CHAPTER 2

ONE DIMENSIONAL UNCONSTRAINED MINIMIZATION

P2.1

(i) From Definition in Eq. (2.7), we need to show that for any x, y and $0 < \alpha < 1$,

$$(\alpha\, x + (1-\alpha)\, y\,)^2 \le \alpha\, x^2 + (1-\alpha)\, y^2.$$

From the triangular inequality,

$$|\alpha\, x + (1-\alpha)\, y| \le \alpha\, |x| + (1-\alpha)\, |y|$$

Squaring,

$$\begin{aligned}(\alpha\, x + (1-\alpha)\, y)^2 &\le \alpha^2\, x^2 + (1-\alpha)^2\, y^2 + 2\alpha(1-\alpha)\, x\, y \\ &= \alpha\, x^2 - \alpha\, x^2 + \alpha^2\, x^2 \\ &\quad + (1-\alpha)\, y^2 - (1-\alpha)\, y^2 + (1-\alpha)^2\, y^2 \\ &\quad + 2\alpha(1-\alpha)\, x\, y \\ &= \alpha\, x^2 + (1-\alpha)\, y^2 \\ &\quad - \alpha\,(1-\alpha)\,(x-y)^2 \\ &\le \alpha\, x^2 + (1-\alpha)\, y^2\end{aligned}$$

as was required to be shown.

(ii) Using the C^1 test for convexity, we need to show that for any x, y,

$$f(x) + f'(x)\,(y-x) \le f(y)$$

or to show that

$$x^2 + 2\,x\,(y-x) \le y^2$$

or

$$-\,(x - y)^2 \le 0$$

which is evident.

(iii) Using the C^2 test for convexity, we need to show that $f''(x) \ge 0$ for all x. This follows since $f''(x) = 2 > 0$. In fact, this shows that $f = x^2$ is a <u>strictly</u> convex function.

P2.2 Necessary condition (for x^* to be a strict local maximum): $f'(x^*) = 0$

Sufficiency condition: $f''(x^*) < 0$.

P2.3

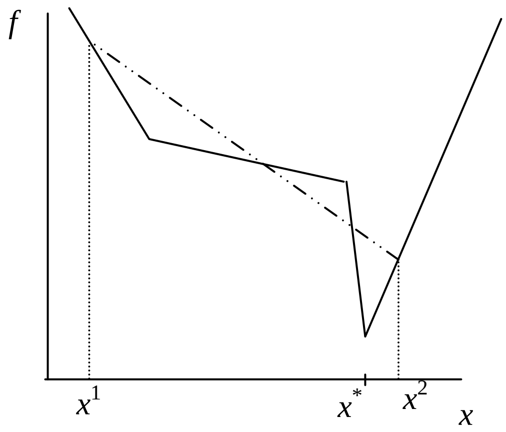


Figure P2.3

f is not convex – for the pair of points x^1 and x^2 as shown, the chord does not lie entirely above the function (which violates Eq. (2.7)).

f is unimodal since there exists a unique point, x^*, where the function monotonically increases on either side of it.

P2.4

$$c' = \frac{-240}{V^2} + 10^{-4}\,(1.5)\,V^{0.5} = 0$$

$$(1.5)\,(10^{-4})\,V^{2.5} = 240$$

$$\textit{Taking}\ \log s, V = 303.14$$

$$c'' = \frac{240}{V^3} + \frac{0.75e-4}{V^{0.5}} > 0$$

Thus $V = 303.14$ is a strict local minimum and the corresponding cost is

$c_{\min} = 1.7685$

P2.5

$$E' = \frac{1.44}{r^2} - \frac{9 \; . \; 5.99e-6}{r^{10}} = 0$$

$$r^8 = 3.7438e-5$$

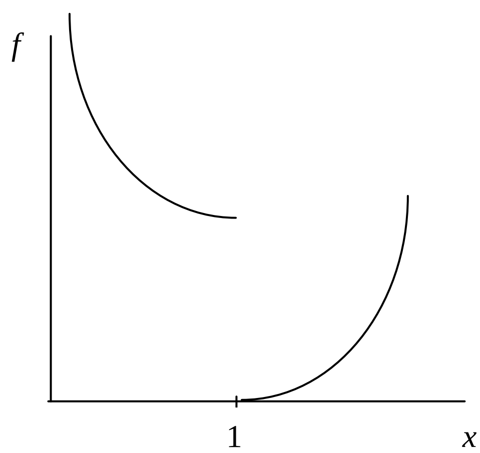
$$solving,\ r = 0.2797$$

$$E'' = \frac{-2.88}{r^3} + \frac{(9)(5.99)(10)e-6}{r^{11}} = -131.6 + 657.7 > 0$$

Thus r = 0.2797 nm is a strict local minimum and the corresponding energy is

$E_{\min} = -4.5852$.

P2.6



Although discontinuous, the function is unimodal. So, yes, Fibonacci and Golden Section methods will work given any interval of uncertainty containing $x = 1$.

P2.7 For $f = x_1^4 + x_2^4$, the Hessian is $H = \begin{bmatrix} 12x_1^2 & 0 \\ 0 & 12x_2^2 \end{bmatrix}$.

Now, $\mathbf{y}^T \mathbf{H}\, \mathbf{y} = 12(x_1^2 y_1^2 + x_2^2 y_2^2) \geq 0$ for all $\mathbf{x}$, hence $\mathbf{H}$ is positive semi-definite, and hence f is convex. Thus, any point satisfying the 1st order necessary condition is a global minimum.

P2.8

(i) *** Correction: n was not specified. Say, n = 10.

Then, the first two points as per Fibonacci are

$$\left[2-\frac{55}{89}(2-0), \frac{55}{89}(2-0)+0\right]=[0.764, 1.236]$$

(ii) As per Golden Section Search: $\left[2-\tau(2-0), \tau(2-0)+0\right]=[0.7639, 1.2361]$

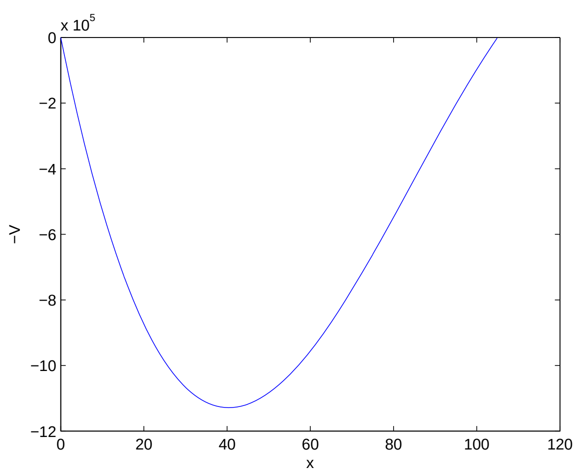
We see that since n is large, the two methods start out nearly the same.

(iii)

	Fibonacci	Golden Section
I_{10} / I_1	$1/F_n = 1/89 = 0.011236$	$\tau^{n-1} = 0.013156$

As per theory, Fibonacci gives smallest interval for fixed n.

P2.9 A) Consider the function $f = -V$, where $V = x\,(210 - 2x)\,(297 - 2x)$ = volume of open box (Example 2.4). Beyond x = 105, the box is not defined as a dimension becomes negative. Function f is twice–differentiable (i.e. in C^2), and so it is convex provided $f'' \geq 0$. We have $f'' = 2028 - 24\,x$ which is non–negative for $x \leq 84.5$. Thus, f is only convex on [0, 84.5]. It is not convex on R^1 nor on [0, 105]. However, the global minimum of f on [0, 105] is well defined. See plot below.



B) With 4 function evaluations, min $f = e^{3x} + 5e^{-2x}$, on the initial interval **[0, 1]** as per: (i) Fibonacci search, (ii) Golden Section search

Fibonacci

$n = 4$ gives $F_{n-1} / F_n = F_3 / F_4 = 3/5 = 0.6$. Thus the first two points are at $x = 0.4$ and $x = 0.6$, respectively. Evaluating f at these points gives $f(0.4) = 5.5668$ and $f(0.6) = 7.5556$. Comparing these two, the new interval is thus **[0, 0.6]**. $F_{n-2} / F_{n-1} = F_2 / F_3 = 2/3$. Thus the new point is at $x = 0.2$ with $x = 0.4$ already in the right location. $f(0.2) = 5.1737 < f(0.4)$. Thus the new interval is **[0, 0.4]**. We finish by choosing $\delta = 0.01$, which gives the two inside points as 0.2 and 0.21. $f(0.21) = 5.1628 < f(0.2) = 5.1737$.The final interval is thus [0.2, 0.4]. The length of this is 0.2, and its ratio to the original length = $0.2/1 = 1/F_n = 1/F_4 = 1/5$, as per theory.

Golden Section

The first two points are at $x = 0.618 = \tau$ and $x = 0.382 = 1- \tau$, respectively. Evaluating f at these points gives $f(0.618) = 7.8386$ and $f(0.382) = 5.4744$. Comparing these two, the new interval is thus **[0, 0.618]**. The new point is at $x = 0.2361$ with $x = 0.382$ already in the right location. $f(0.2361) = 5.1487 < f(0.382)$. Thus the new interval is **[0, 0.382]**. The new point is at $x = 0.1459$ with $x = 0.2361$ already in the right location. $f(0.1459) = 5.2837 > f(0.2361)$. Thus the new interval is **[0.1459, 0.382]**. The length of this is $0.2361 = \tau^{n-1} = \tau^3$ as per theory.

Further, the final interval length in Fibonacci is smaller than in Golden Section as per theory.

P2.10 The code used is

```
function [] = test1()
clear all; close all;
[xopt, fopt, ifl, out] = fminbnd(@(x) getfun(x),0,10)

function [f]= getfun(x)
f = exp(3*x) + 5*exp(-2*x);
```

The flag ifl = 1 means:
“FMINBND converged with a solution X based on OPTIONS.TolX.”

Solution is xopt = 0.2408, fopt = 5.1483.

$$f'(0.2408) = 3\,e^{3(0.2408)} - 10\,e^{-2(0.2408)} = 0.000168 \approx 0$$
$$f''(x) = 9e^{3x} + 20e^{2x} > 0$$

So, f is (strictly) convex and the solution is a (strict) global minimum.

P2.11 Using in-house code:

FIBONACCI

Getfun subroutine: `f = 240/x + 1e-4*x^1.5 + 0.45;`

Initial Interval	Final Interval	X-Value within interval	Function Value within interval	No. of Function Calls
[0, 1000]	[292.1, 303.4]	303.26	1.7695	10
[0, 500]	[297.8, 303.4]	303.37	1.7695	10

P2.12 Results from in-house codes:

GOLDINTV:

Getfun subroutine: `F = -1.44/X + 5.9e-6/X^9;`

Initial Interval	No. of Function Calls	Final Interval	X-Value within interval	Function Value within interval
[0, 100]	13	[0.1919, 0.5025]	0.31056	−4.4172
[0, 10]	13	[0.2942, 0.2631]	0.2823	−4.5828

GOLDLINE:

Getfun subroutine: `F = -1.44/X + 5.9e-6/X^9;`

Initial Point	No. of Function Calls	X-Value within interval	Function Value within interval
0.1	15	0.2793	−4.5853
1	17	0.2790	−4.5853
10	28	0.2790	−4.5853

P2.13 Using MATLAB optimization toolbox ‘fminbnd’ code:

```
function [] = test1()
clear all; close all;
[xopt, fopt, ifl, out] = fminbnd(@(x) getfun(x),0,1000)

function [f]= getfun(x)
f = 240/x + 1e-4*x^1.5 + 0.45;
```

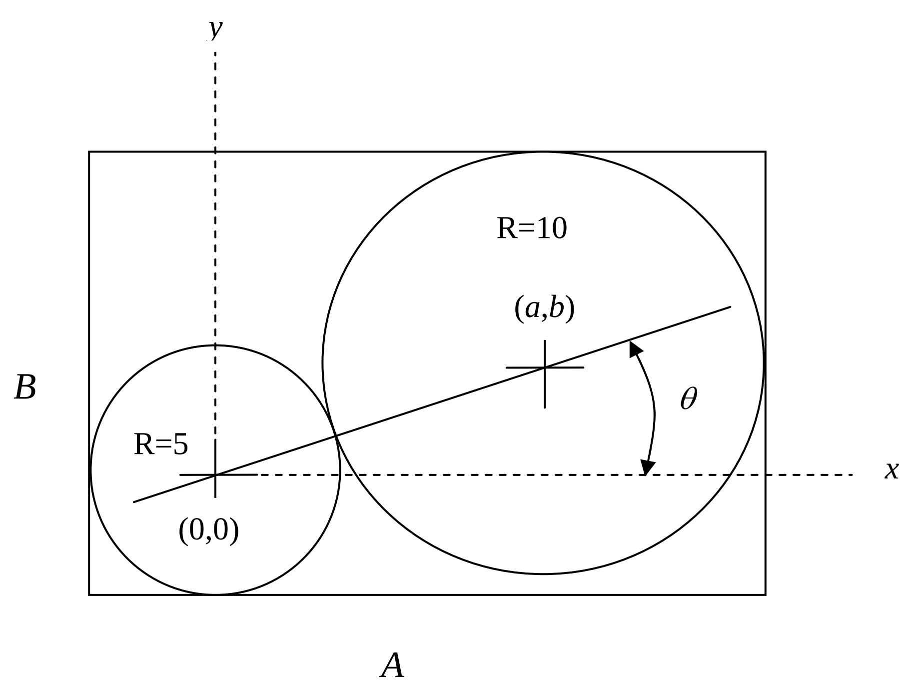
Matlab fminbnd

Getfun subroutine: `f = 240/x + 1e-4*x^1.5 + 0.45;`

Initial Interval	Xopt	Fopt	No. of Function Calls
[0, 1000]	303.14	1.7695	12

P2.14 For each configuration of the disks, we need to evaluate the function value which is the length of the boundary of the circumscribing rectangle for case (a) and the area enclosed by the rectangle for case (b). We proceed as follows.

With the origin set at (0,0) let the center of disk 1 of radius 5cm be set at the origin. Let disk 2 of radius 10 cm be placed so that it is in contact with disk 1 and the line joining the centers is at an angle θ as shown.



Denoting (a, b) as the location of the center of the second circle of radius 10cm, we have

$$(a, b) = (15\cos\theta, 15\sin\theta)$$

Angle θ has the limits

$$0 \le \theta \le 45^0 \qquad (\pi/4 \text{ radians})$$

The next step is to find x_{min}, y_{min}, x_{max}, y_{max} for the configuration. We set their values using the first circle as the base.

$$x_{min} = -5,\ y_{min} = -5,\ x_{max} = 5,\ y_{max} = 5$$

Then it is easy to see from the figure that the minimum and maximum values for the configurations can be obtained using the following 'if' statements.

If $x_{min} > a - 10$ then $x_{min} = a - 10$
If $y_{min} > b - 10$ then $y_{min} = b - 10$
If $x_{max} < a + 10$ then $x_{max} = a + 10$
If $y_{max} < b + 10$ then $y_{max} = b + 10$

Once the these four values are evaluated, the sides of the enclosing rectangle A and B are given by

$$A = x_{max} - x_{min} \qquad B = y_{max} - y_{min}$$

The function values for the two cases are given by

(a) $f = 2*(A + B)$

(b) $f = A*B$)

We now find the minima of f using the program GOLDINTV. We provide here the modifications for the GOLDINTV.BAS program.

In the main program, following two lines are added.

```
PI = 3.14159
A = 0: B = 45 * PI / 180
```

The first line defines π used in the degree-radian conversion and the second line replaces the current interval. Also the following print statement is modified to print degrees converted from radians.

```
PRINT "Coordinate of Point X2 = "; X * 180 / PI
```

In the subroutine `SUB GETFUN (X, F, NF)` following lines are added to define the function

```
SUB GETFUN (X, F, NF)
        'PROBLEM 1  CHAPTER 2
        A = 15 * COS(X)
        B = 15 * SIN(X)
        XMIN = -5: YMIN = -5
        XMAX = 5: YMAX = 5
        IF XMIN > A - 10 THEN XMIN = A - 10
        IF XMAX < A + 10 THEN XMAX = A + 10
        IF YMIN > B - 10 THEN YMIN = B - 10
        IF YMAX < B + 10 THEN YMAX = B + 10
        AA = XMAX - XMIN
        BB = YMAX - YMIN
        F = 2 * (AA + BB)
        'F = AA * BB
```

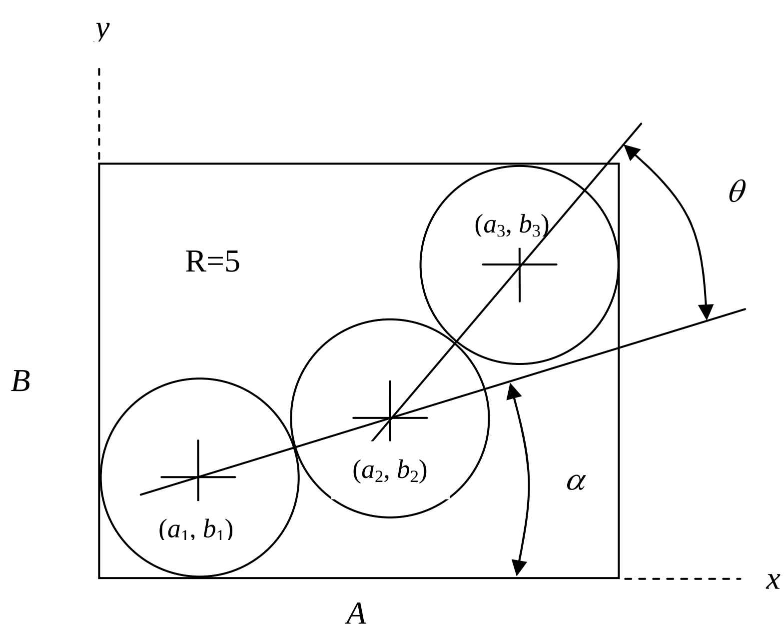
The second function above is a comment statement.

On running the program, following results are obtained

(a) minimum at $\theta = 19.4712^0$ $f_{\min} = 98.284\text{cm}$ 31 function evaluations
(b) minimum at $\theta = 19.4712^0$ $f_{\min} = 582.84\text{cm}^2$ 31 function evaluations

In both cases, the configuration for the minima corresponds to the one where the bottom tangent line to the two circles is parallel to the *x*-axis.

2.15 This is a two variable problem. We choose the two variables as α and θ as shown in the figure.



Because of the symmetry, the limits on these angles may be set as

$$0 \le \alpha \le 30^0 \qquad (\pi/4 \text{ radians})$$
$$0 \le \theta \le 120^0 \qquad (2\pi/3 \text{ radians})$$

We will attempt to solve this problem as a single variable problem by working in 5^0 steps for α and finding an optimum θ for each α. The centers of the three circles are denoted (a_i, b_i), i = 1,2,3. We place the center of the first disk at (5,5), so that

$$(a_1,b_1) = (5,5)$$
$$(a_2,b_2) = (a_1 + 10\cos\alpha,\ b_1 + 10\sin\alpha)$$
$$(a_3,b_3) = (a_2 + 10\cos(\alpha + \theta),\ b_2 + 10\sin(\alpha + \theta))$$

We initialize $x_{\min}$, $y_{\min}$, $x_{\max}$, $y_{\max}$ using the first circle as the base.

$$x_{\min} = 0,\ y_{\min} = 0,\ x_{\max} = 10,\ y_{\max} = 10$$

Then following 'if' statements will result in the final values of $x_{\min}$, $y_{\min}$, $x_{\max}$, $y_{\max}$.

i = 2,3
If $x_{\min} > a_i - 5$ then $x_{\min} = a_i - 5$
If $y_{\min} > b_i - 5$ then $y_{\min} = b_i - 5$
If $x_{\max} < a_i + 5$ then $x_{\max} = a_i + 5$
If $y_{\max} < b_i + 5$ then $y_{\max} = b_i + 5$

Now following the steps of problem 2.1,

$$A = x_{\max} - x_{\min} \qquad B = y_{\max} - y_{\min}$$

The function values for the two cases are given by

(a) $f = 2*(A + B)$

(b) $f = A*B)$

<u>Solution for case (a)</u>
Above relations are then implemented in the program GOLDINT as in Problem 2.1. The results are given below.

α^0	θ^0	$f_{\min}$ cm
0	120	77.3205
5	120	78.0501
10	120	78.4900
20	120	78.6371
25	120	78.4900
30	120	77.3205

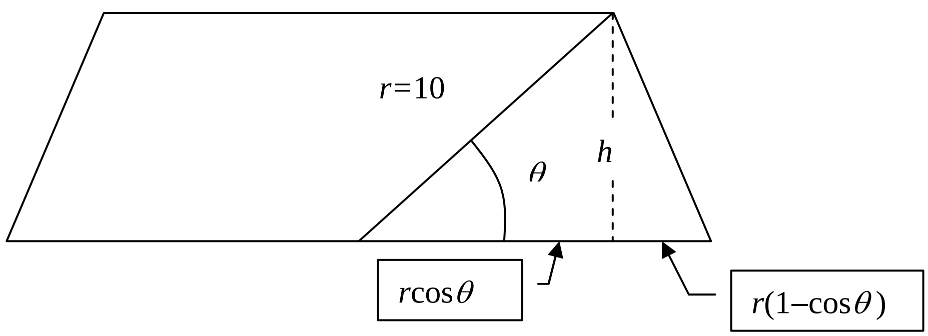
We note here that the minimum is at $\alpha = 0^0$ and $\theta = 120^0$. Also note the symmetry about $\alpha = 15^0$.

Solution for case (b)
The results for minimum area are given below.

α^0	θ^0	f_{min} cm^2
0	120	373.2053
5	120	380.5364
10	120	384.9920
20	120	386.4868
25	120	384.9921
30	120	373.2055

We note here that the minimum is at $\alpha = 0^0$ and $\theta = 120^0$.

2.16 We denote θ as the variable shown in the figure.



The height of the trapezoidal region can be written as

$$h = r \sin\theta$$

The area of the trapezoidal region f can now be written as

$$f = (2r \cos\theta)h + h\, r\, (1-\cos\theta)$$

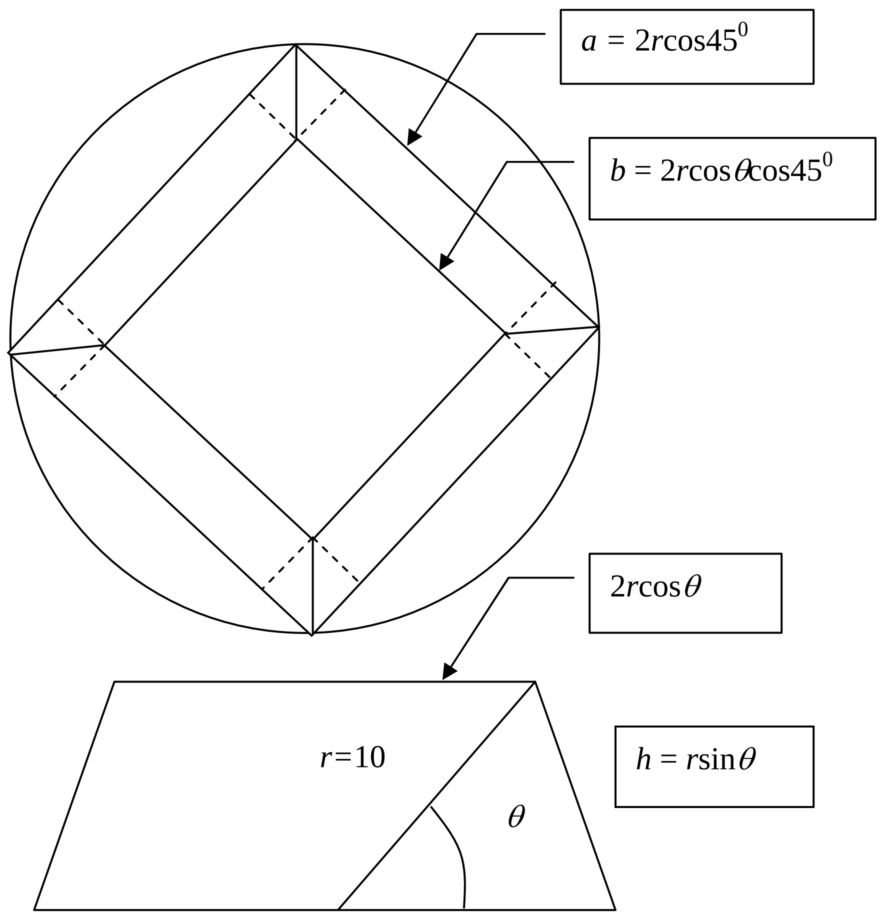
Above relationships can be introduced into the program GOLDINTV with θ ranging 0 to 0.5π and find the optimum solution. In this problem, we can obtain the solution directly by setting $df/d\theta = 0$. This will lead to $2\cos\theta = 1$. Thus

$$\theta = 60^0$$

maximum of $f = 129.9$.

The second derivative of f is negative at $\theta = 60^0$.

2.17 The geometry of the problem is shown in the figure.



We note that the height is h given by

$h = r\ \sin\theta$

The volume is divided into three distinct parts.

(1) Volume of the central square of side b

$$V_1 = b^2 h$$

(2) Volume of four prismatic parts of length b, height h, and width $0.5(a–b)$

$$V_2 = bh(a–b)$$

(3) Volume of four corner pieces which forms a pyramid of height h and base $(a–b)$

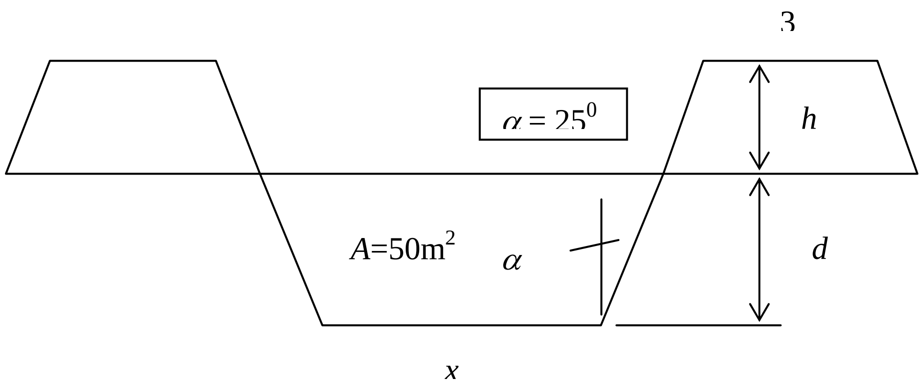
$$V_3 = (1/3)(a–b)^2 h$$

The objective is to maximize $f = V_1 + V_2 + V_3$

Above function has been introduced into the function part of the program GOLDINTV and the limits on θ have been set as 0^0 to 90^0. The results are as follows:

Maximum value of f = 1050.005 at $\theta = 49.723^0$.

2.18 This is a one variable problem, and we define d as the variable as shown in the figure.



Since the dirt dug from the ground is spread on the banks, we have

$$(x + d\tan\alpha)d = 50$$

and $$2\,(3 + h\tan\alpha)h = 50$$

The second equation is a quadratic equation in h. Solving the equation and choosing the meaningful solution, we get

$$h = \frac{\sqrt{9 + 100\tan\alpha} - 3}{2\tan\alpha}$$

From the first equation, we have

$$x = \frac{50}{d} - d\tan\alpha$$

The objective function f to be minimized is the wetted perimeter given by

$$f = x + \frac{(h+d)}{\tan\alpha}$$

On substituting for *h* and *x* from above, and differentiating with respect to *d* and setting it equal to zero (optimality condition), we get

$$d = \sqrt{50\frac{\cos\alpha}{1-\sin\alpha}}$$

Setting $\alpha = 25^0$, $d = 8.8591$, and the corresponding function value is $f = 16.563$.

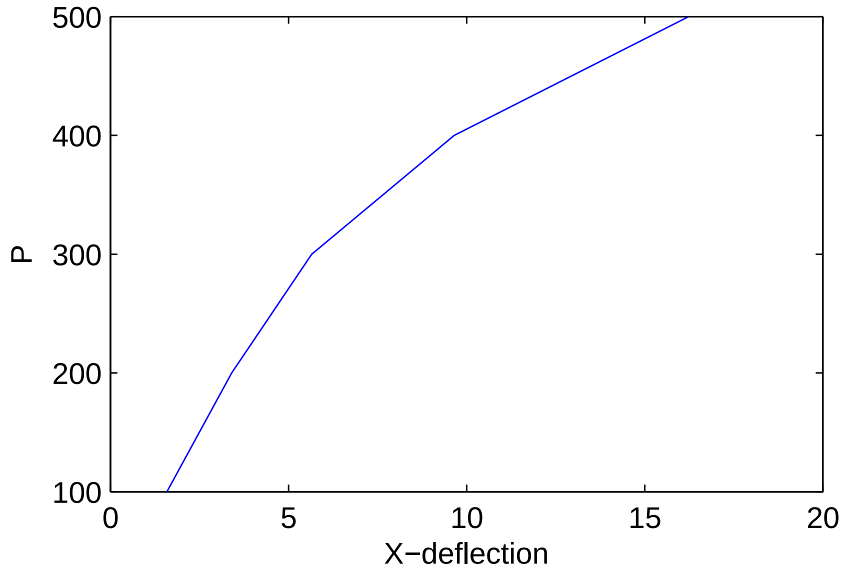
P2.19 Maximizing f = (17-2*x)*(22-2*x)*x*0.1 - 4*x*x*.04 - 4*x*.02 within the interval [0,8.5] gives an optimum height of the box as

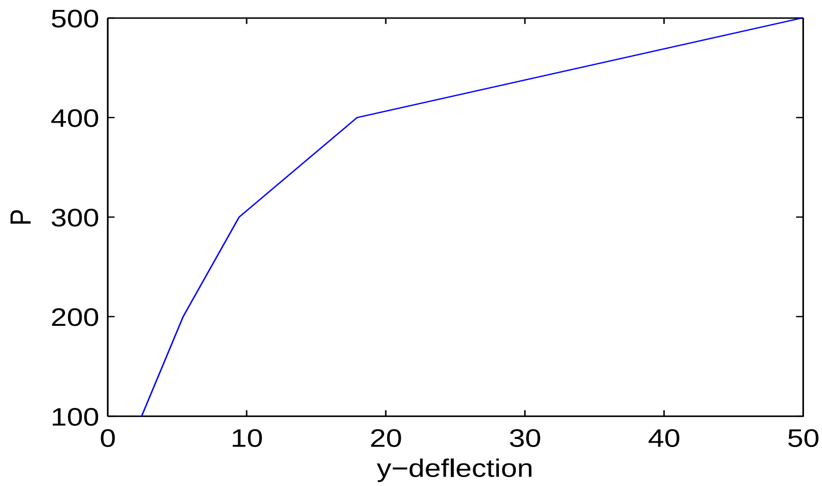
x^* = 3.0415, and profit f^* = $0.51, per box manufactured.

P2.20. A Matlab code for this problem is given below, which also generates the plots below.

```
function [] = FIBONACCI()
clear all; close all;
global P H A L k
%
H=50; L=300; k=100;
A=sqrt(H^2 + (L/2)^2);
xU = 2*A-L;
PP = 100*[1:5];
for i=1:length(PP)
   P = PP(i);
   [xopt, fopt, ifl, out] = fminbnd(@(x) getfun(x),0,xU)
   deltax(i) = xopt;
   deltay(i) = H - sqrt(A^2 - ((L+xopt)/2)^2);
end
plot(deltax,PP)
figure(2)
plot(deltay,PP)

function [f]= getfun(x)
global P H A L k
y = H - sqrt(A^2 - ((L+x)/2)^2);
f = .5*k*x^2 - P*y;
```





P2.21 2.10 The objective function is given. We need to be careful with the units. Note that 1μm = 10^{-3} mm (10^{-6} m). Using these units, the objective function can be defined as follows:

s_1 = 1000000 N/mm
s_2 = 600000 N/mm
E = 210000 N/mm^2 (Note that E for steel is 210 GPa = 210 N/m^2)
d = 75 mm
a = 100 mm

and $I = \pi d^4/64$

$$\text{minimize } f = \frac{a^3}{3EI}\left(1+\frac{1}{\alpha}\right) + \frac{(1+\alpha)^2}{s_1} + \frac{\alpha^2}{s_2}$$

The first term is large for small values of α and the second and third terms are large for large values of α. Choosing an interval of 0.001 to 10, we get

$$f_{\min} = 5.725\times10^{-6} \text{ at } \alpha = 0.4748$$

Note that $b/a = 1/\alpha = 2.11$. The spacing between bearings is larger than the overhang.

Machine tool spindles are made hollow. In this case, the moment of inertia is given by $I = \pi(d_o^4 - d_i^4)/64$.

P2.22 The deflection relationships for the left half of the beam configuration are given.

$$y = \frac{x^4}{24} + c_1 x + c_2 \qquad \text{for } x \le 1 - a$$

$$y = \frac{x^4}{24} - \frac{(x-1+a)^3}{6} + c_1 x + c_2 \qquad \text{for } x \ge 1 - a$$

First the constants c_1 and c_2 are to be determined using the boundary conditions. The deflection is zero at $x = 1 - a$ and the slope y' is zero at $x = 1$.

Using the slope condition, we get

$$c_1 = \frac{a^2}{2} - \frac{1}{6}$$

and using the deflection condition we get

$$c_2 = -\frac{(1-a)^4}{24} - c_1(1-a)$$

The deflection is largest either at $x = 0$ or $x = 1$. Thus the objective is to

$$\text{Minimize } f = \max(\text{abs}(y_{x=0}),\ \text{abs}(y_{x=1}))$$

Which can be put in the form

$$\text{Minimize } f = \max(\text{abs}(c_2),\ \text{abs}(1/24 - a^3/6 + c_1 + c_2))$$

Introducing these into a one dimensional minimization program and setting the interval for x as 0 to 1, we get

$$f_{\min} = 0.004316 \text{ at } a = 0.553702$$

Exact solution can be obtained by setting the deflections at $x = 0$ and $x = 1$ to be equal. This leads to the solution of the cubic equation $4a^3 - 12\,a^2 + 3 = 0$.

P2.23 Using the getfun subroutine below:

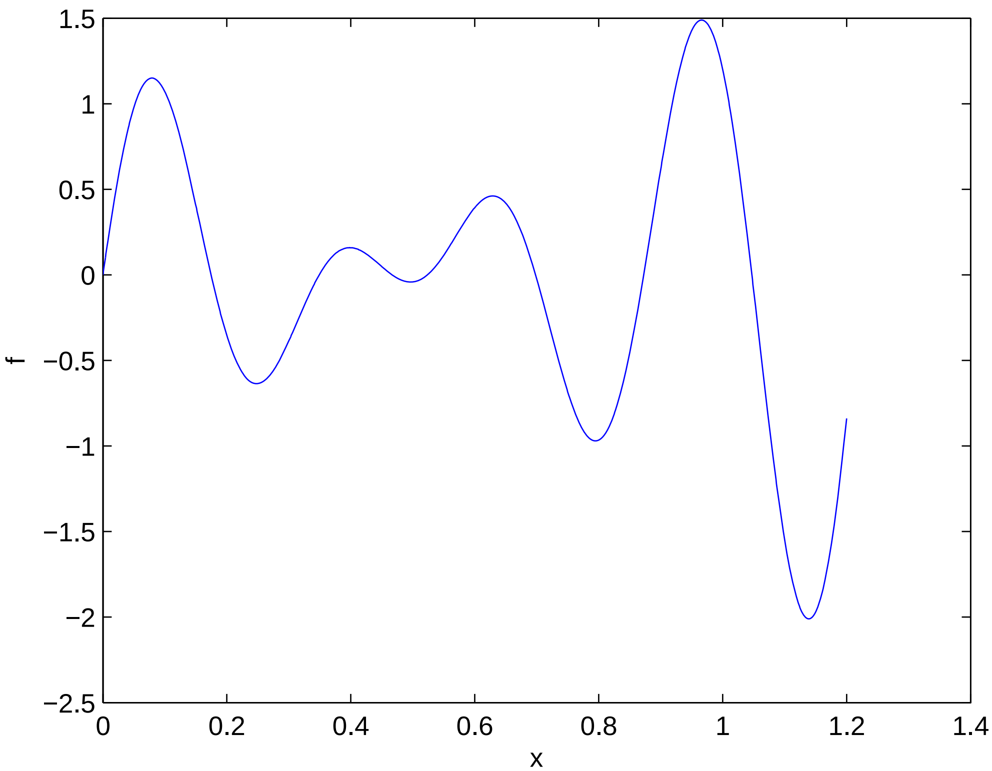
```
function [f]= getfun(x)
Msupport = -.5*(1-x)^2;
Mcentre = -0.5 + x;
f = max(abs(Msupport), abs(Mcentre));
```

we obtain the optimum half-spacing as $x^* = 0.5858$ that minimizes the maximum bending stress in the beam.

P2.24 Results from in-house code SHUBERT.m with Lipshitz constant estimated = 45 are:

Tolerance on objective function = 1.0000E-002
Number of Function Evaluations = 93
xopt, yopt = 0.96613 1.4891
best upper bound = 1.4973E+000
tolerance on x for merging int. of uncertainty = 5.0000E-002
Uncertainty Intervals =
A2 = [0.9126 0.9739]

This agrees with the plot of the function below.



P2.25 The study should pay attention to starting values, initial intervals etc. Number of function calls and robustness may be used as measures of performance.

P2.26 We define the objective function as follows:

$$d = 16$$
$$a = 8$$
$$b = 12$$
$$c = 12$$
$$R_1 = d/a$$
$$R_2 = d/b$$
$$R_3 = (d^2 + a^2 + b^2 - c^2)/(2ab)$$

$$\text{Minimize } f = (R_1\cos\phi - R_2\cos\theta + R_3 - \cos(\phi - \theta))^2$$

θ is taken in steps of 10^0 from 10^0 to 90^0 and ϕ corresponding to the minimum of f is obtained for each of these values. The limits for ϕ are set as 0^0 to 150^0. The minimum value of f is zero. The results are tabulated below:

θ^0	ϕ^0
10	100.372
20	93.922
30	90.606
40	90.116
50	91.857
60	95.264
70	99.895
80	105.426
90	111.625

Alternatively, the Matlab 'fzero' function may be used instead if a minimization routine.

■