big[(y^2 - 2y + 2)\,\mathrm{e}^{y} + C\big]$$

$$\therefore\ \ x = y^2 - 2y + 2 + C\,\mathrm{e}^{-y}$$

Problem 2.60

Solve $\quad y^2\,\mathrm{d}x + (xy + y^2 - 1)\,\mathrm{d}y = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{1}{y}\cdot x = -\frac{y^2-1}{y^2}, \quad P(y) = \frac{1}{y}, \quad Q(y) = -1 + \frac{1}{y^2}$$

$$\int P(y)\,\mathrm{d}y = \int \frac{1}{y}\,\mathrm{d}y = \ln|y|, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{\ln|y|} = y$$

$$x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = \frac{1}{y}\Big[\int\Big(-1+\frac{1}{y^2}\Big)y\,\mathrm{d}y + C\Big]$$

$$= \frac{1}{y}\big(-\tfrac{1}{2}y^2 + \ln|y| + C\big) \implies y^2 + 2xy - 2\ln|y| = C$$

Problem 2.61

Solve $\quad y\,\mathrm{d}x = (\mathrm{e}^y + 2xy - 2x)\,\mathrm{d}y$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{2(1-y)}{y}\cdot x = \frac{\mathrm{e}^y}{y}, \quad P(y) = \frac{2(1-y)}{y}, \quad Q(y) = \frac{\mathrm{e}^y}{y}$$

$$\int P(y)\,\mathrm{d}y = 2\int\Big(\frac{1}{y} - 1\Big)\mathrm{d}y = 2\big(\ln|y| - y\big) = 2\ln|y\,\mathrm{e}^{-y}| = \ln\big(y^2\,\mathrm{e}^{-2y}\big)$$

$$\mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{\ln(y^2\,\mathrm{e}^{-2y})} = y^2\,\mathrm{e}^{-2y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int \frac{\mathrm{e}^y}{y}\cdot y^2\,\mathrm{e}^{-2y}\,\mathrm{d}y = \int y\,\mathrm{e}^{-y}\,\mathrm{d}y = -(y+1)\,\mathrm{e}^{-y}$$

$$x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = \frac{\mathrm{e}^{2y}}{y^2}\big[-\mathrm{e}^{-y}(y+1) + C\big]$$

$$\therefore\quad y^2 x = C\,\mathrm{e}^{2y} - (y+1)\,\mathrm{e}^y$$

Problem 2.62

Solve $\quad (2x+3)\,y' = y + (2x+3)^{1/2}, \quad y(-1) = 0$

$$y' - \frac{1}{2x+3}\cdot y = \frac{1}{\sqrt{2x+3}}, \quad P(x) = -\frac{1}{2x+3}, \quad Q(x) = \frac{1}{\sqrt{2x+3}}$$

$$\int P(x)\,\mathrm{d}x = -\int \frac{1}{2x+3}\,\mathrm{d}x = -\tfrac{1}{2}\ln|2x+3|, \quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{-\frac{1}{2}\ln|2x+3|} = \frac{1}{\sqrt{2x+3}}$$

$$y = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big]$$

$$= \sqrt{2x+3}\Big[\int \frac{1}{\sqrt{2x+3}}\cdot\frac{1}{\sqrt{2x+3}}\,\mathrm{d}x + C\Big] = \sqrt{2x+3}\Big[\tfrac{1}{2}\ln|2x+3| + C\Big]$$

$$y(-1) = 0: \quad 0 = \sqrt{1}\left(\tfrac{1}{2}\ln 1 + C\right) \implies C = 0$$

$$\therefore \quad 2y = \sqrt{2x+3}\,\ln\left|2x+3\right|$$

Problem 2.63

Solve $\quad y\,\mathrm{d}x + (y^2\,\mathrm{e}^y - x)\,\mathrm{d}y = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}y} - \frac{1}{y}\cdot x = -y\,\mathrm{e}^y, \quad P(y) = -\frac{1}{y}, \quad Q(y) = -y\,\mathrm{e}^y$$ ✍ Linear first-order

$$\int P(y)\,\mathrm{d}y = \int -\frac{1}{y}\,\mathrm{d}y = -\ln\left|y\right|, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{-\ln|y|} = \frac{1}{y}$$

$$x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\left[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\right] = y\left[\int -y\,\mathrm{e}^y\cdot\frac{1}{y}\,\mathrm{d}y + C\right]$$

$$= y\left(\int -\mathrm{e}^y\,\mathrm{d}y + C\right) = y(-\mathrm{e}^y + C) \implies x = Cy - y\,\mathrm{e}^y$$

Problem 2.64

Solve $\quad y' = 1 + 3y\tan x$

$$y' - 3\tan x\cdot y = 1, \quad P(x) = -3\tan x, \quad Q(x) = 1$$ ✍ Linear first-order

$$\int P(x)\,\mathrm{d}x = -3\int \tan x\,\mathrm{d}x = 3\ln\left|\cos x\right|, \quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{3\ln|\cos x|} = \cos^3 x$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x = \int 1\cdot\cos^3 x\,\mathrm{d}x = \int (1-\sin^2 x)\,\mathrm{d}(\sin x) = \sin x - \tfrac{1}{3}\sin^3 x$$

$$y = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\left[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\right] = \frac{1}{\cos^3 x}\left(\sin x - \tfrac{1}{3}\sin^3 x + C\right)$$

Problem 2.65

Solve $\quad (1+\cos x)\,y' = \sin x\,(\sin x + \sin x\cos x - y)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\sin x}{1+\cos x}\cdot y = \sin^2 x, \quad P(x) = \frac{\sin x}{1+\cos x}, \quad Q(x) = \sin^2 x$$

$$\int P(x)\,\mathrm{d}x = \int \frac{\sin x}{1+\cos x}\,\mathrm{d}x = -\int \frac{\mathrm{d}(1+\cos x)}{1+\cos x} = -\ln\left|1+\cos x\right|$$

$$\mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{-\ln|1+\cos x|} = \frac{1}{1+\cos x}, \quad \mathrm{e}^{-\int P(x)\,\mathrm{d}x} = 1+\cos x$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x = \int \frac{\sin^2 x}{1+\cos x}\,\mathrm{d}x = \int \frac{1-\cos^2 x}{1+\cos x}\,\mathrm{d}x$$

$$= \int (1-\cos x)\,\mathrm{d}x = x - \sin x$$

$$y = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x + C\Big] = (1+\cos x)(x-\sin x+C)$$

Problem 2.66

Solve $\quad y' = (\sin^2 x - y)\cos x$

Let $u = \sin^2 x - y \implies \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2\sin x\cos x - \dfrac{\mathrm{d}y}{\mathrm{d}x} \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\sin x\cos x - \dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$2\sin x\cos x - \frac{\mathrm{d}u}{\mathrm{d}x} = u\cos x \implies \frac{\mathrm{d}u}{\mathrm{d}x} + \cos x\cdot u = 2\sin x\cos x$$

$$P(x) = \cos x,\quad Q(x) = 2\sin x\cos x,\quad \int P(x)\,\mathrm{d}x = \int \cos x\,\mathrm{d}x = \sin x$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x = \int 2\sin x\cos x\,\mathrm{e}^{\sin x}\mathrm{d}x = 2\int \sin x\,\mathrm{e}^{\sin x}\,\mathrm{d}(\sin x)$$

$$= 2\int t\,\mathrm{e}^t\mathrm{d}t = 2\mathrm{e}^t(t-1) = 2\mathrm{e}^{\sin x}(\sin x-1),\quad t = \sin x$$

$$u = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x + C\Big] = \mathrm{e}^{-\sin x}\big[2\mathrm{e}^{\sin x}(\sin x-1) + C\big]$$

$$\therefore\ \sin^2 x - y = 2(\sin x-1) + C\mathrm{e}^{-\sin x}$$

Alternatively,

$$y' + \cos x\cdot y = \cos x\sin^2 x \qquad \text{✍ Linear first-order}$$

$$P(x) = \cos x,\quad Q(x) = \cos x\sin^2 x,\quad \int P(x)\,\mathrm{d}x = \int\cos x\,\mathrm{d}x = \sin x$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x = \int \cos x\sin^2 x\,\mathrm{e}^{\sin x}\mathrm{d}x = \int \sin^2 x\,\mathrm{e}^{\sin x}\,\mathrm{d}(\sin x)$$

$$= \int t^2\mathrm{e}^t\mathrm{d}t = \mathrm{e}^t(t^2-2t+2) = \mathrm{e}^{\sin x}\big(\sin^2 x - 2\sin x + 2\big),\quad t = \sin x$$

$$y = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x + C\Big]$$

$$= \mathrm{e}^{-\sin x}\Big[\mathrm{e}^{\sin x}\big(\sin^2 x - 2\sin x + 2\big) + C\Big] = \sin^2 x - 2\sin x + 2 + C\mathrm{e}^{-\sin x}$$

Problem 2.67

Solve $\quad xy' - ny - x^{n+2}\mathrm{e}^x = 0,\quad n = \text{constant}$

$$y' - \frac{n}{x}\cdot y = x^{n+1}\mathrm{e}^x,\quad P(x) = -\frac{n}{x},\quad Q(x) = x^{n+1}\mathrm{e}^x \qquad \text{✍ Linear first-order}$$

$$\int P(x)\,\mathrm{d}x = \int -\frac{n}{x}\,\mathrm{d}x = -n\ln|x|,\quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{-n\ln|x|} = x^{-n}$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x = \int x^{n+1}\mathrm{e}^{x}\cdot x^{-n}\,\mathrm{d}x = \int x\,\mathrm{e}^{x}\,\mathrm{d}x = \mathrm{e}^{x}(x-1)$$

$$\therefore\quad y = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = x^{n}\big[\mathrm{e}^{x}(x-1) + C\big]$$

Problem 2.68

Solve $\qquad (1+x)\,\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = x(1+x)^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{1+x}\cdot y = x(1+x), \quad P(x) = -\frac{1}{1+x}, \quad Q(x) = x(1+x)$$

$$\int P(x)\,\mathrm{d}x = -\int \frac{1}{1+x}\mathrm{d}x = -\ln|1+x|, \quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{-\ln|1+x|} = \frac{1}{1+x}$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x = \int x(1+x)\cdot\frac{1}{1+x}\,\mathrm{d}x = \int x\,\mathrm{d}x = \tfrac{1}{2}x^2$$

$$\therefore\quad y = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = (1+x)\big(\tfrac{1}{2}x^2 + C\big)$$

Problem 2.69

Solve $\qquad (1+y)\,\mathrm{d}x + \big[x - y(1+y)^2\big]\mathrm{d}y = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{1}{1+y}\cdot x = y(1+y), \quad P(y) = \frac{1}{1+y}, \quad Q(y) = y(1+y)$$

$$\int P(y)\,\mathrm{d}y = \int \frac{1}{1+y}\,\mathrm{d}y = \ln|1+y|, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{\ln|1+y|} = 1+y$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y = \int y(1+y)^2\,\mathrm{d}y = \tfrac{1}{2}y^2 + \tfrac{2}{3}y^3 + \tfrac{1}{4}y^4$$

$$\therefore\quad x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y + C\Big] = \frac{1}{1+y}\big(\tfrac{1}{4}y^4 + \tfrac{2}{3}y^3 + \tfrac{1}{2}y^2 + C\big)$$

Problem 2.70

Consider the first-order differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \alpha(x)\,F(y) + \beta(x)\,G(y)$. If $\dfrac{G'(y)F(y) - G(y)F'(y)}{F(y)} = a = \text{constant}$, then the transformation $u = \dfrac{G(y)}{F(y)}$ reduces the differential equation to a first-order linear differential equation. Show that the general solution of the differential equation is given by

$$\frac{G(y)}{F(y)} = \exp\Big[\,a\int\beta(x)\,\mathrm{d}x\Big]\Big\{a\int\alpha(x)\,\exp\Big[-a\int\beta(x)\,\mathrm{d}x\Big]\mathrm{d}x + C\Big\}$$

$$u = \frac{G(y)}{F(y)} \implies \frac{du}{dx} = \frac{G'(y)F(y) - G(y)F'(y)}{F^2(y)} \cdot \frac{dy}{dx}$$

$$= \frac{1}{F(y)} \cdot \frac{G'(y)F(y) - G(y)F'(y)}{F(y)} \cdot \frac{dy}{dx} = \frac{a}{F(y)} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{F(y)}{a} \cdot \frac{du}{dx} \implies \frac{F(y)}{a} \cdot \frac{du}{dx} = \alpha(x) F(y) + \beta(x) G(y)$$

$$\frac{du}{dx} = a\alpha(x) + a\beta(x)\, u \qquad ✍ \; G(x) = F(x)\, u$$

$$\frac{du}{dx} - a\beta(x)\, u = a\alpha(x) \qquad ✍ \; \text{Linear first-order}$$

$$P(x) = -a\beta(x), \quad Q(x) = a\alpha(x), \quad u = e^{\int P(x)\,dx}\Big[\int Q(x)\, e^{\int P(x)\,dx} dx + C\Big]$$

$$\therefore \quad \frac{G(y)}{F(y)} = u = e^{a\int \beta(x)\,dx}\Big[a \int \alpha(x)\, e^{-a\int \beta(x)\,dx} dx + C\Big]$$

Problem 2.71

The *Riccati equation* is given by $y' = \alpha(x)y^2 + \beta(x)y + \gamma(x)$.

1. If one solution of this equation, say $y_1(x)$, is known, then the general solution can be found by using the transformation $y = y_1 + \dfrac{1}{u}$, where u is a new dependent variable. Show that u is given by
$$u = e^{-\int P(x)\,dx\,dx}\Big[\int Q(x)\, e^{\int P(x)\,dx\,dx} dx + C\Big],$$
where $P(x) = 2\alpha(x)y_1(x) + \beta(x)$ and $Q(x) = -\alpha(x)$.
2. For the differential equation $y' + y^2 = 1 + x^2$, first guess a solution $y_1(x)$ and then use the result of Part 1 to find the general solution $y(x)$.

---

1. If $y_1(x)$ is a solution, $y_1' = \alpha(x)y_1^2 + \beta(x)y_1 + \gamma(x) \implies y = y_1 + \dfrac{1}{u}$

$$\therefore \; y_1' - \frac{1}{u^2}u' = \alpha(x)\Big(y_1 + \frac{1}{u}\Big)^2 + \beta(x)\Big(y_1 + \frac{1}{u}\Big) + \gamma(x)$$

$$= \alpha(x)y_1^2 + 2\alpha(x)y_1\frac{1}{u} + \alpha(x)\frac{1}{u^2} + \beta(x)y_1 + \beta(x)\frac{1}{u} + \gamma(x)$$

$$-\frac{1}{u^2}u' = 2\alpha(x)y_1\frac{1}{u} + \alpha(x)\frac{1}{u^2} + \beta(x)\frac{1}{u}$$

$$\therefore \; u' + \big[2\alpha(x)y_1 + \beta(x)\big]\cdot u = -\alpha(x), \qquad ✍ \; \text{Linear first-order}$$

$$P(x) = 2\alpha(x)y_1(x) + \beta(x), \quad Q(x) = -\alpha(x)$$

$$u = e^{-\int P(x)\,dx}\Big[\int Q(x)\, e^{\int P(x)\,dx} dx + C\Big]$$

2. $y' + y^2 = 1 + x^2 \implies y' = -y^2 + (1+x^2)$

$y_1 = x$ is a solution, $\alpha(x) = -1$, $\beta(x) = 0$, $\gamma(x) = 1 + x^2$

$$P(x) = 2\alpha(x)\,y_1 + \beta(x) = 2(-1)x + 0 = -2x, \qquad Q(x) = -\alpha(x) = 1$$

$$\int P(x)\,\mathrm{d}x = \int -2x\,\mathrm{d}x = -x^2$$

$$u = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = \mathrm{e}^{x^2}\Big(\int \mathrm{e}^{-x^2}\,\mathrm{d}x + C\Big)$$

$$\therefore\ y = y_1 + \frac{1}{u} \implies y = x + \frac{\mathrm{e}^{-x^2}}{\int \mathrm{e}^{-x^2}\,\mathrm{d}x + C}$$

Problem 2.72

Solve $\quad 3xy' - 3xy^4 \ln x - y = 0$

$$y' - \frac{1}{3x}\cdot y = \ln x \cdot y^4$$ ✍ Bernoulli DE

Case 1. $y = 0$ is a solution.

Case 2. $y \neq 0 \implies \dfrac{1}{y^4}\,y' - \dfrac{1}{3x}\cdot\dfrac{1}{y^3} = \ln x$

Let $u = \dfrac{1}{y^3} \implies \dfrac{\mathrm{d}u}{\mathrm{d}x} = -\dfrac{3}{y^4}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x} \implies -\dfrac{1}{3}\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x} - \dfrac{1}{3x}\,u = \ln x$

$\therefore\ \dfrac{\mathrm{d}u}{\mathrm{d}x} + \dfrac{1}{x}\cdot u = -3\ln x$, $P(x) = \dfrac{1}{x}$, $Q(x) = -3\ln x$ ✍ Linear first-order

$$\int P(x)\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x = \ln x, \qquad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{\ln x} = x$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x = \int -3\ln x\cdot x\,\mathrm{d}x = -\tfrac{3}{2}\int \ln x\,\mathrm{d}(x^2)$$

$$= -\tfrac{3}{2}\Big(x^2\ln x - \int x^2\cdot\frac{1}{x}\,\mathrm{d}x\Big) = -\tfrac{3}{2}\big(x^2\ln x - \tfrac{1}{2}x^2\big)$$

$$\therefore\ u = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = \frac{1}{x}\Big[-\tfrac{3}{2}x^2\big(\ln x - \tfrac{1}{2}\big) + C\Big]$$

$$\therefore\ \frac{1}{y^3} = -\tfrac{3}{4}\,x(2\ln x - 1) + \frac{C}{x}$$

Problem 2.73

Solve $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{4x^3y^2}{x^4y + 2}$

Note that x is regarded as the independent variable and y the dependent variable.

Case 1. $y = 0$ is a solution.

Case 2. $y \neq 0 \implies \dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{x^4 y + 2}{4x^3 y}$

$$\frac{\mathrm{d}x}{\mathrm{d}y} - \frac{1}{4y}\cdot x = \frac{1}{2y^2}\cdot\frac{1}{x^3} \implies 4x^3\frac{\mathrm{d}x}{\mathrm{d}y} - \frac{1}{y}\cdot x^4 = \frac{2}{y^2}$$ ✍ Bernoulli DE

Let $u = x^4 \implies \dfrac{\mathrm{d}u}{\mathrm{d}y} = 4x^3\dfrac{\mathrm{d}x}{\mathrm{d}y} \implies \dfrac{\mathrm{d}u}{\mathrm{d}y} - \dfrac{1}{y}\cdot u = \dfrac{2}{y^2}$

$$P(y) = -\frac{1}{y},\quad Q(y) = \frac{2}{y^2},\quad \int P(x)\,\mathrm{d}y = -\int\frac{1}{y}\mathrm{d}y = -\ln|y|$$

$$\mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{-\ln|y|} = \frac{1}{y},\quad \int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int\frac{2}{y^2}\cdot\frac{1}{y}\,\mathrm{d}y = -\frac{1}{y^2}$$

$$\therefore\ u = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = y\Big(-\frac{1}{y^2} + C\Big)$$

$$\therefore\ x^4 = -\frac{1}{y} + Cy$$

Problem 2.74

Solve $\quad y\,(6y^2 - x - 1)\,\mathrm{d}x + 2x\,\mathrm{d}y = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{x+1}{2x}\cdot y = -\frac{3}{x}\cdot y^3$$ ✍ Bernoulli DE

Case 1. $y = 0$ is a solution.

Case 2. $y \neq 0 \implies \dfrac{1}{y^3}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{x+1}{2x}\cdot\dfrac{1}{y^2} = -\dfrac{3}{x}$

Let $u = \dfrac{1}{y^2},\quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = -\dfrac{2}{y^3}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$\therefore\ \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{x+1}{x}\cdot u = \frac{6}{x},\quad P(x) = \frac{x+1}{x},\quad Q(x) = \frac{6}{x}$$ ✍ Linear first-order

$$\int P(x)\,\mathrm{d}x = \int\Big(1+\frac{1}{x}\Big)\mathrm{d}x = x + \ln|x|,\quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{x+\ln|x|} = x\,\mathrm{e}^x$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\mathrm{d}x = \int\frac{6}{x}\cdot x\,\mathrm{e}^x\,\mathrm{d}x = \int 6\mathrm{e}^x\,\mathrm{d}x = 6\mathrm{e}^x$$

$$\therefore\ u = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = \frac{\mathrm{e}^{-x}}{x}\,(6\mathrm{e}^x + C)$$

$$\frac{1}{y^2} = \frac{1}{x}\,(6 + C\,\mathrm{e}^{-x})$$

Problem 2.75

Solve $\quad (1+x)\,(y'+y^2)-y=0$

$$y'-\frac{1}{1+x}\,y=-y^2 \qquad \text{✍ Bernoulli DE}$$

Case 1. $\quad y=0$ is a solution.

Case 2. $\quad y\neq 0 \implies \dfrac{1}{y^2}y'-\dfrac{1}{1+x}\cdot\dfrac{1}{y}=-1$ ✍ Bernoulli DE

Let $u=\dfrac{1}{y} \implies \dfrac{\mathrm{d}u}{\mathrm{d}x}=-\dfrac{1}{y^2}\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{1+x}\cdot u=1,\quad P(x)=\frac{1}{1+x},\quad Q(x)=1 \qquad \text{✍ Linear first-order}$$

$$\int P(x)\,\mathrm{d}x=\int\frac{1}{1+x}\mathrm{d}x=\ln\left|1+x\right|,\quad \mathrm{e}^{\int P(x)\,\mathrm{d}x}=\mathrm{e}^{\ln|1+x|}=1+x$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x=\int 1\cdot(1+x)\,\mathrm{d}x=x+\tfrac{1}{2}x^2$$

$$\therefore\ u=\mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x+C\Big]=\frac{1}{1+x}\big(\tfrac{1}{2}x^2+x+C\big)=\frac{1}{y}$$

Problem 2.76

Solve $\quad xyy'+y^2-\sin x=0$

$$y'+\frac{1}{x}\cdot y=\frac{\sin x}{x}\cdot\frac{1}{y} \implies yy'+\frac{1}{x}y^2=\frac{\sin x}{x} \qquad \text{✍ Bernoulli DE}$$

Let $\ y^2=u \implies 2yy'=u'$

$$u'+\frac{2}{x}\cdot u=\frac{2\sin x}{x},\quad P(x)=\frac{2}{x},\ Q(x)=\frac{2\sin x}{x} \qquad \text{✍ Linear first-order}$$

$$\int P(x)\,\mathrm{d}x=\int\frac{2}{x}\,\mathrm{d}x=2\ln\left|x\right|,\quad \mathrm{e}^{\int P(x)\,\mathrm{d}x}=\mathrm{e}^{2\ln|x|}=x^2$$

$$u=\mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x+C\Big]=\frac{1}{x^2}\Big(\int\frac{2\sin x}{x}x^2\,\mathrm{d}x+C\Big)$$

$$=\frac{1}{x^2}\Big(\int 2x\,\sin x\mathrm{d}x+C\Big)=\frac{1}{x^2}\big[2\,(\sin x-x\cos x)+C\big]$$

$\therefore\ x^2y^2=2\sin x-2x\cos x+C$

Problem 2.77

Solve $\quad (2x^3-y^4)\,\mathrm{d}x+xy^3\,\mathrm{d}y=0$

$$xy^3y'+2x^3-y^4=0 \implies y^3y'-\frac{1}{x}y^4=-2x^3 \qquad \text{✍ Bernoulli DE}$$

Let $u = y^4 \implies \dfrac{\mathrm{d}u}{\mathrm{d}x} = 4y^3 \dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$\therefore \ \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4}{x} \cdot u = -8x^2, \ \ P(x) = -\frac{4}{x}, \ \ Q(x) = -8x^2 \quad \text{✍ Linear first-order}$$

$$\int P(x)\,\mathrm{d}x = \int -\frac{4}{x}\,\mathrm{d}x = -4\ln|x|, \quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{-4\ln|x|} = \frac{1}{x^4}$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x = \int -8x^2 \cdot \frac{1}{x^4}\,\mathrm{d}x = -8\int \frac{1}{x^2}\,\mathrm{d}x = \frac{8}{x}$$

$$\therefore \ u = \mathrm{e}^{\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = x^4\Big(\frac{8}{x} + C\Big)$$

$$\therefore \ y^4 = 8x^3 + Cx^4$$

Problem 2.78

Solve $\quad y' - y\tan x + y^2 \cos x = 0$

$$y' - \tan x \cdot y = -\cos x \cdot y^2 \qquad \text{✍ Bernoulli DE}$$

Case 1. $\quad y = 0$ is a solution.

Case 2. $\quad y \neq 0 \implies \dfrac{1}{y^2}y' - \tan x \cdot \dfrac{1}{y} = -\cos x$

Let $u = \dfrac{1}{y} \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{y^2} \cdot \dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$\therefore \ \frac{\mathrm{d}u}{\mathrm{d}x} + \tan x \cdot u = \cos x, \ \ P(x) = \tan x, \ Q(x) = \cos x \quad \text{✍ Linear first-order}$$

$$\int P(x)\,\mathrm{d}x = \int \tan x\,\mathrm{d}x = -\ln\big|\cos x\big|, \quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{-\ln|\cos x|} = \frac{1}{\cos x}$$

$$u = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = \cos x\Big(\int \cos x \cdot \frac{1}{\cos x}\,\mathrm{d}x + C\Big)$$

$$\therefore \ \frac{1}{y} = \cos x\,(x + C)$$

Problem 2.79

Solve $\quad 6y^2\,\mathrm{d}x - x(2x^3 + y)\,\mathrm{d}y = 0$

Case 1. $\quad y = 0$ is a solution.

Case 2. $\quad y \neq 0 \implies \dfrac{\mathrm{d}x}{\mathrm{d}y} - \dfrac{1}{6y} \cdot x = \dfrac{1}{3y^2} \cdot x^4 \qquad$ ✍ Bernoulli DE

$$x^{-4}\frac{\mathrm{d}x}{\mathrm{d}y} - \frac{1}{6y}x^{-3} = \frac{1}{3y^2}. \qquad \text{Let } u = x^{-3} \implies \frac{\mathrm{d}u}{\mathrm{d}y} = -3x^{-4}\frac{\mathrm{d}x}{\mathrm{d}y}$$

$$\therefore \ \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{1}{2y} \cdot u = -\frac{1}{y^2}, \ \ P(x) = \frac{1}{2y}, \ \ Q(x) = -\frac{1}{y^2} \quad \text{✍ Linear first-order}$$

$$\int P(y)\,\mathrm{d}y = \int \frac{1}{2y}\,\mathrm{d}y = \tfrac{1}{2}\ln|y|, \qquad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{\frac{1}{2}\ln|y|} = \sqrt{y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y = \int -\frac{1}{y^2}\cdot y^{\frac{1}{2}}\,\mathrm{d}y = -\int y^{-\frac{3}{2}}\,\mathrm{d}y = 2y^{-\frac{1}{2}}$$

$$\therefore\ \ u = \frac{1}{x^3} = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y + C\Big] = \frac{1}{\sqrt{y}}\Big(\frac{2}{\sqrt{y}} + C\Big)$$

$$\therefore\ \ \frac{y}{x^3} = 2 + C\sqrt{y} \implies y - 2x^3 = Cx^3\sqrt{y} \implies (y-2x^3)^2 = Cx^6 y$$

Problem 2.80

Solve $\quad x\,y'^3 - y\,y'^2 + 1 = 0$

$$\therefore\ \ y = xy' + \frac{1}{(y')^2} \qquad \text{✍ Clairaut's DE}$$

Let $y' = p, \quad y = xp + \dfrac{1}{p^2}$

Differentiate with respect to x: $\quad p = p + x\dfrac{\mathrm{d}p}{\mathrm{d}x} + \dfrac{\mathrm{d}}{\mathrm{d}x}\Big(\dfrac{1}{p^2}\Big)$

$$x\,\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}p}\Big(\frac{1}{p^2}\Big)\frac{\mathrm{d}p}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}p}{\mathrm{d}x}\Big(x - \frac{2}{p^3}\Big) = 0$$

Case 1. $\quad \dfrac{\mathrm{d}p}{\mathrm{d}x} = 0 \implies p = C \implies y = Cx + \dfrac{1}{C^2}$ ✍ General solution

Case 2. $\quad x = \dfrac{2}{p^3}, \quad y = xp + \dfrac{1}{p^2}$ ✍ Singular solution

or $\quad p = \Big(\dfrac{2}{x}\Big)^{\frac{1}{3}} \implies y = x\Big(\dfrac{2}{x}\Big)^{\frac{1}{3}} + \Big(\dfrac{x}{2}\Big)^{\frac{2}{3}} = 2^{\frac{1}{3}}x^{\frac{2}{3}} + \Big(\dfrac{x}{2}\Big)^{\frac{2}{3}} = \tfrac{3}{2}\cdot 2^{\frac{1}{3}}\cdot x^{\frac{2}{3}}$

$$y^3 = \tfrac{27}{8}\cdot 2\cdot x^2 \implies 4y^3 = 27x^2$$

Problem 2.81

Solve $\quad y = xy' + y'^3$

Let $\quad y' = p \implies y = xp + p^3$ ✍ Clairaut's DE

Differentiate with respect to x: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = p + x\dfrac{\mathrm{d}p}{\mathrm{d}x} + 3p^2\dfrac{\mathrm{d}p}{\mathrm{d}x} \implies (x+3p^2)\,\dfrac{\mathrm{d}p}{\mathrm{d}x} = 0$

Case 1. $\quad \dfrac{\mathrm{d}p}{\mathrm{d}x} = 0 \implies p = C \implies y = Cx + C^3$ ✍ General solution

Case 2. $\quad x + 3p^2 = 0 \implies x = -3p^2$

$$y = xp + p^3 \implies y = (-3p^2)p + p^3 = -2p^3$$

$$x^3 = -27p^6, \quad y^2 = 4p^6 \implies \frac{x^3}{y^2} = \frac{-27p^6}{4p^6}$$

$$\therefore \; 4x^3 = -27y^2 \qquad \text{✍ Singular solution}$$

Problem 2.82

Solve $\quad x(y'^2 - 1) = 2y'$

Let $y' = p \implies x(p^2-1) = 2p$

Since $p^2-1=0$ is not a solution, $p^2 \neq 1 \implies x = \dfrac{2p}{p^2-1} = g(p)$

Differentiate with respect to y : $\quad \dfrac{dx}{dy} = g_y + g_p \dfrac{dp}{dy}$

$$\therefore \; \frac{1}{p} = \frac{2\left[1\cdot(p^2-1) - p\cdot 2p\right]}{(p^2-1)^2}\cdot\frac{dp}{dy} = -2\,\frac{p^2+1}{(p^2-1)^2}\cdot\frac{dp}{dy}$$

$$\therefore \; y = \int -2p\frac{p^2+1}{(p^2-1)^2}\,dp + C = -\int \frac{p^2+1}{(p^2-1)^2}\,d(p^2) + C, \quad p^2 = u$$

$$= -\int \frac{u+1}{(u-1)^2}\,du + C = -\int \frac{(u-1)+2}{(u-1)^2}\,du + C$$

$$= -\int \left[\frac{1}{u-1} + \frac{2}{(u-1)^2}\right]du + C = -\left[\ln|u-1| - \frac{2}{u-1}\right] + C$$

$$\therefore \; x = \frac{2p}{p^2-1}, \quad y = -\ln|p^2-1| + \frac{2}{p^2-1} + C$$

Problem 2.83

Solve $\quad xy'(y'+2) = y$

Let $p = y' \implies xp(p+2) = y = f(x,p)$

Differentiate with respect to x:

$$\frac{dy}{dx} = f_x + f_p\frac{dp}{dx} \implies p = p(p+2) + x(2p+2)\frac{dp}{dx}$$

$$2x(p+1)\frac{dp}{dx} + p(p+1) = 0 \implies (p+1)\left(2x\frac{dp}{dx} + p\right) = 0$$

Case 1. $\quad p+1=0 \implies p=-1 \implies y = xp(p+2) = -x$

Case 2. $\quad 2x\dfrac{dp}{dx} = -p \qquad$ ✍ Variable separable

$$\therefore \; \int \frac{2}{p}\,dp = -\int \frac{1}{x}\,dx + C \implies 2\ln|p| = -\ln|x| + 2\ln|C|$$

$$p^2 = \frac{C^2}{x} \implies p = \pm\frac{C}{\sqrt{x}}$$

$$\therefore\ y = x(p^2+2p) = x\left(\frac{C^2}{x} \pm \frac{2C}{\sqrt{x}}\right) = \pm 2C\sqrt{x} + C^2$$

✍ Since C is an arbitrary constant, the general solution can be written as

$$y = 2C\sqrt{x} + C^2$$

Problem 2.84

Solve $\quad x = y'\sqrt{y'^2+1}$

Let $y' = p \implies x = p\sqrt{p^2+1} = g(y, p)$

Differentiate with respect to y: $\quad \dfrac{\mathrm{d}p}{\mathrm{d}y} = g_y + g_p\dfrac{\mathrm{d}p}{\mathrm{d}y}$

$$\frac{1}{p} = \left(\sqrt{p^2+1} + p\cdot\frac{p}{\sqrt{p^2+1}}\right)\frac{\mathrm{d}p}{\mathrm{d}y}$$ ✍ Variable separable

$$\mathrm{d}y = \left(\sqrt{p^2+1} + \frac{p^2}{\sqrt{p^2+1}}\right)p\,\mathrm{d}p$$

$$\therefore\ y = \tfrac{1}{2}\int\left[\sqrt{p^2+1} + \frac{(p^2+1)-1}{\sqrt{p^2+1}}\right]\mathrm{d}(p^2+1) + C$$

$$= \tfrac{1}{2}\int\left(2\sqrt{p^2+1} - \frac{1}{\sqrt{p^2+1}}\right)\mathrm{d}(p^2+1) + C$$

$$= \tfrac{1}{2}\left[2\,\frac{(p^2+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(p^2+1)^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = \tfrac{2}{3}(p^2+1)^{\frac{3}{2}} - (p^2+1)^{\frac{1}{2}} + C$$

$$\therefore\ x = p\sqrt{p^2+1}, \quad y = \tfrac{1}{3}\sqrt{p^2+1}\,(2p^2-1) + C$$

Problem 2.85

Solve $\quad 2y'^2(y - xy') = 1$

Let $y' = p \implies 2p^2(y - xp) = 1$

$\because\ p \neq 0 \implies y = xp + \dfrac{1}{2p^2} = f(x, p)$

Differentiate with respect to x: $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = f_x + f_p\dfrac{\mathrm{d}p}{\mathrm{d}x}$

$$\therefore\ p = p + \left(x - \frac{1}{p^3}\right)\frac{\mathrm{d}p}{\mathrm{d}x} \implies \left(x - \frac{1}{p^3}\right)\frac{\mathrm{d}p}{\mathrm{d}x} = 0$$

Case 1. $x = \dfrac{1}{p^3} \implies y = xp + \dfrac{1}{2p^2} = \dfrac{1}{p^3}\cdot p + \dfrac{1}{2p^2} = \dfrac{3}{2p^2}$

$$x^2 = \frac{1}{p^6}, \quad y^3 = \frac{27}{8p^6} \implies \frac{1}{p^6} = \frac{8}{27}y^3 = x^2$$

$$\therefore\ 8y^3 = 27x^2 \qquad \text{✍ Singular solution}$$

Case 2. $\dfrac{dp}{dx} = 0 \implies p = C$

$$\therefore\ y = xp + \frac{1}{2p^2} = Cx + \frac{1}{2C^2} \qquad \text{✍ General solution}$$

Problem 2.86

Solve $\quad y = 2xy' + y^2y'^3$

Let $y' = p \implies y = 2xp + y^2p^3$

Case 1. $p = 0 \implies y = 0$

Case 2. $p \neq 0 \implies x = \dfrac{y - y^2p^3}{2p} = \dfrac{y}{2p} - \dfrac{y^2p^2}{2} = g(y, p)$

Differentiate with respect to $y \implies \dfrac{dx}{dy} = g_y + g_p\dfrac{dp}{dy}$

$$\frac{1}{p} = \frac{1}{2p} - yp^2 + \left(-\frac{y}{2p^2} - y^2p\right)\frac{dp}{dy}$$

Multiply by $2p^2$: $\quad \dfrac{1}{2p}\cdot 2p^2 + yp^2\cdot 2p^2 = -(y + 2y^2p^3)\dfrac{dp}{dy}$

$$p(1+2yp^3) + y(1+2yp^3)\frac{dp}{dy} = 0 \implies (1+2yp^3)\left(p + y\frac{dp}{dy}\right) = 0$$

Case 2.1 $\quad 1 + 2yp^3 = 0 \implies p^3 = -\dfrac{1}{2y} \implies p = -\left(\dfrac{1}{2y}\right)^{\frac{1}{3}}$

$$\therefore\ y = 2xp + y^2p^3 = 2x\left[-\left(\frac{1}{2y}\right)^{\frac{1}{3}}\right] + y^2\left(-\frac{1}{2y}\right)$$

$$\frac{3}{2}y = -2x\left(\frac{1}{2y}\right)^{\frac{1}{3}} \implies \frac{27y^3}{8} = -8x^3\cdot\frac{1}{2y} \implies 27y^4 + 32x^3 = 0$$

Case 2.2 $\quad p + y\dfrac{dp}{dy} = 0 \implies \displaystyle\int\frac{dy}{y}\,dy = -\int\frac{dp}{p}\,dp + C$

$$\therefore\ \ln|y| = -\ln|p| + C \implies y = \frac{C}{p} \quad\text{or}\quad p = \frac{C}{y}$$

$$y = 2xp + y^2p^3 = 2x\cdot\frac{C}{y} + y^2\left(\frac{C}{y}\right)^3 \implies y^2 = 2Cx + C^3$$

Problem 2.87

Solve $\quad y'^3 + y^2 = xyy'$

Let $\; y' = p \implies p^3 + y^2 = xyp$

Case 1. $\; p = 0 \implies y = 0$

Case 2. $\; p \neq 0$ and $y \neq 0 \implies x = \dfrac{p^3 + y^2}{yp} = \dfrac{p^2}{y} + \dfrac{y}{p} = g(y, p)$

Differentiate with respect to y: $\quad \dfrac{dx}{dy} = g_y + g_p \dfrac{dp}{dy}$

$$\frac{1}{p} = \left(-\frac{p^2}{y^2} + \frac{1}{p}\right) + \left(\frac{2p}{y} - \frac{y}{p^2}\right)\frac{dp}{dy}$$

Multiply y^2p^2: $\quad -p^4 + (2yp^3 - y^3)\dfrac{dp}{dy} = 0$

$$\therefore\; (2yp^3 - y^3)\,dp - p^4 dy = 0, \quad M = 2yp^3 - y^3, \quad N = -p^4$$

$$\frac{\partial M}{\partial y} = 2p^3 - 3y^2, \qquad \frac{\partial N}{\partial p} = -4p^3$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial p} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(2p^3 - y^2)}(-4p^3 - 2p^3 + 3y^2) = -\frac{3}{y}$$

$$\mu(y) = \exp\left(\int -\frac{3}{y}\,dy\right) = e^{-3\ln|y|} = \frac{1}{y^3}$$

Multiply the DE by $\mu(y) \implies \left(2\dfrac{p^3}{y^2} - 1\right)dp - \dfrac{p^4}{y^3}dy = 0$

$$2\frac{p^3}{y^2}dp \;-\; \frac{p^4}{y^3}dy \;-\; 1\cdot dp = 0 \implies \frac{p^4}{2y^2} - p = C$$

$$\int dp \qquad \boxed{\frac{p^4}{2y^2}} \qquad \frac{\partial}{\partial y} \qquad \int dp \qquad \boxed{-p}$$

General solution: $\quad p^3 + y^2 = xyp, \quad \dfrac{p^4}{2y^2} - p = C$

Problem 2.88

Solve $\quad 2xy' - y = y'\ln(yy')$

Let $\; y' = p \implies 2xp - y = p\ln(yp)$

$\because\; p \neq 0 \implies x = \dfrac{p\ln(yp) + y}{2p} = \frac{1}{2}\ln(yp) + \dfrac{y}{2p} = g(y, p)$

Differentiate with respect to y : $\dfrac{\mathrm{d}x}{\mathrm{d}y} = g_y + g_p \cdot \dfrac{\mathrm{d}p}{\mathrm{d}y}$

$$\therefore\ \frac{1}{p} = \frac{1}{2}\frac{1}{yp}\cdot p + \frac{1}{2p} + \left(\frac{1}{2}\frac{1}{yp}\cdot y - \frac{y}{2p^2}\right)\frac{\mathrm{d}p}{\mathrm{d}y}$$

$$\frac{1}{p} = \frac{1}{y} + \left(\frac{1}{p} - \frac{y}{p^2}\right)\frac{\mathrm{d}p}{\mathrm{d}y} \implies (p-y)\left(\frac{1}{y} + \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}y}\right) = 0$$

Case 1. $p = y \implies x = \dfrac{p\ln(yp)+y}{2p} = \dfrac{y\ln(y^2)+y}{2y} \implies 2x = 2\ln|y|+1$

Case 2. $\dfrac{1}{y} + \dfrac{1}{p}\cdot\dfrac{\mathrm{d}p}{\mathrm{d}y} = 0$ ✍ Variable separable

$$\int \frac{1}{p}\,\mathrm{d}p = -\int \frac{1}{y}\,\mathrm{d}y + C \implies \ln|p| = -\ln|y| + \ln|C| \implies p = \frac{C}{y},\ C>0$$

$$\therefore\ x = \tfrac{1}{2}\ln\left(y\cdot\tfrac{C}{y}\right) + \frac{y}{2\cdot\frac{C}{y}} = \tfrac{1}{2}\ln C + \frac{y^2}{2C} \implies y^2 = 2Cx - C\ln C.$$

Problem 2.89

Solve $\quad y = xy' - x^2y'^3$

Let $y' = p \implies y = xp - x^2p^3 = f(x,p)$

Differentiate with respect to x : $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f_x + f_p \cdot \dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$p = p - 2xp^3 + (x - 3x^2p^2)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

Divide by x: $\quad -2p^3\,\mathrm{d}x + (1-3xp^2)\,\mathrm{d}p = 0$

$$M = -2p^3,\quad N = 1-3xp^2,\quad \frac{\partial M}{\partial p} = -6p^2,\quad \frac{\partial N}{\partial x} = -3p^2$$

Case 1. $p = 0 \implies y = 0$

Case 2. $p \neq 0$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial p}\right) = \frac{1}{-2p^3}(-3p^2+6p^2) = -\frac{3}{2p}$$ ✍ A function of p only

$$\mu(p) = \exp\left(\int -\frac{3}{2p}\,\mathrm{d}p\right) = \mathrm{e}^{-\frac{3}{2}\ln|p|} = |p|^{-\frac{3}{2}},$$

Multiply the DE by $\mu(p)$: $\quad -2p^{\frac{3}{2}}\,\mathrm{d}x + \left(p^{-\frac{3}{2}} - 3xp^{\frac{1}{2}}\right)\mathrm{d}p = 0$

$$-2p^{\frac{3}{2}}\,\mathrm{d}x \quad - \quad 3p^{\frac{1}{2}}x\,\mathrm{d}p \quad + \quad p^{-\frac{3}{2}}\,\mathrm{d}p = 0$$

$$\int \mathrm{d}x \quad \boxed{-2p^{\frac{3}{2}}x} \quad \frac{\partial}{\partial p} \qquad\qquad \int \mathrm{d}p \downarrow \quad \boxed{-2p^{-\frac{1}{2}}}$$

$$\therefore\quad -2p^{\frac{3}{2}}x - 2p^{-\frac{1}{2}} = C \implies -2p^{-\frac{1}{2}}(p^2x+1) = C$$

General solution: $\quad xp^2 = C\sqrt{|p|} - 1, \quad y = xp - x^2p^3$

Problem 2.90

Solve $\quad y(y-2xy')^3 = y'^2$

Let $y'=p \implies y(y-2xp)^3 = p^2$

Case 1. $\quad p=0 \implies y=0$

Case 2. $\quad p\neq 0 \ (y\neq 0)$: $\; x = \dfrac{y - p^{\frac{2}{3}}y^{-\frac{1}{3}}}{2p} = \dfrac{y}{2p} - \dfrac{1}{2}p^{-\frac{1}{3}}y^{-\frac{1}{3}} = g(y,p)$

Differentiate with respect to y: $\quad \dfrac{\mathrm{d}x}{\mathrm{d}y} = g_y + g_p\,\dfrac{\mathrm{d}p}{\mathrm{d}y}$

$$\frac{1}{p} = \left(\frac{1}{2p} + \frac{1}{6}p^{-\frac{1}{3}}y^{-\frac{4}{3}}\right) + \left(-\frac{y}{2p^2} + \frac{1}{6}p^{-\frac{4}{3}}y^{-\frac{1}{3}}\right)\cdot\frac{\mathrm{d}p}{\mathrm{d}y}$$

$$\frac{1}{2p} = \frac{1}{6}p^{-\frac{1}{3}}y^{-\frac{4}{3}} + \left(-\frac{y}{2p^2} + \frac{1}{6}p^{-\frac{4}{3}}y^{-\frac{1}{3}}\right)\cdot\frac{\mathrm{d}p}{\mathrm{d}y}$$

$$\frac{p}{2}\left(\frac{1}{p^2} - \frac{1}{3}p^{-\frac{4}{3}}y^{-\frac{4}{3}}\right) + \frac{y}{2}\left(\frac{1}{p^2} - \frac{1}{3}p^{-\frac{4}{3}}y^{-\frac{4}{3}}\right)\frac{\mathrm{d}p}{\mathrm{d}y} = 0$$

$$\frac{1}{2}\left(\frac{1}{p^2} - \frac{1}{3}p^{-\frac{4}{3}}y^{-\frac{4}{3}}\right)\left(p + y\,\frac{\mathrm{d}p}{\mathrm{d}y}\right) = 0$$

Case 2.1 $\quad \dfrac{1}{p^2} - \dfrac{1}{3}p^{-\frac{4}{3}}y^{-\frac{4}{3}} = 0 \implies 3y^{\frac{4}{3}} = p^{\frac{2}{3}} \implies p = \pm 3^{\frac{3}{2}}y^2$

$$\therefore\; 2x = \frac{y}{p} - p^{-\frac{1}{3}}y^{-\frac{1}{3}} = \frac{y}{\pm 3^{\frac{3}{2}}y^2} - \left(\pm 3^{-\frac{1}{2}}y^{-\frac{2}{3}}\right)\cdot y^{-\frac{1}{3}}$$

$$= \pm\left(\frac{1}{3\sqrt{3}y} - \frac{1}{\sqrt{3}y}\right) = \pm\left(-\frac{2}{3\sqrt{3}y}\right)$$

Square both sides: $\quad x^2 = \dfrac{1}{27y^2} \implies 27x^2y^2 = 1$

Case 2.2 $\quad p + y\dfrac{\mathrm{d}p}{\mathrm{d}y} = 0 \qquad$ ✍ Varibale separable

$$\int\frac{1}{p}\,\mathrm{d}p = -\int\frac{1}{y}\,\mathrm{d}y + C \implies \ln|p| = -\ln|y| + \ln|C| \implies p = \frac{C}{y}$$

$$x = \frac{y}{2p} - \frac{1}{2}p^{-\frac{1}{3}}y^{-\frac{1}{3}} = \frac{y}{2\cdot\dfrac{C}{y}} - \frac{1}{2}\left(\frac{C}{y}\right)^{-\frac{1}{3}}y^{-\frac{1}{3}} = \frac{y^2}{2C} - \frac{1}{2C^{\frac{1}{3}}}$$

$$\therefore\ x = \frac{y^2}{2C^3} - \frac{1}{2C} \implies y^2 = 2C^3x + C^2$$

Since $y=0$ is a special case when $C=0$, the solutions are

$$y^2 = 2C^3x + C^2; \qquad 27x^2y^2 = 1$$

Problem 2.91

Solve $\quad y + xy' = 4\sqrt{y'}$

Let $y'=p \implies y = -xp + 4\sqrt{p} = f(x,p), \quad p>0$

Differentiate with respect to x: $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = f_x + f_p\dfrac{\mathrm{d}y}{\mathrm{d}p}$

$$\therefore\ p = -p + \Big(-x + \frac{2}{\sqrt{p}}\Big)\frac{\mathrm{d}p}{\mathrm{d}x} \implies 2p\frac{\mathrm{d}x}{\mathrm{d}p} = -x + \frac{2}{\sqrt{p}}$$

Case 1. $p=0 \implies y=0$

Case 2. $\dfrac{\mathrm{d}x}{\mathrm{d}p} + \dfrac{1}{2p}\cdot x = p^{-\frac{3}{2}},\ P(p) = \dfrac{1}{2p},\ Q(p) = p^{-\frac{3}{2}}$ Linear first-order

$$\int P(p)\,\mathrm{d}p = \int \frac{1}{2p}\,\mathrm{d}p = \frac{1}{2}\ln p, \quad \mathrm{e}^{\int P(p)\,\mathrm{d}p} = \mathrm{e}^{\frac{1}{2}\ln p} = \sqrt{p}$$

$$\int Q(p)\,\mathrm{e}^{\int P(p)\,\mathrm{d}p}\mathrm{d}p = \int p^{-\frac{3}{2}}\cdot p^{\frac{1}{2}}\,\mathrm{d}p = \int p^{-1}\mathrm{d}p = \ln p$$

$$\therefore\ x = \mathrm{e}^{-\int P(p)\,\mathrm{d}p}\Big[\int Q(p)\,\mathrm{e}^{\int P(p)\,\mathrm{d}p}\mathrm{d}p + C\Big] = \frac{1}{\sqrt{p}}(\ln p + C)$$

$$\therefore\ x = \frac{1}{\sqrt{p}}(\ln p + C), \quad y = -xp + 4\sqrt{p} = \sqrt{p}(4 - \ln p - C)$$

Problem 2.92

Solve $\quad 2xy' - y = \ln y', \quad y'>0$

Let $y'=p \implies y = 2xp - \ln p = f(x,p)$

Differentiate with respect to x: $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial p}\cdot\dfrac{\mathrm{d}p}{\mathrm{d}x}$

$$\therefore\ p = 2p + \Big(2x - \frac{1}{p}\Big)\frac{\mathrm{d}p}{\mathrm{d}x} \implies p\frac{\mathrm{d}x}{\mathrm{d}p} + 2x - \frac{1}{p} = 0$$

$p\neq 0 \implies \dfrac{\mathrm{d}x}{\mathrm{d}p} + \dfrac{2}{p}\cdot x = \dfrac{1}{p^2},\ P(p) = \dfrac{2}{p},\ Q(p) = \dfrac{1}{p^2}$ Linear first-order

$$\int p(p)\,\mathrm{d}p = \int \frac{2}{p}\,\mathrm{d}p = 2\ln p, \quad \mathrm{e}^{\int P(p)\,\mathrm{d}p} = \mathrm{e}^{2\ln p} = p^2$$

$$\int Q(p)\, e^{\int P(p)\,dp} dp = \int \frac{1}{p^2} \cdot p^2 dp = p$$

$$\therefore\; x = e^{-\int P(p)\,dp}\Big[\int Q(p)\, e^{\int P(p)\,dp} dp + C\Big] = \frac{1}{p^2}(p+C) = \frac{1}{p} + \frac{C}{p^2}$$

$$\therefore\; y = 2xp - \ln p = 2\Big(\frac{1}{p} + \frac{C}{p^2}\Big)p - \ln p = 2\Big(1 + \frac{C}{p}\Big) - \ln p$$

General solution: $\quad x = \dfrac{1}{p} + \dfrac{C}{p^2}, \quad y = 2\Big(1 + \dfrac{C}{p}\Big) - \ln p$

Problem 2.93

Solve $\qquad y'' = 2yy'^3$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{dy}{dx} \implies y'' = \dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx} = u \cdot \dfrac{du}{dy}$

$$\therefore\; u \cdot \frac{du}{dy} = 2yu^3 \implies u\Big(\frac{du}{dy} - 2yu^2\Big) = 0$$

Case 1. $\quad u = 0: \quad \dfrac{dy}{dx} = 0 \implies y = C$

Case 2. $\quad \dfrac{du}{dy} - 2yu^2 = 0 \qquad$ ✍ Variable separable

$$\int \frac{1}{u^2}\, du = \int 2y\, dy + C \implies -\frac{1}{u} = y^2 + C \implies u = -\frac{1}{y^2 + C}$$

$$\therefore\; \frac{dy}{dx} = -\frac{1}{y^2 + C}$$ ✍ Variable separable

$$\int (y^2 + C_1)\, dy = -\int dx + C_2 \implies \tfrac{1}{3}y^3 + C_1 y = -x + C_2$$

$$\therefore\; x + \tfrac{1}{3}y^3 + C_1 y = C_2 \qquad \text{or} \qquad 3x + y^3 + C_1 y = C_2$$

Problem 2.94

Solve $\qquad yy'' = y'^2 - y'^3$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{dy}{dx} \implies y'' = \dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx} = u \cdot \dfrac{du}{dy}$

$$\therefore\; y \cdot u \frac{du}{dy} = u^2 - u^3$$ ✍ Variable separable

Case 1. $u=0 \implies \dfrac{dy}{dx}=0 \implies y=C$

Case 2. $u\neq 0 \implies \displaystyle\int \frac{du}{u-u^2}=\int\frac{dy}{y}+C_1$

$$\int\Big(\frac{1}{u}+\frac{1}{1-u}\Big)du=\ln|y|+C_1 \implies \ln|u|-\ln|1-u|=\ln|y|+C_1$$

$$\ln\Big|\frac{u}{1-u}\Big|=\ln|C_1y| \implies \frac{u}{1-u}=C_1y \implies u=\frac{C_1y}{1+C_1y}$$

$$\frac{dy}{dx}=\frac{C_1y}{1+C_1y}$$ ✍ Variable separable

$$\int\Big(\frac{1}{C_1y}+1\Big)dy=\int dx+C_2 \implies \frac{1}{C_1}\ln|y|+y=x+C_2$$

General solutions: $y=C,\quad \dfrac{1}{C_1}\ln|y|+y=x+C_2$

Problem 2.95

Solve $\quad xy'''=(1-x)\,y''$

The DE is of the type y' absent. Let $y''=u,\ \ y'''=u'$.

$$x\frac{du}{dx}=(1-x)\,u \implies \int\frac{1}{u}du=\int\Big(\frac{1}{x}-1\Big)dx+C$$ ✍ Variable separable

$$\ln|u|=\ln|x|-x+C \implies \ln|u|=\ln|C_1xe^{-x}| \implies u=C_1xe^{-x}$$

$$\frac{d^2y}{dx^2}=C_1xe^{-x}$$ ✍ Immediately integrable

$$\frac{dy}{dx}=C_1\int xe^{-x}dx+C_2=-C_1e^{-x}(x+1)+C_2$$

$$\begin{aligned}y&=-C_1\Big(\int xe^{-x}dx+\int e^{-x}dx\Big)+C_2x+C_3\\&=-C_1\big[-e^{-x}(x+1)-e^{-x}\big]+C_2x+C_3\end{aligned}$$

$$\therefore\ y=C_1(x+2)e^{-x}+C_2x+C_3$$

Problem 2.96

Solve $\quad y''=e^x y'^2$

The DE is of the type y absent. Let $u=y',\ \ u'=y''$.

$$\frac{du}{dx}=e^xu^2$$ ✍ Variable separable

Case 1. $u=0 \implies y'=0 \implies y=C$ is a solution

Case 2. $u \neq 0 \implies \displaystyle\int \frac{1}{u^2}\,\mathrm{d}u = \int \mathrm{e}^x\,\mathrm{d}x + C_1 \implies -\frac{1}{u} = \mathrm{e}^x + C_1$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = u = -\frac{1}{\mathrm{e}^x + C_1}$$ ✍ Variable separable

$$y = -\int \frac{\mathrm{e}^{-x}}{1 + C_1\,\mathrm{e}^{-x}}\,\mathrm{d}x + C_2 = \frac{1}{C_1}\int \frac{1}{1+C_1\,\mathrm{e}^{-x}}\,\mathrm{d}(1+C_1\mathrm{e}^{-x}) + C_2$$

$$= \frac{1}{C_1}\ln\left|1+C_1\,\mathrm{e}^{-x}\right| + C_2$$

Problem 2.97

Solve $\quad yy'' + y'^2 = 0$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} \implies y'' = \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x} = u\cdot\dfrac{\mathrm{d}u}{\mathrm{d}y}$

$$yu\frac{\mathrm{d}u}{\mathrm{d}y} + u^2 = 0 \implies u\Big(y\frac{\mathrm{d}u}{\mathrm{d}y} + u\Big) = 0$$

Case 1. $u = 0 \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies y = C$

Case 2. $y\dfrac{\mathrm{d}u}{\mathrm{d}y} = -u$ ✍ Variable separable

$$\int \frac{1}{u}\,\mathrm{d}u = \int -\frac{1}{y}\,\mathrm{d}y + C \implies \ln|u| = -\ln|y| + \ln|C| \implies u = \frac{C_1}{y}$$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{C_1}{y}$$ ✍ Variable separable

$$\int y\,\mathrm{d}y = \int C_1\,\mathrm{d}x + C_2 \implies \tfrac{1}{2}y^2 = C_1x + C_2$$

Alternatively,

$$(yy')' = 0 \implies yy' = C_1 \implies y\frac{\mathrm{d}y}{\mathrm{d}x} = C_1$$

$$\int y\,\mathrm{d}y = \int C_1\,\mathrm{d}x + C_2 \implies \tfrac{1}{2}y^2 = C_1x + C_2$$

Problem 2.98

Solve $\quad 1 + y'^2 = 2yy''$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{dy}{dx} \implies y'' = \dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx} = u \cdot \dfrac{du}{dy}$

$$\therefore\ 1 + u^2 = 2y \cdot u \frac{du}{dy} \qquad \text{✍ Variable separable}$$

$$\because\ y \neq 0 \implies \int \frac{2u}{1+u^2}\,du = \int \frac{1}{y}\,dy + C \implies \ln(1+u^2) = \ln|y| + C_1$$

$$1 + u^2 = C_1 y \implies u = \frac{dy}{dx} = \pm\sqrt{C_1 y - 1} \qquad \text{✍ Variable separable}$$

$$\pm\int \frac{1}{\sqrt{C_1 y - 1}}\,dy = \int dx + C_2 \implies \pm\frac{2}{C_1}\sqrt{C_1 y - 1} = x + C_2$$

$$\therefore\ 4(C_1 y - 1) = C_1^2 (x + C_2)^2$$

Problem 2.99

Solve $\quad xy'' = y'(\ln y' - \ln x)$

The DE is of the type y absent. Let $u = y'$, $u' = y''$.

$$xu' = u(\ln u - \ln x) \implies u' = \frac{u}{x} \ln \frac{u}{x} \qquad \text{✍ Homogeneous DE}$$

$$\therefore\ v = \frac{u}{x} \implies u = xv \implies \frac{du}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v \ln v \implies x\frac{dv}{dx} = v(\ln v - 1) \qquad \text{✍ Variable separable}$$

Case 1. $\ln v - 1 = 0 \implies v = e \implies u = vx = ex$

$$\therefore\ u = \frac{dy}{dx} = ex \implies y = \tfrac{1}{2}ex^2 + C$$

Case 2. $\ln v - 1 \neq 0$

$$\int \frac{1}{v(\ln v - 1)}\,dv = \int \frac{1}{x}\,dx + C_1 \implies \int \frac{1}{(\ln v - 1)}\,d(\ln v - 1) = \ln x + C_1$$

$$\ln\big|\ln v - 1\big| = \ln(C_1 x) \implies \ln v - 1 = C_1 x \implies v = \frac{u}{x} = e^{C_1 x + 1}$$

$$u = \frac{dy}{dx} = x e^{C_1 x + 1} \qquad \text{✍ Immediately integrable}$$

$$\therefore\ y = e\int x e^{C_1 x}\,dx + C_2 = e\,\frac{C_1 x - 1}{C_1^2} e^{C_1 x} + C_2 = \frac{1}{C_1^2}(C_1 x - 1) e^{C_1 x + 1} + C_2$$

Problem 2.100

Solve $\quad 3yy'y'' - y'^3 + 1 = 0$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{dy}{dx} \implies y'' = \dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx} = u \cdot \dfrac{du}{dy}$

$$\therefore\ 3y \cdot u \cdot u \frac{du}{du} = u^3 = 1 \qquad \text{✍ Variable separable}$$

Case 1. $u^3 - 1 = 0 \implies u = \dfrac{dy}{dx} = 1 \implies y = x$

Case 2. $u^3 - 1 \neq 0$

$$\therefore \int \frac{3u^2\, du}{u^3 - 1} = \int \frac{dy}{y} + C_1 \implies \ln\left|u^3 - 1\right| = \ln\left|y\right| + \ln\left|C_1\right|$$

$$\therefore\ u^3 - 1 = C_1 y \implies \frac{dy}{dx} = u = (C_1 y + 1)^{\frac{1}{3}} \qquad \text{✍ Variable separable}$$

$$\int (C_1 y + 1)^{-\frac{1}{3}}\, dy = \int dx + C_2 \implies \frac{1}{C_1} \int (C_1 y + 1)^{-\frac{1}{3}}\, d(C_1 y + 1) = x + C_2$$

$$\therefore\ \frac{1}{C_1} \cdot \frac{3}{2} (C_1 y + 1)^{\frac{2}{3}} = x + C_2 \implies 3(C_1 y + 1)^{\frac{2}{3}} - 2C_1 x = C_2$$

Problem 2.101

Solve $\quad y'' - y'^2 - 1 = 0$

The DE is of the type y absent. Let $u = y'$, $u' = y'' \implies u' - u^2 - 1 = 0$.

$$\frac{du}{dx} = u^2 + 1 \qquad \text{✍ Variable separable}$$

$$\int \frac{du}{u^2 + 1} = \int dx + C_1 \implies \tan^{-1} u = x + C_1 \implies u = \tan(x + C_1)$$

$$\therefore\ \frac{dy}{dx} = \tan(x + C_1) \qquad \text{✍ Immediately integrable}$$

$$\therefore\ y = \int \tan(x + C_1)\, dx + C_2 = -\ln\left|\cos(x + C_1)\right| + C_2$$

Problem 2.102

Solve $\quad x^3 y'' - x^2 y' = 3 - x^2$

The DE is of the type y absent. Let $u = y'$, $u' = y'' \implies x^3 u' - x^2 u = 3 - x^2$.

$$\therefore\ u' - \frac{1}{x} u = \frac{3 - x^2}{x^3}, \quad P(x) = -\frac{1}{x}, \quad Q(x) = \frac{3 - x^2}{x^3} \qquad \text{✍ Linear first-order}$$

$$\int P(x)\, dx = \int -\frac{1}{x}\, dx = -\ln\left|x\right|, \quad e^{\int P(x)\, dx} = e^{-\ln|x|} = \frac{1}{x}$$

$$\int Q(x)\,e^{\int P(x)\,dx}\,dx = \int \frac{3-x^2}{x^3}\cdot\frac{1}{x}\,dx = \int (3x^{-4}-x^{-2})\,dx = -x^{-3}+x^{-1}$$

$$u = e^{-\int P(x)\,dx}\left[\int Q(x)\,e^{\int P(x)\,dx}\,dx + C\right] = x(-x^{-3}+x^{-1}+C)$$

$$= -x^{-2}+1+Cx = y' \qquad \text{✍ Immediately integrable}$$

Integrate: $\quad y = x^{-1}+x+\frac{1}{2}Cx^2+D \implies y = \frac{1}{x}+C_1x^2+x+C_2$

Problem 2.103

Solve $\quad 2y'' = y'^3\sin 2x,\ \ y(0)=1,\ \ y'(0)=1$

The DE is of the type y absent. Let $u=y',\ \ u'=y'' \implies 2\dfrac{du}{dx} = u^3\sin 2x$

$$\int \frac{2}{u^3}\,du = \int \sin 2x\,dx + C \implies -\frac{1}{u^2} = -\frac{1}{2}\cos 2x + C$$

$x=0,\ y'=u=1:\quad -1 = -\frac{1}{2}+C \implies C = -\frac{1}{2}$

$$\therefore\ -\frac{1}{u^2} = -\frac{1+\cos 2x}{2} = -\cos^2 x \implies u = \pm\frac{1}{\cos x}$$

Again, since $x=0,\ u=1 \implies u = \dfrac{1}{\cos x}$

$$\therefore\ \frac{dy}{dx} = \sec x \qquad \text{✍ Immediately integrable}$$

$$y = \int \sec x\,dx + C_1 = \ln\left|\sec x+\tan x\right| + C_1$$

$x=0,\ y=1:\quad 1 = \ln|1+0|+C_1 \implies C_1 = 1$

$$\therefore\ y = 1+\ln\left|\sec x+\tan x\right|$$

Problem 2.104

Solve $\quad x\dfrac{d^2y}{dx^2} = 2-\dfrac{dy}{dx}$

The DE is of the type y absent. Let $u=y',\ u'=y'' \implies x\dfrac{du}{dx} = 2-u$

Case 1. $\quad u=2 \implies y'=2 \implies y=2x+C_0$

Case 2. $\quad u\neq 2$. The differential equation is variable separable.

$$\therefore\ x\frac{du}{dx} = 2-u \qquad \text{✍ Variable separable}$$

$$\int \frac{\mathrm{d}u}{u-2} = \int -\frac{\mathrm{d}x}{x} + C \implies \ln|u-2| = -\ln|x| + \ln|C_1| = \ln\left|\frac{C_1}{x}\right|$$

$$\therefore\ u-2 = \frac{C_1}{x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{C_1}{x} + 2$$ ✍ Immediately integrable

$$y = \int \left(\frac{C_1}{x} + 2\right) \mathrm{d}x + C_2 = C_1 \ln|x| + 2x + C_2$$

✍ Case 1 is included in Case 2 for $C_1 = 0$.

Problem 2.105

Solve $\quad y'' = 3\sqrt{y}, \quad y(0) = 1, \quad y'(0) = 2$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} \implies y'' = \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}y} \cdot \dfrac{\mathrm{d}y}{\mathrm{d}x} = u \cdot \dfrac{\mathrm{d}u}{\mathrm{d}y}$

$$\therefore\ u \frac{\mathrm{d}u}{\mathrm{d}y} = 3y^{\frac{1}{2}}$$ ✍ Variable separable

$$\int u \, \mathrm{d}u = 3 \int y^{\frac{1}{2}} \, \mathrm{d}y + C_1 \implies \tfrac{1}{2} u^2 = 2y^{\frac{3}{2}} + C_1$$

$x=0,\ y=1,\ y'=u=2$: $\quad \frac{1}{2} \cdot 2^2 = 2 \cdot 1 + C_1 \implies C_1 = 0$

$$\therefore\ u^2 = 4y^{\frac{3}{2}} \implies u = \frac{\mathrm{d}y}{\mathrm{d}x} = \pm 2y^{\frac{3}{4}}$$ ✍ Variable separable

$$\int y^{-\frac{3}{4}} \, \mathrm{d}y = \pm 2 \int \mathrm{d}x + C_2 \implies 4y^{\frac{1}{4}} = \pm 2x + C_2$$

$x=0,\ y=1$: $\quad 4 \cdot 1 = \pm 2 \cdot 0 + C_2 \implies C_2 = 4$

$$\therefore\ 4y^{\frac{1}{4}} = \pm 2x + 4 \implies y = \left(\pm \frac{x}{2} + 1\right)^4$$

Problem 2.106

Solve $\quad x \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}y}{\mathrm{d}x} + x \sin\left(\dfrac{1}{x} \cdot \dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$

The DE is of the type y absent. Let $u = y',\ u' = y'' \implies x \dfrac{\mathrm{d}u}{\mathrm{d}x} = u + x \sin\left(\dfrac{u}{x}\right)$

$$\therefore\ \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u}{x} + \sin\left(\frac{u}{x}\right)$$ ✍ Homogeneous DE

Let $v = \dfrac{u}{x} \implies u = xv \implies \dfrac{\mathrm{d}u}{\mathrm{d}x} = v + x \dfrac{\mathrm{d}v}{\mathrm{d}x}$

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + \sin v \implies x\frac{\mathrm{d}v}{\mathrm{d}x} = \sin v$$ ✍ Varibale separable

Case 1. $\sin v \neq 0$

$$\int \frac{\mathrm{d}v}{\sin v} = \int \frac{\mathrm{d}x}{x} + C \implies \ln\left|\tan\frac{v}{2}\right| = \ln|x| + \ln|C_1| \implies \tan\frac{v}{2} = C_1 x$$

$$v = 2\tan^{-1}(C_1 x), \quad v = \frac{u}{x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{1}{x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 2\tan^{-1}(C_1 x)$$

$$y = 2\int x\tan^{-1}(C_1 x)\,\mathrm{d}x + C_2 = \int \tan^{-1}(C_1 x)\,\mathrm{d}(x^2) + C_2$$

$$= x^2\tan^{-1}(C_1 x) - \int x^2\frac{C_1}{1+C_1^2x^2}\,\mathrm{d}x + C_2$$

$$= x^2\tan^{-1}(C_1 x) - \frac{1}{C_1}\int \mathrm{d}x + \frac{1}{C_1^2}\int \frac{C_1}{1+C_1^2x^2}\,\mathrm{d}x + C_2$$

$$= x^2\tan^{-1}(C_1 x) - \frac{x}{C_1} + \frac{1}{C_1^2}\tan^{-1}(C_1 x) + C_2$$

$$\therefore\ y = \left(x^2 + \frac{1}{C_1^2}\right)\tan^{-1}(C_1 x) - \frac{x}{C_1} + C_2$$

Case 2. $\sin v = 0 \implies v = k\pi,\ k = 0, \pm1, \pm2, \ldots \implies \frac{1}{x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = k\pi$

$$\therefore\ y = k\pi\int x\,\mathrm{d}x + C = \tfrac{1}{2}k\pi x^2 + C, \quad k = 0, \pm1, \pm2, \ldots$$

Problem 2.107

Solve $\quad yy'' = y'^2(1 - y'\sin y - yy'\cos y)$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x} \implies y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = u\cdot\frac{\mathrm{d}u}{\mathrm{d}y}$

$$\therefore\ y\cdot u\frac{\mathrm{d}u}{\mathrm{d}y} = u^2(1 - u\sin y - yu\cos y)$$

Case 1. $u = 0 \implies y' = 0 \implies y = C$

Case 2. $u\, Neq 0 \implies y\frac{\mathrm{d}u}{\mathrm{d}y} = u(1 - u\sin y - yu\cos y)$

$$\therefore\ \frac{\mathrm{d}u}{\mathrm{d}y} - \frac{1}{y}\cdot u = -\frac{\sin y + y\cos y}{y}u^2$$ ✍ Bernoulli DE

$$\frac{1}{u^2}\cdot\frac{\mathrm{d}u}{\mathrm{d}y} - \frac{1}{y}\cdot\frac{1}{u} = -\frac{\sin y + y\cos y}{y}$$

Let $v = \dfrac{1}{u} \implies \dfrac{\mathrm{d}v}{\mathrm{d}y} = -\dfrac{1}{u^2}\cdot\dfrac{\mathrm{d}u}{\mathrm{d}y}$

$$\therefore\ \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{1}{y}\cdot v = \frac{\sin y + y\cos y}{y}, \quad P(y) = \frac{1}{y}, \ Q(y) = \frac{\sin y + y\cos y}{y}$$

$$\int P(y)\,\mathrm{d}y = \int \frac{1}{y}\,\mathrm{d}y = \ln|y|, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{\ln|y|} = y$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int \frac{\sin y + y\cos y}{y}\cdot y\,\mathrm{d}y = \int (\sin y + y\cos y)\,\mathrm{d}y$$

$$= -\cos y + (y\sin y + \cos y) = y\sin y$$

$$\therefore\ v = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = \frac{1}{y}(y\sin y + C) = \sin y + \frac{C}{y}$$

$$\therefore\ v = \frac{1}{u} = \frac{\mathrm{d}x}{\mathrm{d}y} = \sin y + \frac{C}{y}$$ ✍ Immediately integrable

$$\int \mathrm{d}x = \int \Big(\sin y + \frac{C_1}{y}\Big)\,\mathrm{d}y + C_2 \implies x = -\cos y + C_1\ln|y| + C_2$$

Problem 2.108

Solve $\quad y'' + xy' = x$

The DE is of the type y absent. Let $u = y'$, $u' = y''$:

$$\therefore\ u' + x\cdot u = x, \quad P(x) = x, \quad Q(x) = x$$ ✍ Linear first-order

$$\int P(x)\,\mathrm{d}x = \int x\,\mathrm{d}x = \tfrac{1}{2}x^2$$

$$\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x = \int x\,\mathrm{e}^{\frac{1}{2}x^2}\,\mathrm{d}x = \int \mathrm{e}^{\frac{1}{2}x^2}\,\mathrm{d}\big(\tfrac{1}{2}x^2\big) = \mathrm{e}^{\frac{1}{2}x^2}$$

$$\therefore\ u = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = \mathrm{e}^{-\frac{1}{2}x^2}\big(\mathrm{e}^{\frac{1}{2}x^2} + C\big) = 1 + C\,\mathrm{e}^{-\frac{1}{2}x^2}$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + C_1\,\mathrm{e}^{-\frac{1}{2}x^2} \implies y = x + C_1\int \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x + C_2$$

Problem 2.109

Solve $\quad xy'' - y'^3 - y' = 0$

The DE is of the type y absent. Let $u = y'$, $u' = y''$:

$$\therefore\ xu' - u^3 - u = 0 \implies u' - \frac{1}{x}\cdot u = \frac{1}{x}\cdot u^3$$ ✍ Bernoulli DE

Case 1. $\quad u = 0 \implies y' = 0 \implies y = C$

Case 2. $\quad u \neq 0 \implies \dfrac{1}{u^3}u' - \dfrac{1}{x}\cdot\dfrac{1}{u^2} = \dfrac{1}{x}$

Let $v = \dfrac{1}{u^2}, \quad \dfrac{dv}{dx} = -\dfrac{2}{u^3}\dfrac{du}{dx} \implies \dfrac{dv}{dx} + \dfrac{2}{x}\cdot v = -\dfrac{2}{x}$

$$P(x) = \frac{2}{x}, \quad Q(x) = -\frac{2}{x}, \quad \int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x|, \quad e^{\int P(x)\,dx} = e^{2\ln|x|} = x^2$$

$$\therefore \quad v = e^{-\int P(x)\,dx}\Big[\int Q(x)\,e^{\int P(x)\,dx}\,dx + C\Big] = \frac{1}{x^2}\Big[\int -\frac{2}{x}\cdot x^2\,dx + C\Big]$$

$$= \frac{1}{x^2}\,(-x^2 + C)$$

$$\therefore \quad \frac{1}{u^2} = \frac{C_1 - x^2}{x^2} \implies u = \frac{dy}{dx} = \pm\frac{x}{\sqrt{C_1 - x^2}}$$ ✍ Immediately integrable

$$y = \pm\int \frac{x}{\sqrt{C_1 - x^2}}\,dx + C_2 = \pm\sqrt{C_1 - x^2} + C_2$$

$$\therefore \quad (y - C_2)^2 = C_1 - x^2 \implies x^2 + (y - C_2)^2 = C_1$$

Problem 2.110

Solve $\quad y\,(1 - \ln y)\,y'' + (1 + \ln y)\,y'^2 = 0$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{dy}{dx} \implies y'' = \dfrac{du}{dx} = \dfrac{du}{dy}\cdot\dfrac{dy}{dx} = u\cdot\dfrac{du}{dy}$

$$y\,(1 - \ln y)\,u\,\frac{du}{dy} + (1 + \ln y)\,u^2 = 0 \implies u\Big[y\,(1 - \ln y)\,\frac{du}{dy} + (1 + \ln y)\,u\Big] = 0$$

Case 1. $\quad u = 0 \implies \dfrac{dy}{dx} = 0 \implies y = C$

Case 2. $\quad u \neq 0 \implies y(1 - \ln y)\,\dfrac{du}{dy} = -(1 + \ln y)u$ ✍ Variable separable

$$\therefore \quad \int \frac{du}{u} = -\int \frac{1 + \ln y}{y(1 - \ln y)}\,d(\ln y) + C_0$$

$$\ln|u| = -\int \frac{1 + \ln y}{1 - \ln y}\,d(\ln y) + C_0, \quad z = \ln y$$

$$= -\int \frac{1 + z}{1 - z}\,dz + C_0 = \int\Big(1 + \frac{2}{z - 1}\Big)\,dz + C_0 = z + 2\ln|z - 1| + C_0$$

$$= \ln y + \ln|\ln y - 1|^2 + \ln|C_1| = \ln|C_1\,y\,(\ln y - 1)^2|$$

$$\therefore \quad u = \frac{dy}{dx} = C_1\,y\,(\ln y - 1)^2$$ ✍ Variable separable

$$\int \frac{dy}{y\,(\ln y - 1)^2} = C_1\int dx + C_2 \implies \int \frac{d(\ln y - 1)}{(\ln y - 1)^2} = C_1 x + C_2$$

$$\therefore \quad -\frac{1}{\ln y - 1} = C_1 x + C_2 \qquad \text{or} \qquad (C_1 x + C_2)(\ln y - 1) + 1 = 0$$

Problem 2.111

Solve $\quad xy^2(xy' + y) = 1$

$$x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy^3 = 1 \implies \underbrace{(xy^3 - 1)}_{M}\,\mathrm{d}x + \underbrace{x^2y^2}_{N}\,\mathrm{d}y = 0$$

$$\frac{\partial M}{\partial y} = 3xy^2, \quad \frac{\partial N}{\partial x} = 2xy^2, \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3xy^2 - 2xy^2}{x^2y^2} = \frac{1}{x}$$

$$\mu(x) = \exp\left(\int \frac{1}{x}\,\mathrm{d}x\right) = \mathrm{e}^{\ln x} = x$$

Multiply the DE by $\mu(x)$: $\quad (x^2y^3 - x)\,\mathrm{d}x + x^3y^2\,\mathrm{d}y = 0$

$$\begin{array}{ccccccc} & x^2y^3\,\mathrm{d}x & + & x^3y^2\,\mathrm{d}y & - & x\,\mathrm{d}x & = 0 \\ \int \mathrm{d}x & \boxed{\tfrac{1}{3}x^3y^3} & & \frac{\partial}{\partial y} & & \int \mathrm{d}x \quad \boxed{-\tfrac{1}{2}x^2} & \end{array}$$

$$\therefore \quad \tfrac{1}{3}x^3y^3 - \tfrac{1}{2}x^2 = C \implies 2x^3y^3 - 3x^2 = C$$

Problem 2.112

Solve $\quad 5y + y'^2 = x(x + y')$

Let $\quad y' = p \implies y = \tfrac{1}{5}(x^2 + xp - p^2) = f(x, p)$

Differentiating with respect to x: $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = f_x + f_p\dfrac{\mathrm{d}p}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{5}(2x + p) + \tfrac{1}{5}(x - 2p)\frac{\mathrm{d}p}{\mathrm{d}x} \implies 5p = 2x + p + (x - 2p)\frac{\mathrm{d}p}{\mathrm{d}x}$$

$$2x - 4p + (x - 2p)\frac{\mathrm{d}p}{\mathrm{d}x} = 0 \implies (x - 2p)\left(2 + \frac{\mathrm{d}p}{\mathrm{d}x}\right) = 0$$

Case 1. $\quad x - 2p = 0 \implies p = \dfrac{x}{2}$

$$\therefore \quad y = \tfrac{1}{5}(x^2 + xp - p^2) = \tfrac{1}{5}\left(x^2 + x\cdot\frac{x}{2} - \frac{x^2}{4}\right) = \frac{x^2}{4} \implies 4y = x^4$$

Case 2. $\quad \dfrac{\mathrm{d}p}{\mathrm{d}x} = -2 \implies p = -2x + C$

$$y = \tfrac{1}{5}\left[x^2 + (-2x^2 + Cx) + (4x^2 - 4Cx + C^2)\right] = \tfrac{1}{5}(3x^2 - 3Cx + C^2)$$

$$\therefore \quad 5y = 3x^2 - 3Cx + C^2$$

Problem 2.113

Solve $\quad y' = \dfrac{y+2}{x+1} + \tan\dfrac{y-2x}{x+1}$

Let $X = x+1, \quad Y = y+2 \implies y-2x = (Y-2)-2(X-1) = Y-2X$

$$\therefore \quad \frac{dY}{dX} = \frac{Y}{X} + \tan\frac{Y-2X}{X} \qquad \text{✍ Homogeneous}$$

Let $\dfrac{Y}{X} = u \implies Y = Xu \implies \dfrac{dY}{dX} = u + X\dfrac{du}{dX}$

$$\therefore \; u + X\frac{du}{dX} = u + \tan(u-2) \implies X\frac{du}{dX} = \tan(u-2)$$

Case 1. $\tan(u-2) = 0 \implies u-2 = n\pi, \quad n = 0, \pm1, \pm2, \dots$

$$u-2 = \frac{Y}{X} - 2 = \frac{y-2x}{x+1} = n\pi$$

Case 2. $u-2 \neq n\pi$

$$\int \cot(u-2)\,du = \int \frac{1}{X}\,dX + C \implies \ln\left|\sin(u-2)\right| = \ln|X| + \ln|C|$$

$$\therefore \; \sin(u-2) = CX \implies \sin\frac{y-2x}{x+1} = C(x+1)$$

✍ Case 1 is included in Case 2 for $C = 0$.

Problem 2.114

Solve $\quad y''(e^x+1) + y' = 0$

The DE is of the type y absent. Let $y' = u, \; y'' = u' \implies \dfrac{du}{dx}(e^x+1) = -u$

Case 1. $u = 0 \implies y' = 0 \implies y = C$

Case 2. $u \neq 0: \quad \displaystyle\int \frac{du}{u} = \int -\frac{dx}{e^x+1} + C$ ✍ Variable separable

$$-\int \frac{dx}{e^x+1} = -\int \frac{e^{-x}}{e^{-x}+1}\,dx = \int \frac{1}{e^{-x}+1}\,d(e^{-x}+1) = \ln(e^{-x}+1)$$

$$\therefore \; \ln|u| = \ln(e^{-x}+1) + \ln|C_1| \implies u = \frac{dy}{dx} = C_1(e^{-x}+1)$$

$$\therefore \; y = C_1(-e^{-x}+x) + C_2$$

Problem 2.115

Solve $\quad xy' = y - x e^{y/x}$

$$y' = \frac{y}{x} - \mathrm{e}^{\frac{y}{x}} \qquad \text{✍ Homogeneous DE}$$

$$\text{Let } u = \frac{y}{x} \implies y = xu \implies \frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\therefore\ u + x\frac{\mathrm{d}u}{\mathrm{d}x} = u - \mathrm{e}^{u} \implies x\frac{\mathrm{d}u}{\mathrm{d}x} = -\mathrm{e}^{u} \qquad \text{✍ Variable separable}$$

$$-\int \mathrm{e}^{-u}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x + C \implies \mathrm{e}^{-u} = \ln|x| + \ln\big|C\big|$$

$$\mathrm{e}^{-u} = \ln|Cx| \implies u = \frac{y}{x} = -\ln\big|\ln|Cx|\big| \implies y = -x\ln\big|\ln|Cx|\big|$$

Problem 2.116

Solve $\quad (1 + y^2\sin 2x)\,\mathrm{d}x - 2\,y\cos^2 x\,\mathrm{d}y = 0$

$$M = 1 + y^2\sin 2x, \qquad N = -2y\cos^2 x$$

$$\frac{\partial M}{\partial y} = 2y\sin 2x, \qquad \frac{\partial N}{\partial x} = -2y\cdot 2\cos x(-\sin x) = 2y\sin 2x$$

$$\therefore\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{✍ Exact DE}$$

$$1\cdot\mathrm{d}x - 2y\cos^2 x\,\mathrm{d}y \quad + \quad 2y^2\cos x\sin x\,\mathrm{d}x = 0$$

$$\int\mathrm{d}x \downarrow \boxed{x} \qquad \int\mathrm{d}y\ \boxed{-y^2\cos^2 x}\ \frac{\partial}{\partial x}$$

General solution: $\quad x - y^2\cos^2 x = C$

Problem 2.117

Solve $\quad (2\sqrt{xy} - y)\,\mathrm{d}x - x\,\mathrm{d}y = 0, \quad x > 0, \quad y > 0$

$$\left(2\sqrt{\frac{y}{x}} - \frac{y}{x}\right) - \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \text{✍ Homogeneous DE}$$

$$\text{Let } u = \frac{y}{x} \implies y = xu \implies \frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$(2\sqrt{u} - u) - u - x\frac{\mathrm{d}u}{\mathrm{d}x} = 0 \implies -x\frac{\mathrm{d}u}{\mathrm{d}x} = 2\sqrt{u}\big(\sqrt{u} - 1\big)$$

Case 1. $\quad u = 0 \implies y = 0$. Not a solution since $y > 0$.

Case 2. $\quad u = 1 \implies y = x$

Case 3. $\quad u \neq 0,\ u \neq 1 \implies \displaystyle\int \frac{\mathrm{d}u}{2\sqrt{u}(\sqrt{u} - 1)} = \int -\frac{\mathrm{d}x}{x} + C$

$$\int \frac{\mathrm{d}(\sqrt{u} - 1)}{\sqrt{u} - 1} = -\ln|x| + \ln|C| \implies \ln\big|\sqrt{u} - 1\big| = \ln\left|\frac{C}{x}\right|$$

$$\therefore\ \sqrt{u}-1=\frac{C}{x} \implies \sqrt{\frac{y}{x}}-1=\frac{C}{x} \implies \sqrt{xy}-x=C$$

✍ $y=x$ is included in the general solution $\sqrt{xy}-x=C$ for $C=0$.

Problem 2.118

Solve $\quad y''+y'^2=2\mathrm{e}^{-y}$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u=y'=\dfrac{\mathrm{d}y}{\mathrm{d}x} \implies y''=\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{\mathrm{d}u}{\mathrm{d}y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}=u\cdot\dfrac{\mathrm{d}u}{\mathrm{d}y}$

$$u\frac{\mathrm{d}u}{\mathrm{d}y}+u^2=2\mathrm{e}^{-y}$$ ✍ Bernoulli DE

Let $v=u^2 \implies \dfrac{\mathrm{d}v}{\mathrm{d}y}=2u\cdot\dfrac{\mathrm{d}u}{\mathrm{d}y} \implies \dfrac{\mathrm{d}v}{\mathrm{d}y}+2\cdot v=4\mathrm{e}^{-y}$ ✍ Linear first-order

$$P(y)=2,\quad Q(y)=4\mathrm{e}^{-y},\quad \int P(y)\,\mathrm{d}y=2y,\quad \mathrm{e}^{\int P(y)\,\mathrm{d}y}=\mathrm{e}^{2y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y=\int 4\mathrm{e}^{-y}\cdot\mathrm{e}^{2y}\,\mathrm{d}y=4\int \mathrm{e}^{y}\,\mathrm{d}y=4\mathrm{e}^{y}$$

$$\therefore\ v=u^2=\mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\mathrm{d}y+C\Big]=\mathrm{e}^{-2y}\big(4\mathrm{e}^{y}+C\big)$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=u=\pm\mathrm{e}^{-y}\sqrt{4\mathrm{e}^{y}+C_1}$$ ✍ Variable separable

$$\int\frac{\mathrm{e}^{y}\,\mathrm{d}y}{\sqrt{4\mathrm{e}^{y}+C_1}}=\int\pm\,\mathrm{d}x+C_2 \implies \tfrac{1}{2}\sqrt{4\mathrm{e}^{y}+C_1}=\pm x+C_2$$

Square both sides: $\frac{1}{4}\big(4\mathrm{e}^{y}+C_1\big)=(x+C_2)^2 \implies \mathrm{e}^{y}+C_1=(x+C_2)^2$

Problem 2.119

Solve $\quad y'=\mathrm{e}^{xy'/y}$

Let $y'=p \implies p=\mathrm{e}^{xp/y} \implies \ln p=\dfrac{xp}{y} \implies x=\dfrac{y}{p}\ln p=g(y,p),\quad p>0$

Differentiate with respect to y: $\quad \dfrac{\mathrm{d}x}{\mathrm{d}y}=g_y+g_p\dfrac{\mathrm{d}p}{\mathrm{d}y}$

$$\frac{1}{p}=\frac{1}{p}\ln p+\Big(-\frac{y}{p^2}\ln p+\frac{y}{p^2}\Big)\frac{\mathrm{d}p}{\mathrm{d}y} \implies (1-\ln p)\Big(\frac{y}{p}\cdot\frac{\mathrm{d}p}{\mathrm{d}y}-1\Big)=0$$

Case 1. $\quad \ln p=1 \implies p=\mathrm{e} \implies x=\dfrac{y}{\mathrm{e}}\ln\mathrm{e}=\dfrac{y}{\mathrm{e}} \implies y=\mathrm{e}x$

Case 2. $\dfrac{y}{p} \cdot \dfrac{\mathrm{d}p}{\mathrm{d}y} - 1 = 0$ ✍ Variable separable

$$\int \frac{1}{p}\,\mathrm{d}p = \int \frac{1}{y}\,\mathrm{d}y + C \implies \ln|p| = \ln|y| + \ln|C| \implies p = Cy$$

$$\therefore\ x = \frac{y}{Cy}\ln|Cy| \implies Cx = \ln|Cy|$$

Problem 2.120

Solve $(2x^3y^2 - y)\,\mathrm{d}x + (2x^2y^3 - x)\,\mathrm{d}y = 0$

$$2x^2y^2(x\,\mathrm{d}x + y\,\mathrm{d}y) - (y\,\mathrm{d}x + x\,\mathrm{d}y) = 0 \implies x^2y^2\,\mathrm{d}(x^2+y^2) - \mathrm{d}(xy) = 0$$

$$\mathrm{d}(x^2+y^2) - \frac{1}{(xy)^2}\,\mathrm{d}(xy) = 0 \implies \mathrm{d}(x^2+y^2) + \mathrm{d}\Big(\frac{1}{xy}\Big) = 0$$

$$\therefore\ x^2 + y^2 + \frac{1}{xy} = C$$

Problem 2.121

Solve $(y - 1 - xy)\,\mathrm{d}x + x\,\mathrm{d}y = 0$

$$M = y - 1 - xy, \quad N = x, \quad \frac{\partial M}{\partial y} = 1 - x, \quad \frac{\partial N}{\partial x} = 1$$ ✍ Not exact DE

$$\frac{1}{N}\Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big) = \frac{1}{x}\big[(1-x) - 1\big] = -1 \implies \mu(x) = \mathrm{e}^{-\int \mathrm{d}x} = \mathrm{e}^{-x}$$

Multiply the DE by $\mu(x)$: $(y\mathrm{e}^{-x} - \mathrm{e}^{-x} - xy\mathrm{e}^{-x})\,\mathrm{d}x + x\mathrm{e}^{-x}\,\mathrm{d}y = 0$

$$x\mathrm{e}^{-x}\,\mathrm{d}y \quad + \quad (y\mathrm{e}^{-x} - xy\mathrm{e}^{-x})\,\mathrm{d}x \ - \ \mathrm{e}^{-x}\,\mathrm{d}x = 0$$

$$\int \mathrm{d}x \quad \boxed{xy\mathrm{e}^{-x}} \quad \frac{\partial}{\partial x} \qquad\qquad \int \mathrm{d}x \downarrow \ \boxed{\mathrm{e}^{-x}}$$

General solution: $xy\mathrm{e}^{-x} + \mathrm{e}^{-x} = C$ or $xy + 1 = C\mathrm{e}^{x}$

Problem 2.122

Solve $xy' - y = x\tan\dfrac{y}{x}$

$y' - \dfrac{y}{x} = \tan\dfrac{y}{x}$. Let $u = \dfrac{y}{x} \implies y = xu \implies y' = u + xu'$ ✍ Homogeneous

$$\therefore\ u + xu' - u = \tan u \implies x\frac{\mathrm{d}u}{\mathrm{d}x} = \tan u$$

Case 1. $u = 0 \implies y = 0$

Case 2. $\sin u \neq 0 \ (u \neq 0): \implies \displaystyle\int \cot u \, du = \int \frac{1}{x} \, dx + C$

$$\ln|\sin u| = \ln|x| + \ln|C| \implies \sin u = Cx \implies \sin\frac{y}{x} = Cx$$

✍ Case 1 is included in Case 2 for $C=0$.

Problem 2.123

Solve $\quad y' + \dfrac{y}{x} = e^{xy}$

Let $u = xy \implies u' = xy' + y \implies y' = \dfrac{u'}{x} - \dfrac{y}{x}$

$$\therefore \quad \left(\frac{u'}{x} - \frac{y}{x}\right) + \frac{y}{x} = e^u \implies \frac{1}{x}\frac{du}{dx} = e^u$$ ✍ Variable separable

$$\int e^{-u} \, du = \int x \, dx + C \implies -e^{-u} = \tfrac{1}{2}x^2 + C \implies -e^{-xy} = \tfrac{1}{2}x^2 + C$$

Problem 2.124

Solve $\quad yy'' - yy' = (y')^2$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{dy}{dx} \implies y'' = \dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx} = u \cdot \dfrac{du}{dy}$

$$\therefore \quad y \cdot u \frac{du}{dy} - y \cdot u = u^2 \implies u\left(y\frac{du}{dy} - u - y\right) = 0$$

Case 1. $u = 0 \implies y' = 0 \implies y = C$

Case 2. $\dfrac{du}{dy} - \dfrac{1}{y} \cdot u = 1, \quad P(y) = -\dfrac{1}{y}, \quad Q(y) = 1$ ✍ Linear first-order

$$\int P(y) \, dy = \int -\frac{1}{y} \, dy = -\ln|y|, \quad e^{\int P(y) \, dy} = e^{-\ln|y|} = \frac{1}{y}$$

$$\int Q(y) e^{\int P(y) \, dy} \, dy = \int 1 \cdot \frac{1}{y} \, dy = \ln|y|$$

$$u = \frac{dy}{dx} = e^{-\int P(y) \, dy}\left[\int Q(y) e^{\int P(y) \, dy} \, dy + C\right] = y(\ln|y| + C)$$

$$\int \frac{1}{y(\ln|y| + C_1)} \, dy = \int dx + C_2 \implies \int \frac{1}{(\ln|y| + C_1)} \, d(\ln|y| + C_1) = x + C_2$$

$$\therefore \quad \ln\Big|\ln|y| + C_1\Big| = x + C_2$$

Problem 2.125

Solve $\quad 2y\,dx - x\big[\ln(x^2y) - 1\big]dy = 0$

Multiply the DE by x: $\quad (2xy\,dx + x^2dy) - x^2\ln(x^2y)\,dy = 0$

$$\therefore\ d(x^2y) - x^2\ln(x^2y)\,dy = 0 \implies \underbrace{\frac{1}{\ln(x^2y)}\,d(x^2y)}_{\text{Group 1}} - \underbrace{x^2\,dy}_{\text{Group 2}} = 0$$

Group 1. $\quad \mu_1 = \ln(x^2y), \quad u_1 = x^2y$

Group 2. $\quad \mu_2 = \dfrac{1}{x^2}, \quad u_2 = y$

$$\underbrace{\ln(x^2y)}_{\mu_1}\cdot\underbrace{\frac{1}{\ln(x^2y)\cdot x^2y}}_{g_1(u_1)} = \underbrace{\frac{1}{x^2}}_{\mu_2}\cdot\underbrace{\frac{1}{y}}_{g_2(u_2)} = \frac{1}{x^2y} = \mu$$

Multiply the DE by $\mu = \dfrac{1}{x^2y}$: $\quad \dfrac{1}{x^2y\cdot\ln(x^2y)}\,d(x^2y) - \dfrac{1}{y}\,dy = 0$

$$d\big[\ln\big|\ln(x^2y)\big|\big] - d\big(\ln|y|\big) = 0 \implies \ln\big|\ln(x^2y)\big| - \ln|y| = C \implies \ln(x^2y) = Cy$$

Problem 2.126

Solve $\quad y' = \dfrac{1}{xy + x^3y^3}$

$$\frac{dx}{dy} = xy + x^3y^3 \implies \frac{dx}{dy} - y\cdot x = y^3\cdot x^3 \qquad \text{✍ Bernoulli DE}$$

$$x^{-3}\frac{dx}{dy} - y\cdot x^{-2} = y^3$$

Let $\ u = x^{-2} \implies \dfrac{du}{dy} = -2x^{-3}\cdot\dfrac{dx}{dy} \implies \dfrac{du}{dy} + 2y\cdot u = -2y^3$

$$P(y) = 2y,\ \ Q(y) = -2y^3,\ \ \int P(y)\,dy = \int 2y\,dy = y^2,\ \ e^{\int P(y)\,dy} = e^{y^2}$$

$$\int Q(y)\,e^{\int P(y)\,dy}dy = \int -2y^3e^{y^2}dy = -\int 2y\cdot y^2e^{y^2}dy$$

$$= -\int y^2e^{y^2}d(y^2) = -\int ze^zdz = -e^z(z-1) = e^{y^2}(1-y^2), \quad z = y^2$$

$$\therefore\ u = e^{-\int P(y)\,dy}\Big[\int Q(y)\,e^{\int P(y)\,dy}dy + C\Big] = e^{-y^2}\Big[e^{y^2}(1-y^2) + C\Big]$$

$$\therefore\ \frac{1}{x^2} = 1 - y^2 + Ce^{-y^2}$$

Problem 2.127

Solve $\quad y' = 2\left(\dfrac{y+2}{x+y-1}\right)^2$

Point of intersection of two lines

$$\begin{cases} y+2=0 \\ x+y-1=0 \end{cases} \implies x=3, \quad y=-2$$

Let $\;x = X+3, \quad y = Y-2, \quad y' = \dfrac{\mathrm{d}Y}{\mathrm{d}X}$

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = 2\left[\frac{(Y-2)+2}{(X+3)+(Y-2)-1}\right]^2 = 2\left(\frac{Y}{X+Y}\right)^2 = 2\left(\frac{\dfrac{Y}{X}}{1+\dfrac{Y}{X}}\right)^2$$

Let $\;u = \dfrac{Y}{X} \implies Y = Xu \implies \dfrac{\mathrm{d}Y}{\mathrm{d}X} = u + X\dfrac{\mathrm{d}u}{\mathrm{d}X}$

$$u + X\frac{\mathrm{d}u}{\mathrm{d}X} = 2\left(\frac{u}{1+u}\right)^2 \implies X\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{2u^2}{(1+u)^2} - u = -\frac{u(1+u^2)}{(1+u)^2}$$

Case 1. $\quad u=0 \implies \dfrac{y+2}{x-3} = 0 \implies y+2=0$

Case 2. $\quad u \neq 0 \implies \displaystyle\int \frac{(1+u)^2}{u(1+u^2)}\mathrm{d}u = -\int \frac{\mathrm{d}X}{X} + D$

$$\frac{1+2u+u^2}{u(1+u^2)} = \frac{A}{u} + \frac{Bu+C}{1+u^2} = \frac{A + Au^2 + Bu^2 + Cu}{u(1+u^2)}$$

$$A=1,\; B=0,\; C=2 \implies \int\left(\frac{1}{u} + \frac{2}{1+u^2}\right)\mathrm{d}u = -\int \frac{\mathrm{d}X}{X} + D$$

$$\ln|u| + 2\tan^{-1}u = -\ln|X| + \ln|C| \implies 2\tan^{-1}u = \ln\left|\frac{C}{uX}\right|$$

$$2\tan^{-1}\frac{y+2}{x-3} = \ln\left|\frac{C}{y+2}\right| \implies y+2 = C\exp\left(-2\tan^{-1}\frac{y+2}{x-3}\right)$$

✍ $y+2=0$ is included in the general solution with $C=0$.

Problem 2.128

Solve $\quad (\mathrm{e}^x + 3y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0$

$$M = \mathrm{e}^x + 3y^2, \quad N = 2xy, \quad \frac{\partial M}{\partial y} = 6y, \quad \frac{\partial N}{\partial x} = 2y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(6y-2y) = \frac{2}{x}$$ ✍ A function of x only

$$\mu(x) = \exp\Big(\int \frac{2}{x}\,\mathrm{d}x\Big) = \mathrm{e}^{2\ln|x|} = x^2$$

Multiply the DE by $\mu(x) = x^2$: $\quad (x^2\mathrm{e}^x + 3x^2y^2)\,\mathrm{d}x + 2x^3y\,\mathrm{d}y = 0$

$$\therefore \qquad x^2\mathrm{e}^x\mathrm{d}x \quad + 3x^2y^2\mathrm{d}x \quad + \quad 2x^3y\,\mathrm{d}y = 0$$

$$\int \mathrm{d}x \downarrow \quad \boxed{\mathrm{e}^x(x^2-2x+2)} \qquad \int \mathrm{d}x \quad \boxed{x^3y^2} \quad \frac{\partial}{\partial x}$$

General solution: $\quad \mathrm{e}^x(x^2-2x+2) + x^3y^2 = C$

Problem 2.129

Solve $\quad (xy + 2x^3y)\,\mathrm{d}x + x^2\mathrm{d}y = 0$

Method 1. Method of inspection

$$(xy\,\mathrm{d}x + x^2\,\mathrm{d}y) + 2x^3y\,\mathrm{d}x = 0 \implies x(y\,\mathrm{d}x + x\,\mathrm{d}y) + 2x^2(xy)\,\mathrm{d}x = 0$$

$$\frac{1}{xy}\,\mathrm{d}(xy) + 2x\,\mathrm{d}x = 0 \implies \mathrm{d}\big(\ln|xy|\big) + \mathrm{d}(x^2) = 0$$

General solution: $\quad \ln|xy| + x^2 = C$

Method 2. Integrating factor

$$M = xy + 2x^3y, \quad N = x^2, \quad \frac{\partial M}{\partial y} = x + 2x^3, \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{1}{N}\Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big) = \frac{1}{x^2}(x + 2x^3 - 2x) = 2x - \frac{1}{x}$$

$$\mu(x) = \exp\Big(\int\Big(2x - \frac{1}{x}\Big)\mathrm{d}x\Big) = \mathrm{e}^{x^2 - \ln|x|} = \frac{1}{x}\,\mathrm{e}^{x^2}$$

Multiply the DE by $\mu(x)$: $\quad y\mathrm{e}^{x^2}\,\mathrm{d}x + 2x^2y\mathrm{e}^{x^2}\,\mathrm{d}x + x\mathrm{e}^{x^2}\,\mathrm{d}y = 0$

$$x\mathrm{e}^{x^2}\,\mathrm{d}y \quad + \quad (y\mathrm{e}^{x^2} + 2x^2y\mathrm{e}^{x^2})\,\mathrm{d}x = 0$$

$$\int \mathrm{d}y \quad \boxed{xy\mathrm{e}^{x^2}} \quad \frac{\partial}{\partial x}$$

General solution: $\quad xy\mathrm{e}^{x^2} = C$

Problem 2.130

Solve $\quad xy'^2 - 2yy' + 4x = 0$

Let $y'=p \implies xp^2-2yp+4x=0$. Since $y'=p=0$ is not a solution, solve for y:

$$y=\frac{1}{2p}\,(xp^2+4x)=\frac{xp}{2}+\frac{2x}{p}=f(x,p)$$

Differentiate with respect to x:

$$p=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\cdot\frac{\mathrm{d}p}{\mathrm{d}x}=\left(\frac{p}{2}+\frac{2}{p}\right)+\left(\frac{x}{2}-\frac{2x}{p^2}\right)\frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\left(-\frac{p}{2}+\frac{2}{p}\right)+\frac{x}{p}\left(\frac{p}{2}-\frac{2}{p}\right)\frac{\mathrm{d}p}{\mathrm{d}x}=0 \implies \left(-\frac{p}{2}+\frac{2}{p}\right)\left(1-\frac{x}{p}\cdot\frac{\mathrm{d}p}{\mathrm{d}x}\right)=0$$

Case 1. $-\dfrac{p}{2}+\dfrac{2}{p}=0 \implies p^2-4=0 \implies p=\pm 2$

$$y=\frac{1}{2p}\,(xp^2+4x)=\pm\frac{1}{4}\,(x\cdot 4+4x)=\pm 2x$$

Case 2. $1-\dfrac{x}{p}\cdot\dfrac{\mathrm{d}p}{\mathrm{d}x}=0$ ✍ Variable separable

$$\int\frac{1}{p}\,\mathrm{d}p=\int\frac{1}{x}\,\mathrm{d}x+C \implies \ln|p|=\ln|x|+\ln|C|$$

$$p=Cx \implies y=\frac{xp}{2}+\frac{2x}{p}=\frac{x\cdot Cx}{2}+\frac{2x}{Cx}=\frac{C}{2}x^2+\frac{2}{C}$$

Problem 2.131

Solve $\quad y'''=2\,(y''-1)\cot x$

The DE is of the type y' absent. Let $y''=u,\ \ y'''=u'$.

$$\therefore\quad \frac{\mathrm{d}u}{\mathrm{d}x}=2\,(u-1)\cot x \qquad \text{✍ Variable separable}$$

Case 1. $u=1 \implies y''=1$ ✍ Immediately integrable

$$\therefore\ y'=x+C_1 \implies y=\tfrac{1}{2}x^2+C_1x+C_0$$

Case 2. $u\neq 1 \implies \displaystyle\int\frac{\mathrm{d}u}{u-1}=\int 2\cot x\,\mathrm{d}x+C$

$$\ln|u-1|=2\ln|\sin x|+\ln|C| \implies u-1=C\sin^2 x$$

$$\therefore\ y''=C\,\frac{1-\cos 2x}{2}+1=\frac{C+2}{2}-\frac{C}{2}\cos 2x \qquad \text{✍ Immediately integrable}$$

$$y'=\frac{C+2}{2}\,x-\frac{C}{4}\sin 2x+C_1 \implies y=\frac{C+2}{4}\,x^2+\frac{C}{8}\cos 2x+C_1x+C_0$$

✍ The solution in Case 1 is included in Case 2 with $C=0$.

Problem 2.132

Solve $(y+3x^4y^2)\,\mathrm{d}x + (x+2x^2y^3)\,\mathrm{d}y = 0$

$$(y\,\mathrm{d}x + x\,\mathrm{d}y) + x^2y^2(3x^2\,\mathrm{d}x + 2y\,\mathrm{d}y) = 0$$

Case 1. $y=0$ is a solution

Case 2. $y \neq 0$: $\dfrac{1}{(xy)^2}\,\mathrm{d}(xy) + \mathrm{d}(x^3) + \mathrm{d}(y^2) = 0$

$$-\mathrm{d}\Big(\frac{1}{xy}\Big) + \mathrm{d}(x^3) + \mathrm{d}(y^2) = 0 \implies -\frac{1}{xy} + x^3 + y^2 = C$$

Problem 2.133

Solve $xy' = y + \sqrt{x^2-y^2}, \quad x>0, \quad |y| \leqslant |x|$

$$y' = \frac{y}{x} + \sqrt{1-\Big(\frac{y}{x}\Big)^2} \qquad \text{✍ Homogeneous DE}$$

Let $u = \dfrac{y}{x} \implies y = xu \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = u + x\dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore \quad u + x\frac{\mathrm{d}u}{\mathrm{d}x} = u + \sqrt{1-u^2} \qquad \text{✍ Variable separable}$$

Case 1. $1-u^2=0 \implies u = \pm 1 \implies y = \pm x$

Case 2. $1-u^2 \neq 0 \implies \displaystyle\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \int \frac{\mathrm{d}x}{x} + C$

$$\sin^{-1}u = \ln x + C \implies \sin^{-1}\frac{y}{x} = \ln x + C$$

Problem 2.134

Solve $2y\,(x\,\mathrm{e}^{x^2} + y\sin x\cos x)\,\mathrm{d}x + (2\,\mathrm{e}^{x^2} + 3y\sin^2 x)\,\mathrm{d}y = 0$

$$M = 2y\,(x\,\mathrm{e}^{x^2} + y\sin x\cos x), \qquad N = 2\,\mathrm{e}^{x^2} + 3y\sin^2 x$$

$$\frac{\partial M}{\partial y} = 2x\,\mathrm{e}^{x^2} + 4y\sin x\cos x, \qquad \frac{\partial N}{\partial x} = 4x\,\mathrm{e}^{x^2} + 6y\sin x\cos x$$

$$\frac{1}{M}\Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) = \frac{1}{2y\,(x\,\mathrm{e}^{x^2} + y\sin x\cos x)}(2x\,\mathrm{e}^{x^2} + 2y\sin x\cos x) = \frac{1}{y}$$

$$\mu(y) = \exp\Big(\int \frac{1}{y}\,\mathrm{d}y\Big) = \mathrm{e}^{\ln y} = y$$

Multiply by $\mu(y)$: $(2xy^2\,\mathrm{e}^{x^2} + 2y^3\sin x\cos x)\,\mathrm{d}x + (2y\,\mathrm{e}^{x^2} + 3y^2\sin^2 x)\,\mathrm{d}y = 0$

$$2y\mathrm{e}^{x^2}\,\mathrm{d}y \quad + \quad 2xy^2\mathrm{e}^{x^2}\,\mathrm{d}x + 3y^2\sin^2 x\,\mathrm{d}y \quad + \quad 2y^3\sin x\cos x\,\mathrm{d}x = 0$$

$$\int \mathrm{d}y \quad \boxed{y^2\mathrm{e}^{x^2}} \quad \frac{\partial}{\partial x} \qquad\qquad \int \mathrm{d}y \quad \boxed{y^3\sin^2 x} \quad \frac{\partial}{\partial x}$$

General solution: $\quad y^2\mathrm{e}^{x^2} + y^3\sin^2 x = C$

Problem 2.135

Solve $\quad \cos y\,\mathrm{d}x + \sin y\,(x - \sin y\cos y)\,\mathrm{d}y = 0$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \tan y\cdot x = \sin^2 y, \quad P(y) = \tan y, \quad Q(y) = \sin^2 y \quad ✍ \text{ Linear first-order}$$

$$\int P(y)\,\mathrm{d}y = \int \tan y\,\mathrm{d}y = -\ln\left|\cos y\right|, \quad \mathrm{e}^{\int P(x)\,\mathrm{d}x} = \mathrm{e}^{-\ln|\cos y|} = \frac{1}{\cos y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int \sin^2 y\cdot\frac{1}{\cos y}\,\mathrm{d}y = \int \frac{1-\cos^2 y}{\cos y}\,\mathrm{d}y$$

$$= \int (\sec y - \cos y)\,\mathrm{d}y = \ln\left|\sec y + \tan y\right| - \sin y$$

$$\therefore \quad x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = \cos y\left(\ln\left|\sec y + \tan y\right| - \sin y + C\right)$$

Problem 2.136

Solve $\quad y^3\,\mathrm{d}x + (3x^2 - 2xy^2)\,\mathrm{d}y = 0$

$$\underbrace{\left(y^3\mathrm{d}x - 2xy^2\mathrm{d}y\right)}_{\text{Group 1}} \quad + \quad \underbrace{3x^2\mathrm{d}y}_{\text{Group 2}} = 0$$

Group 1: $\quad xy^3\Big(\frac{1}{x}\,\mathrm{d}x - \frac{2}{y}\,\mathrm{d}y\Big) = xy^3(\mathrm{d}\ln|x| - 2\,\mathrm{d}\ln|y|) = xy^3\,\mathrm{d}\ln\left|\frac{x}{y^2}\right|$

$$\therefore \quad \mu_1 = \frac{1}{xy^3}, \quad v_1 = \ln\left|\frac{x}{y^2}\right|; \qquad \mu_2 = \frac{1}{x^2}, \quad v_2 = y$$

$$\mu_1\,g_1(v_1) = \mu_2\,g_2(v_2) \implies \frac{1}{xy^3}\,g_1\Big(\ln\left|\frac{x}{y^2}\right|\Big) = \frac{1}{x^2}\,g_2(y)$$

$$\frac{1}{xy^3}\Big(\frac{x}{y^2}\Big)^{-1} = \frac{1}{x^2}\cdot\frac{1}{y}, \quad g_1(v_1) = (\mathrm{e}^{v_1})^{-1}, \quad g_2(v_2) = \frac{1}{v_2} \implies \mu = \frac{1}{x^2 y}$$

Case 1. $\quad y = 0$ is a solution of the DE.

Case 2. $\quad y \neq 0$. Multiply the DE by $\mu = \dfrac{1}{x^2 y}$: $\quad \dfrac{y^2}{x^2}\,\mathrm{d}x + \Big(\dfrac{3}{y} - \dfrac{2y}{x}\Big)\,\mathrm{d}y = 0$

$$-\frac{2y}{x}\,\mathrm{d}y \quad + \quad \frac{y^2}{x^2}\,\mathrm{d}x + \quad \frac{3}{y}\,\mathrm{d}y = 0$$

$$\int \mathrm{d}y \quad \boxed{-y^2/x} \quad \frac{\partial}{\partial x} \qquad \int \mathrm{d}y \downarrow \quad \boxed{3\ln\left|y\right|}$$

$$\therefore \quad -\frac{y^2}{x} + 3\ln\left|y\right| = C \implies y^3 = C\mathrm{e}^{y^2/x}$$

✍ $y=0$ is included in the general solution with $C=0$.

Problem 2.137

Solve $\qquad (y'+1)\ln\dfrac{y+x}{x+3} = \dfrac{y+x}{x+3}$

Point of intersection of the two lines

$$x+3=0, \quad y+x=0 \implies x=-3, \quad y=3.$$

Let $\;x=X-3,\; y=Y+3 \implies y' = \dfrac{\mathrm{d}Y}{\mathrm{d}X}$

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}X}+1\right)\ln\frac{(Y+3)+(X-3)}{(X-3)+3} = \frac{(Y+3)+(X-3)}{(X-3)+3}$$

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}X}+1\right)\ln\left(\frac{Y}{X}+1\right) = \frac{Y}{X}+1 \qquad \text{✍ Homogeneous}$$

Let $\;u=\dfrac{Y}{X} \implies Y = Xu \implies \dfrac{\mathrm{d}Y}{\mathrm{d}X} = u + X\dfrac{\mathrm{d}u}{\mathrm{d}X}$

$$u + X\frac{\mathrm{d}u}{\mathrm{d}X} + 1 = \frac{u+1}{\ln(u+1)} \implies X\frac{\mathrm{d}u}{\mathrm{d}X} = (u+1)\,\frac{1-\ln(u+1)}{\ln(u+1)}$$

Case 1. $\quad 1-\ln(u+1) = 0 \implies u+1 = \mathrm{e} \implies \dfrac{y+x}{x+3} = \mathrm{e}$

Case 2. $\quad \displaystyle\int \frac{\ln(u+1)}{(u+1)\big[\ln(u+1)-1\big]}\,\mathrm{d}u = -\int\frac{\mathrm{d}X}{X} + C$

$$\text{LHS} = \int\frac{\ln(u+1)}{\ln(u+1)-1}\,\mathrm{d}\big[\ln(u+1)\big] = \int\frac{v}{v-1}\,\mathrm{d}v, \quad v = \ln(u+1)$$

$$= \int\frac{(v-1)+1}{v-1}\,\mathrm{d}v = \int\left(1+\frac{1}{v-1}\right)\mathrm{d}v = v + \ln\left|v-1\right|$$

$$= \ln(u+1) + \ln\left|\ln(u+1)-1\right| = \ln\left|(u+1)\big[\ln(u+1)-1\big]\right|$$

$$\therefore \quad \ln\left|(u+1)\big[\ln(u+1)-1\big]\right| = \ln\left|\frac{C}{X}\right| \implies (u+1)\big[\ln(u+1)-1\big] = \frac{C}{X}$$

$$\ln(u+1) = 1 + \frac{C}{(u+1)X} \implies \ln\frac{y+x}{x+3} = 1 + \frac{C}{x+y}$$

✍ $\dfrac{y+x}{x+3}=\mathrm{e}$ is included in the general solution with $C=0$.

Problem 2.138

Solve $\quad 2x^3yy'+3x^2y^2+7=0$

$$y'+\frac{3}{2x}\cdot y=-\frac{7}{2x^3}\cdot\frac{1}{y} \implies 2yy'+\frac{3}{x}y^2=-\frac{7}{x^3} \quad \text{✍ Bernoulli DE}$$

Let $u=y^2 \implies \dfrac{\mathrm{d}u}{\mathrm{d}x}=2y\dfrac{\mathrm{d}y}{\mathrm{d}x} \implies \dfrac{\mathrm{d}u}{\mathrm{d}x}+\dfrac{3}{x}\cdot u=-\dfrac{7}{x^3}$ ✍ Linear first-order

$$P(x)=\frac{3}{x},\quad Q(x)=-\frac{7}{x^3},\quad \int P(x)\,\mathrm{d}x=\int\frac{3}{x}\,\mathrm{d}x=3\ln|x|$$

$$\mathrm{e}^{\int P(x)\,\mathrm{d}x}=\mathrm{e}^{3\ln|x|}=x^3,\quad \int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x=\int-\frac{7}{x^3}\cdot x^3\,\mathrm{d}x=-7x$$

$$\therefore\ u=\mathrm{e}^{-\int P(x)\,\mathrm{d}x}\left[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x+C\right]=\frac{1}{x^3}(-7x+C)$$

$$\therefore\ y^2=-\frac{7}{x^2}+\frac{C}{x^3} \implies x^3y^2+7x=C$$

Problem 2.139

Solve $\quad \left(x-y\cos\dfrac{y}{x}\right)\mathrm{d}x+x\cos\dfrac{y}{x}\,\mathrm{d}y=0$

$$\cos\frac{y}{x}\neq 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1-\dfrac{y}{x}\cos\dfrac{y}{x}}{\cos\dfrac{y}{x}} \quad \text{✍ Homogeneous}$$

Let $u=\dfrac{y}{x} \implies y=xu \implies \dfrac{\mathrm{d}y}{\mathrm{d}x}=u+x\dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\therefore\ u+x\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{1}{\cos u}+u \quad \text{✍ Variable separable}$$

$$\int\cos u\,\mathrm{d}u=-\int\frac{\mathrm{d}x}{x}+C \implies \sin u=-\ln|x|+\ln|C| \implies \sin\frac{y}{x}=\ln\left|\frac{C}{x}\right|$$

Problem 2.140

Solve $\quad x^2(x\,\mathrm{d}y-y\,\mathrm{d}x)=(x+y)\,y\,\mathrm{d}x$

Divided the DE by x^2y^2: $\quad \dfrac{x\,\mathrm{d}y-y\,\mathrm{d}x}{y^2}=\dfrac{(x+y)\,y}{x^2\cdot y^2}\mathrm{d}x$

$$-\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{y^2}=\frac{1}{x^2}\left(\frac{x}{y}+1\right)\mathrm{d}x \implies \mathrm{d}\left(\frac{x}{y}\right)=-\frac{1}{x^2}\left(\frac{x}{y}+1\right)\mathrm{d}x$$

$$\int \frac{1}{\frac{x}{y}+1}\,\mathrm{d}\Big(\frac{x}{y}+1\Big) = -\int \frac{1}{x^2}\,\mathrm{d}x + C \implies \ln\Big|\frac{x}{y}+1\Big| = \frac{1}{x} + C$$

Problem 2.141

Solve $\quad (y^4+xy)\,\mathrm{d}x + (xy^3-x^2)\,\mathrm{d}y = 0$

$$(y^4\mathrm{d}x+xy^3\mathrm{d}y)+(xy\,\mathrm{d}x-x^2\mathrm{d}y)=0 \implies y^3(y\,\mathrm{d}x+x\,\mathrm{d}y)+x(y\,\mathrm{d}x-x\,\mathrm{d}y)=0$$

Case 1. $\quad y=0$ is a solution of the differential equation.

Case 2. $\quad y\neq 0: \quad (y\,\mathrm{d}x+x\,\mathrm{d}y) + \dfrac{x}{y}\cdot\dfrac{y\,\mathrm{d}x-x\,\mathrm{d}y}{y^2} = 0$

$$\mathrm{d}(xy) + \frac{x}{y}\,\mathrm{d}\Big(\frac{x}{y}\Big) = 0 \implies \mathrm{d}(xy) + \tfrac{1}{2}\,\mathrm{d}\Big(\frac{x}{y}\Big)^2 = 0 \implies xy + \tfrac{1}{2}\Big(\frac{x}{y}\Big)^2 = C$$

Problem 2.142

Solve $\quad (x^2 + 3\ln y)\,\mathrm{d}x - \dfrac{x}{y}\,\mathrm{d}y = 0$

$$M(x) = x^2 + 3\ln y, \quad N(x) = -\frac{x}{y}, \quad \frac{\partial M}{\partial y} = \frac{3}{y}, \quad \frac{\partial N}{\partial x} = -\frac{1}{y}$$

$$\frac{1}{N}\Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big) = \frac{1}{-\frac{x}{y}}\Big(\frac{3}{y}+\frac{1}{y}\Big) = -\frac{4}{x}$$ ✍ A function of x only

$$\mu(x) = \exp\Big[\int \frac{1}{N}\Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big)\mathrm{d}x\Big] = \exp\Big(-\int \frac{4}{x}\,\mathrm{d}x\Big) = \mathrm{e}^{-4\ln x} = \frac{1}{x^4}$$

Multiplying the DE by $\mu(x) = \dfrac{1}{x^4} \implies \Big(\dfrac{1}{x^2} + \dfrac{3\ln y}{x^4}\Big)\mathrm{d}x - \dfrac{1}{x^3 y}\,\mathrm{d}y = 0$

$$\frac{1}{x^2}\,\mathrm{d}x - \frac{1}{x^3 y}\,\mathrm{d}y + \frac{3\ln y}{x^4}\,\mathrm{d}x = 0$$

$$\int \mathrm{d}x \downarrow \quad \boxed{-\frac{1}{x}} \qquad \int \mathrm{d}y \searrow \quad \boxed{-\frac{\ln y}{x^3}} \quad \nearrow \frac{\partial}{\partial x}$$

$$\therefore \quad -\frac{1}{x} - \frac{1}{x^3}\ln y = C \implies x^2 + \ln y = Cx^3$$

Problem 2.143

Solve $\quad xy'' = y' + x$

The DE is of the type y' absent. Let $y''=u,\ \ y'''=u'$.

$$\therefore\ \ x\,u' = u + x \implies u' = \frac{u}{x} + 1 \qquad \text{✍ Homogeneous DE}$$

$$\text{Let}\ \ v = \frac{u}{x} \implies u = xv \implies \frac{\mathrm{d}u}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\therefore\ v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + 1 \implies \int \mathrm{d}v = \int \frac{\mathrm{d}x}{x} + C \qquad \text{✍ Variable separable}$$

$$v = \ln|x| + C \implies \frac{u}{x} = \ln|x| + C \implies u = \frac{\mathrm{d}y}{\mathrm{d}x} = x\,(\ln|x| + C)$$

$$y = \int x\,(\ln|x| + C_1)\,\mathrm{d}x + C_2 = \int (\ln|x| + C_1)\cdot\tfrac{1}{2}\mathrm{d}(x^2) + C_2$$

$$= \tfrac{1}{2}\Big[x^2(\ln|x|+C_1) - \int x^2\cdot\frac{1}{x}\,\mathrm{d}x\Big] + C_2 = \tfrac{1}{2}\Big[x^2(\ln|x|+C_1) - \tfrac{1}{2}x^2\Big] + C_2$$

$$\therefore\ y = \frac{x^2}{4}(2\ln|x| + C_1) + C_2$$

Problem 2.144

Solve $\quad y\,\mathrm{d}x + (xy - x - y^3)\,\mathrm{d}y = 0$

Case 1. $\ y=0\ $ is a solution.

Case 2. $\ y\neq 0 \implies \dfrac{\mathrm{d}x}{\mathrm{d}y} + \dfrac{y-1}{y}\cdot x = y^2, \quad P(y) = \dfrac{y-1}{y}, \quad Q(y) = y^2$

$$\int P(y)\,\mathrm{d}y = \int\Big(1 - \frac{1}{y}\Big)\mathrm{d}y = y - \ln|y|, \quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{y-\ln|y|} = \frac{1}{y}\mathrm{e}^{y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int y^2\cdot\frac{1}{y}\mathrm{e}^{y}\,\mathrm{d}y = \int y\,\mathrm{e}^{y}\,\mathrm{d}y = \mathrm{e}^{y}(y-1)$$

$$\therefore\ \ x = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big]$$

$$= y\,\mathrm{e}^{-y}\big[\mathrm{e}^{y}(y-1) + C\big] = y\,(y-1) + C\,y\,\mathrm{e}^{-y}$$

Problem 2.145

Solve $\quad y + 2y^3y' = (x + 4y\ln y)\,y'$

$$y\,\mathrm{d}x + (2y^3 - x - 4y\ln y)\,\mathrm{d}y = 0, \quad M = y, \quad N = 2y^3 - x - 4y\ln y$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -1 \qquad \text{✍ Not exact DE}$$

$$\frac{1}{M}\Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) = \frac{-1-1}{y} = -\frac{2}{y}, \quad \mu(y) = \exp\Big(-\int\frac{2}{y}\,\mathrm{d}y\Big) = \mathrm{e}^{-2\ln y} = \frac{1}{y^2}$$

Multiply the DE by $\mu(y)$: $\quad \dfrac{1}{y}\,\mathrm{d}x + \Big(2y - \dfrac{x}{y^2} - \dfrac{4\ln y}{y}\Big)\mathrm{d}y = 0$

$$\frac{1}{y}\,\mathrm{d}x \quad - \quad \frac{x}{y^2}\,\mathrm{d}y \quad + \quad \left(2y - \frac{4\ln y}{y}\right)\mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{x/y} \quad \frac{\partial}{\partial y} \qquad\qquad \int \mathrm{d}x \quad \boxed{y^2 - 2\,(\ln y)^2}$$

General solution: $\quad \dfrac{x}{y} + y^2 - 2\,(\ln y)^2 = C$

Problem 2.146

Solve $\quad y \ln x \ \ln y \, \mathrm{d}x + \mathrm{d}y = 0$

$$y \ln x \ln y \,\mathrm{d}x = -\mathrm{d}y \implies \int \ln x \,\mathrm{d}x = \int -\frac{1}{y \ln y}\,\mathrm{d}y + C \quad ✍ \text{ Separable}$$

$$x \ln x - x = -\int \frac{1}{\ln y}\,\mathrm{d}(\ln y) + C \implies x \ln x - x = -\ln\big|\ln y\big| + C$$

Problem 2.147

Solve $\quad (2x\sqrt{x} + x^2 + y^2)\,\mathrm{d}x + 2y\sqrt{x}\,\mathrm{d}y = 0$

$$2\sqrt{x}\,(x\,\mathrm{d}x + y\,\mathrm{d}y) + (x^2+y^2)\,\mathrm{d}x = 0 \implies 2\,\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2+y^2} + \frac{1}{\sqrt{x}}\,\mathrm{d}x = 0$$

$$2\,\mathrm{d}\left[\tfrac{1}{2}\ln(x^2+y^2)\right] + \mathrm{d}(2\sqrt{x}) = 0 \implies \ln(x^2+y^2) + 2\sqrt{x} = C$$

Problem 2.148

Solve $\quad \big[2x + y\cos(xy)\big]\,\mathrm{d}x + x\cos(xy)\,\mathrm{d}y = 0$

$$M = 2x + y\cos(xy), \qquad N = x\cos(xy)$$

$$\frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy) = \frac{\partial N}{\partial x} \qquad ✍ \text{ Exact DE}$$

$$2x\,\mathrm{d}x \quad + \quad y\cos(xy)\,\mathrm{d}x \quad + \quad x\cos(xy)\,\mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{x^2} \qquad\qquad \int \mathrm{d}x \quad \boxed{\sin(xy)} \quad \frac{\partial}{\partial y}$$

General solution: $\quad x^2 + \sin(xy) = C$

Problem 2.149

Solve $\quad yy'' - y^2y' - y'^2 = 0$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} \implies y'' = \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x} = u\cdot\dfrac{\mathrm{d}u}{\mathrm{d}y}$

$$\therefore\quad y\cdot u\frac{\mathrm{d}u}{\mathrm{d}y} - y^2u - u^2 = 0$$

Case 1. $u=0 \implies y'=0 \implies y=C$

Case 2. $u\neq 0 \implies y\dfrac{\mathrm{d}u}{\mathrm{d}y} - y^2 - u = 0$ ✍ $y=0$ is included in Case 1

$$\therefore\ \frac{\mathrm{d}u}{\mathrm{d}y} - \frac{1}{y}\cdot u = y,\quad P(y) = -\frac{1}{y},\quad Q(y) = y$$ ✍ Linear first-order

$$\int P(y)\,\mathrm{d}y = \int -\frac{1}{y}\,\mathrm{d}y = -\ln|y|,\quad \mathrm{e}^{\int P(y)\,\mathrm{d}y} = \mathrm{e}^{-\ln|y|} = \frac{1}{y}$$

$$\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y = \int y\cdot\frac{1}{y}\,\mathrm{d}y = y$$

$$\therefore\ u = \mathrm{e}^{-\int P(y)\,\mathrm{d}y}\Big[\int Q(y)\,\mathrm{e}^{\int P(y)\,\mathrm{d}y}\,\mathrm{d}y + C\Big] = y(y+C) = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\because\ y\neq C \implies \int\frac{\mathrm{d}y}{y(y+C_1)} = \int\mathrm{d}x + C_2$$

$$\frac{1}{C_1}\int\Big(\frac{1}{y} - \frac{1}{y+C_1}\Big)\mathrm{d}y = x + C_2 \implies \frac{1}{C_1}\ln\left|\frac{y}{y+C_1}\right| = x + C_2$$

Problem 2.150

Solve $\quad 2y' + x = 4\sqrt{y}$

Let $y = u^2 \implies y' = 2u\cdot u' \implies 2\cdot 2u\cdot u' + x = 4u$

$$4\frac{u}{x}u' + 1 = 4\frac{u}{x}$$ ✍ Homogeneous

Let $\dfrac{u}{x} = v \implies u = xv \implies u' = v + xv'$

$$\therefore\ 4v(v+xv') + 1 = 4v \implies 4xv\cdot v' = -(2v-1)^2$$

Case 1. $2v-1=0 \implies 2\dfrac{u}{x} - 1 = 0 \implies 2\dfrac{\sqrt{y}}{x} - 1 = 0 \implies 2\sqrt{y} = x$

Case 2. $2v-1\neq 0 \implies 2\displaystyle\int\frac{2v-1+1}{(2v-1)^2}\,\mathrm{d}v = -\int\frac{\mathrm{d}x}{x} + C$

$$\int\Big[\frac{1}{(2v-1)} + \frac{1}{(2v-2)^2}\Big]\mathrm{d}(2v-1) = -\ln|x| + C$$

$$\ln|2v-1| - \frac{1}{2v-1} = -\ln|x| - \ln|C|$$

$$\ln\left|Cx\left(2\frac{\sqrt{y}}{x}-1\right)\right| = \frac{1}{2\frac{\sqrt{y}}{x}-1} \implies (2\sqrt{y}-x)\,\ln\left|C(2\sqrt{y}-x)\right| = x$$

Problem 2.151

Solve $\quad 2y'^3 - 3y'^2 + x = y$

Let $y' = p \implies y = 2p^3 - 3p^2 + x = f(x,p)$

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx} = 1 + (6p^2 - 6p)\frac{dp}{dx}$$

$$p = 1 + 6p(p-1)\frac{dp}{dx} \implies (p-1)\left(1 - 6p\frac{dp}{dx}\right) = 0$$

Case 1. $\quad p=1 \implies y = 2p^3 - 3p^2 + x = 2\cdot 1^3 - 3\cdot 1^2 + x = x - 1$

Case 2. $\quad 1 - 6p\dfrac{dp}{dx} = 0$ ✍ Variable separable

$$\int 6p\,dp = \int dx + C \implies 3p^2 = x + C \implies p^2 = \frac{x+C}{3}$$

$$y = 2p^3 - 3p^2 + (3p^2 - C) = 2p^3 - C \implies p^3 = \frac{y+C}{2}$$

$$\therefore\; p^6 = \left(\frac{x+C}{3}\right)^3 = \left(\frac{y+C}{2}\right)^2 \implies 4(x+C)^3 = 27(y+C)^2$$

Problem 2.152

Solve $\quad y' - 6x\,e^{x-y} - 1 = 0$

Let $\;u = x - y \implies \dfrac{du}{dx} = 1 - \dfrac{dy}{dx} \implies y' = 1 - \dfrac{du}{dx}$

$$\therefore\; 1 - \frac{du}{dx} - 6x\,e^u - 1 = 0 \implies \frac{du}{dx} = -6x\,e^u$$ ✍ Variable separable

$$\int -e^{-u}\,du = \int 6x\,dx + C \implies e^{-u} = 3x^2 + C \implies e^{-(x-y)} = 3x^2 + C$$

Problem 2.153

Solve $\quad (1+y^2)y'' + y'^3 + y' = 0$

The DE is of the type x absent. Let y be the new independent variable.

The new dependent variable is $u = y' = \dfrac{dy}{dx} \implies y'' = \dfrac{du}{dx} = \dfrac{du}{dy}\cdot\dfrac{dy}{dx} = u\cdot\dfrac{du}{dy}$

$$(1+y^2)\,u\frac{du}{dy} + u^3 + u^2 = 0$$ ✍ Variable separable

Case 1. $u=0 \implies \dfrac{dy}{dx}=0 \implies y=C$

Case 2. $u\neq 0 \implies \displaystyle\int \frac{du}{u^2+1} = -\int \frac{dy}{1+y^2}+C \implies \tan^{-1}u = -\tan^{-1}y + C$

$$u=\tan(-\tan^{-1}y+C)=\frac{\tan(-\tan^{-1}y)+\tan C}{1-\tan(-\tan^{-1}y)\tan C}$$

$$\frac{dy}{dx}=\frac{-y+C_1}{1+C_1 y} \implies -\int \frac{C_1 y+1}{y-C_1}dy=\int dx + C_2 \quad \text{✍ Variable separable}$$

$$-\int\Big(C_1+\frac{1+C_1^2}{y-C_1}\Big)dy = x+C_2 \implies -C_1 y-(1+C_1^2)\ln\big|y-C_1\big| = x+C_2$$

Problem 2.154

Solve $(y\sin x+\cos^2 x)\,dx-\cos x\,dy=0$

$$\frac{dy}{dx}-\tan x\cdot y=\cos x,\quad P(x)=-\tan x,\quad Q(x)=\cos x \quad \text{✍ Linear first-order}$$

$$\int P(x)\,dx=\int -\tan x\,dx=\ln\big|\cos x\big|,\quad e^{\int P(x)\,dx}=e^{\ln|\cos x|}=\cos x$$

$$\int Q(x)\,e^{\int P(x)\,dx}dx=\int \cos x\cdot\cos x\,dx=\int\frac{1+\cos 2x}{2}dx=\frac{x}{2}+\frac{\sin 2x}{4}$$

$$\therefore\ y=e^{-\int P(x)\,dx}\Big[\int Q(x)\,e^{\int P(x)\,dx}dx+C\Big]=\frac{1}{\cos x}\Big(\frac{x}{2}+\frac{\sin 2x}{4}+C\Big)$$

Problem 2.155

Solve $y(6y^2-x-1)\,dx+2x\,dy=0$

$$\frac{dy}{dx}-\frac{x+1}{2x}\cdot y=-\frac{3}{x}\cdot y^3 \quad \text{✍ Bernoulli DE}$$

Case 1. $y=0$ is a solution.

Case 2. $y\neq 0 \implies y^{-3}\dfrac{dy}{dx}-\dfrac{x+1}{2x}y^{-2}=-\dfrac{3}{x}$

Let $u=y^{-2}$, $\dfrac{du}{dx}=-2y^{-3}\dfrac{dy}{dx}$

$$\therefore\ \frac{du}{dx}+\frac{x+1}{x}u=\frac{6}{x},\quad P(x)=\frac{x+1}{x},\quad Q(x)=\frac{6}{x} \quad \text{✍ Linear first-order}$$

$$\int P(x)\,dx=\int\frac{x+1}{x}dx=x+\ln|x|,\quad e^{\int P(x)\,dx}=e^{x+\ln|x|}=x\,e^x$$

$$\int Q(x)\,e^{\int P(x)\,dx}dx=\int\frac{6}{x}\cdot x e^x\,dx=6e^x$$

$$u = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\Big[\int Q(x)\,\mathrm{e}^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\Big] = \frac{1}{x\,\mathrm{e}^x}\,(6\mathrm{e}^x + C) = \frac{6}{x} + \frac{C}{x\,\mathrm{e}^x}$$

$$\therefore \quad \frac{x}{y^2} = 6 + C\mathrm{e}^{-x}$$

Problem 2.156

Solve $\quad y'\,(x - \ln y') = 1$

Let $y' = p \implies p\,(x - \ln p) = 1 \implies x = \dfrac{1}{p} + \ln p = g(p)$

Differentiate with respect to y: $\dfrac{\mathrm{d}x}{\mathrm{d}y} = g_y + g_p\,\dfrac{\mathrm{d}p}{\mathrm{d}y} \implies \dfrac{1}{p} = \Big(-\dfrac{1}{p^2} + \dfrac{1}{p}\Big)\dfrac{\mathrm{d}p}{\mathrm{d}y}$

$$\int \mathrm{d}y = \int\Big(1 - \frac{1}{p}\Big)\,\mathrm{d}p + C$$ ✍ Variable separable

$$y = p - \ln p + C \implies x = \frac{1}{p} + \ln p, \quad y = p - \ln p + C$$

Problem 2.157

Solve $\quad (1 + \cos x)\,y' + \sin x\,(\sin x + \sin x\cos x - y) = 0$

$$\underbrace{(\sin^2 x + \sin^2 x\cos x - y\sin x)}_{M}\,\mathrm{d}x + \underbrace{(1 + \cos x)}_{N}\,\mathrm{d}y = 0$$

$$\frac{\partial M}{\partial y} = -\sin x, \quad \frac{\partial N}{\partial x} = -\sin x \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ ✍ Exact DE

$$(\sin^2 x + \sin^2 x\cos x)\,\mathrm{d}x - \sin x \cdot y\,\mathrm{d}x \quad + \quad \cos x\,\mathrm{d}y \; + \mathrm{d}y = 0$$

$$\int \mathrm{d}x \quad \boxed{y\cos x} \quad \frac{\partial}{\partial y}$$

$$\int(\sin^2 x + \sin^2 x\cos x)\,\mathrm{d}x = \int \frac{1 - \cos 2x}{2}\,\mathrm{d}x + \int \sin^2 x\,\mathrm{d}(\sin x)$$
$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{3}\sin^3 x$$

$$\therefore \quad \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{3}\sin^3 x + y\cos x + y = C$$

Problem 2.158

Solve $\quad x\,\mathrm{d}x + \sin^2\Big(\dfrac{y}{x}\Big)\,(y\,\mathrm{d}x - x\,\mathrm{d}y) = 0$

$$\Big(x + y\,\sin^2\frac{y}{x}\Big)\,\mathrm{d}x = x\,\sin^2\frac{y}{x}\,\mathrm{d}y \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \dfrac{y}{x}\sin^2\dfrac{y}{x}}{\dfrac{y}{x}}$$ ✍ Homogeneous

Let $u = \dfrac{y}{x} \implies y = xu \implies \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

$$\therefore\ u + x\frac{du}{dx} = \frac{1+u\sin^2 u}{\sin^2 u} \implies x\frac{du}{dx} = \frac{1}{\sin^2 u}$$

✍ Variable separable

$$\int \frac{1-\cos 2u}{2}\,du = \int \frac{1}{x}\,dx + C \implies \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u = \ln|x| + C$$

$$\therefore\ \frac{y}{2x} - \frac{1}{4}\sin\frac{2y}{x} = \ln\left|\frac{y}{x}\right| + C$$

Problem 2.159

Solve $\underbrace{(2xy^4e^y + 2xy^3 + y)}_{M}\,dx + \underbrace{(x^2y^4e^y - x^2y^2 - 3x)}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1, \qquad \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-4(2xy^3e^y + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)} = -\frac{4}{y}$$

$$\mu(y) = \exp\left(-\int \frac{4}{y}\,dy\right) = e^{-4\ln|y|} = \frac{1}{y^4}$$

Multiply the DE by $\mu(y)$: $\left(2xe^y + \dfrac{2x}{y} + \dfrac{1}{y^3}\right)dx + \left(x^2e^y - \dfrac{x^2}{y^2} - \dfrac{3x}{y^4}\right)dy = 0$

$$2xe^y\,dx + x^2e^y\,dy + \frac{2x}{y}\,dx - \frac{x^2}{y^2}\,dy + \frac{1}{y^3}\,dx - \frac{3x}{y^4}\,dy = 0$$

$$\int dx \quad \boxed{x^2e^y} \quad \frac{\partial}{\partial y} \qquad \int dx \quad \boxed{\frac{x^2}{y}} \quad \frac{\partial}{\partial y} \qquad \int dx \quad \boxed{\frac{x}{y^3}} \quad \frac{\partial}{\partial y}$$

General solution: $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = C$

Problem 2.160

Solve $(xy^3 - 1)\,dx + x^2y^2\,dy = 0$

$$(xy^3dx + x^2y^2dy) - dx = 0 \implies xy^2(y\,dx + x\,dy) - dx = 0$$

Multiply the DE by x: $x^2y^2\,d(xy) - x\,dx = 0$

$$\tfrac{1}{3}d(xy)^3 - \tfrac{1}{2}d(x^2) = 0 \implies \tfrac{1}{3}(xy)^3 - \tfrac{1}{2}(x^2) = 0 \implies 2x^3y^3 - 3x^2 = C$$

✍ $\mu(x) = x$ is an integrating factor.