Industrial Organization: Markets and Strategies
Paul Belleflamme and Martin Peitz
published by Cambridge University Press

Part II. Market power

Exercises & Solutions

Exercise 1 *Monopoly with quality choice*

Consider a monopolist who sells batteries. Each battery works for h hours and then needs to be replaced. Therefore, if a consumer buys q batteries, he gets $H = qh$ hours of operation. Assume that the demand for batteries can be derived from the preferences of a representative consumer whose indirect utility function is $v = u(H) - pq$, where p is the price of a battery. Suppose that u is strictly increasing and strictly concave. The cost of producing batteries is $C(q) = qc(h)$, where c is strictly increasing and strictly convex.

1. Derive the inverse demand function for batteries and denote it by $P(q)$.

2. Suppose that the monopolist chooses q and h to maximize his profit. Write down the first-order conditions for profit maximization assuming that the problem has an interior solution, and explain the meaning of these conditions.

3. Write down the total surplus in the market for batteries (i.e., the sum of consumer surplus and profits) as a function of H and h. Derive the first-order conditions for the socially optimal q and h assuming that there is an interior solution. Explain in words the economic meaning of these conditions.

4. Compare the solution that the monopolists arrives at with the social optimum. Prove that the monopolist provides the socially optimal level of h. Give an intuition for this result.

Solutions to Exercise 1

1. The inverse demand for batteries is obtained by solving the following problem:

$$\max_q u(qh) - pq.$$

The first-order conditions for this problem can be written as

$$P(q) = hu'(qh)$$

which is the inverse demand function for batteries.

2. The monopolist's maximization problem is given by

$$\max_{q,h} qP(q) - qc(h),$$

where $P(q)$ is given in part 1. The first-order conditions for an interior solution are:

$$\begin{aligned} h(u'(H) + Hu''(H)) &= c(h) \text{ and} \\ q(u'(H) + Hu''(H)) &= qc'(h) \end{aligned}$$

where $h(u'(H) + Hu''(H))$ is the marginal revenue from selling an extra battery. First, let us interpret the expressions inside the brackets: $u\prime(H)$ is the revenue from selling an extra hour of operation, while $u''(H)$ is the discount in the price per hour of operation that the monopolist must give in order to induce consumers to buy an extra unit. This discount has to be multiplied by H since the discount is given to all inframarginal units. Hence, $u'(H) + Hu''(H)$ is the marginal revenue from selling an extra hour of operation. Since the monopolist sells hours of operation in packets of h units each (a battery provides h hours of operation), $u'(H) + Hu''(H)$ has to be multiplied by h to give the marginal revenue from selling an extra battery. $c(h)$ is the marginal cost of batteries. Therefore the first equation is the familiar monopoly pricing condition that says that at the optimum, the monopolist produces up to the point where his marginal revenue equals his marginal cost. Similarly, the second equation indicates that at the optimum, the marginal revenue from extending the life of each battery by one hour must equal to marginal cost (note that the marginal cost has to be multiplied by the number of batteries that the monopolist produces since the cost of each one of them increases by $c'(h)$).

3. The total surplus in the market for batteries is given by $u(H) - qc(h)$. The first-order conditions for the social optimum (assuming that an interior solution to this problem exists) are

$$\begin{aligned} hu'(H) &= c(h) \text{ and} \\ qu'(H) &= qc'(h) \end{aligned}$$

where $hu'(H)$ is the marginal utility of consumers from having an extra battery. It equals the consumers' willingness-to-pay for an extra hour of operation multiplied by h which is the number of hours of operation they get when they buy an extra battery. The right-hand side of the first equation is the marginal cost of producing an extra battery; the first equation states that, at the social optimum, the marginal utility of consumers has to be equal to the marginal cost of production. The second equation says that, at the social optimum, the marginal utility of consumers from having batteries which provide one more hour each, has to be equal to the marginal cost of extending the life of each battery by one hour.

4. To compare the solutions in parts 2 and 3, we divide the first-order conditions for the monopoly problem by one another:

$$\frac{h}{q} = \frac{c(h)}{qc'(h)} \Leftrightarrow c'(h) = \frac{c(h)}{h}.$$

This equation shows that the monopolist produces h by equating the marginal cost of h with the average cost of h. This implies in turn that h is produced at the minimum average cost.

By dividing the first-order conditions for social optimum by one another we get the same condition which implies that the monopolist chooses the socially optimal level of h. Moreover, since the condition that determines h is independent of q, it follows that this result is true even though the monopolist provides too little quantity.

Exercise 2 *Price competition [included in the 2nd edition of the book]*

Consider a duopoly in which homogeneous consumers of mass 1 have unit demand. Their valuation for good $i = 1, 2$ is $v(\{i\}) = v_i$ with $v_1 > v_2$. Marginal cost of production is assumed to be zero. Suppose that firms compete in prices.

1. Suppose that consumers make a discrete choice between the two products. Characterize the Nash equilibrium.

2. Suppose that consumers can now also decide to buy both products. If they do so they are assumed to have a valuation $v(\{1, 2\}) = v_{12}$ with $v_1 + v_2 > v_{12} > v_1$. Firms still compete in prices (each firm sets the price for its product—there is no additional price for the bundle) Characterize the Nash equilibrium.

3. Compare regimes from parts (1) and (2) with respect to consumer surplus. Comment on your results.

Solutions to Exercise 2

1. Nash equilibrium given by $p_1 = v_1 - v_2$, $p_2 = 0$, $\pi_1 = v_1 - v_2$ and $\pi_2 = 0$.

2. Nash equilibrium given by $p_1 = v_{12} - v_2$, $p_2 = v_{12} - v_1$, $\pi_1 = v_{12} - v_2$ and $\pi_2 = v_{12} - v_1$.

3. Consumer surplus in 1) is $CS_a = v_2$, whereas it is given by $CS_b = v_1 + v_2 - v_{12}$ in 2). As $v_{12} > v_1$ by assumption, consumer welfare is strictly greater in 1) than in 2). In 2) the nature of competition changes because consumers have positive valuation to buy both products; this relaxes competition.

Exercise 3 *Cournot competition*

Two firms (firm 1 and firm 2) compete in a market for a homogenous good by setting quantities. The demand is given by $Q(p) = 2 - p$. The firms have constant marginal cost $c = 1$.

1. Draw the two firms' reaction function. Find the equilibrium quantities and calculate equilibrium profits.

2. Suppose now that there are n firms where $n \geq 2$. Calculate equilibrium quantities and profits.

Solutions to Exercise 3

This is the standard Cournot model, just in case students have forgotten about their intermediate micro class.

Exercise 4 *Equilibrium uniqueness in the Cournot model*

Consider an oligopoly with n firms that produce homogeneous goods and compete à la Cournot. Inverse demand is given by $P(Q)$ with $P'(Q) < 0$, and each firm i has a cost function of $C_i(q_i)$ with $C_i'(q_i) > 0$ and $C_i''(q_i) \geq 0$. Denote $q_{-i} = \sum_{j \neq i} q_j$.

1. Compute the first- and second order condition of firm i. Under which conditions is the profit function of firm i, π_i, strictly concave?
2. Compute the slope of the best-reply function of firm i, $\frac{dq_i}{dq_{-i}}$. In which interval is this slope?

A sufficient condition for uniqueness of a Cournot equilibrium is (see, e.g., Tirole (1999), page 226)

$$\frac{\partial^2 \pi_i}{\partial q_i^2} + (n-1)\left|\frac{\partial^2 \pi_i}{\partial q_i \partial q_{-i}}\right| < 0,$$

3. Suppose that demand is concave and that marginal costs are constant. For which number of n is the condition above satisfied?
4. Suppose that $P(Q) = a - b\sum_{i=1}^{n} q_i$ and $C_i(q_i) = cq_i$, for all $i \in \{1, ..., n\}$. Is there a unique equilibrium for any n?

Solutions to Exercise 4
see the 1999-book by Vives

Exercise 5 *Industries with price or quantity competition*

Which model, the Cournot or the Bertrand model, would you think provides a better first approximation to each of the following industries/markets: the oil refining industry, farmer markets, cleaning services. Discuss!

Exercise 6 *An investment game*

Consider a duopoly market with a continuum of homogeneous consumers of mass 1. Consumers derive utility $v_i \in \{v^H, v^L\}$ for product i depending on whether the product is of high or low quality. Firms play the following 2-stage game: At stage 1, firms simultaneously invest in quality: The more a firm invests the higher is its probability λ_i of obtaining a high-quality product. The associated investment cost is denoted by $I(\lambda_i)$ and satisfies standard properties that ensure an interior solution: $I(\lambda_i)$ is continuous for $\lambda_i \in [0,1)$, $I'(\lambda_i) > 0$ and $I''(\lambda_i) > 0$ for $\lambda_i \in (0,1)$, and $\lim_{\lambda \downarrow 0} I'(\lambda_i) = 0, \lim_{\lambda \uparrow 1} I'(\lambda_i) = \infty$. Before the beginning of stage 2 qualities become publicly observable—i.e., all uncertainty is resolved. At stage 2, firms simultaneously set prices.

1. For any given (λ_1, λ_2), what are the expected equilibrium profits? In case of multiple equilibria select the (from the view point of the firms) Pareto-dominant equilibrium.

2. Are investments strategic complements or substitutes? Explain your finding.
3. Provide the equilibrium condition at the investment stage.
4. How do equilibrium investments change as $v^H - v^L \equiv \Delta$ is increased?

Solutions to Exercise 6

1. Bertrand competition: If $v_1 = v_2$, $\pi_1^* = \pi_2^* = 0$. If $v_i > v_j$, $\pi_i^* = \Delta$ and $\pi_j = 0$, $i \neq j$. The expected equilibrium profit is $E\pi_i^* = \lambda_i(1-\lambda_j)\Delta$.
2. At stage 1, each firm solves $\max_{\lambda_i} \lambda_i(1-\lambda_j)\Delta - I(\lambda_i)$. First-order condition of profit maximization is
$$(1-\lambda_j)\Delta = I'(\lambda_i)$$
Since $I(\lambda_i)$ is strictly convex $I'(\lambda_i)$ is monotone and thus invertible.
$$\lambda_i = (I')^{-1}[(1-\lambda_j)\Delta]$$
Since $I'(\lambda_i)$ is increasing the best response of firm i is decreasing in λ_j and investment decisions are strategic substitutes.
3. The equilibrium investment decisions $\lambda_1^* = \lambda_2^* \equiv \lambda^*$ are characterized by
$$(1-\lambda^*)\Delta = I'(\lambda^*).$$
There is a unique solution to this equation.
4. Rewriting the above equation as
$$\Delta = \frac{I'(\lambda^*)}{1-\lambda^*}$$
we see that the numerator on the right-hand side is increasing in λ^* while the denominator on that side is decreasing in λ^*. Thus, an increase in Δ implies a larger λ^*. Due to the nature of Bertrand competition only the quality difference but not the absolute levels of qualities affect investment incentives.

Exercise 7 *Hotelling model*

Reconsider the simple Hotelling model in which consumers are uniformly distributed on the unit interval and firms are located at the extremes of this interval. Now take consumers' participation constraint explicitly into account. Derive the equilibrium depending on the parameter τ. [Be careful to distinguish between different regimes with respect to competition between firms!]

Exercise 8 *Price and quantity competition*

Reconsider the duopoly model with linear individual demand and differentiated products. Show that profits under quantity competition are higher than under price competition if products are substitutes and that the reverse holds if products are complements.

Exercise 9 *Asymmetric duopoly [included in the 2nd edition of the book]*

Consider two quantity-setting firms that produce a homogenous good and choose their quantities simultaneously. The inverse demand function for the good is given by $P = a - q_1 - q_2$, where q_1 and q_2 are the outputs of firms 1 and 2 respectively. The cost functions of the two firms are $C_1(q_1) = c_1 q_1$ and $C_2(q_2) = c_2 q_2$, where $c_1 < a$ and $c_2 < (a + c_1)/2$.

1. Compute the Nash equilibrium of the game. What are the market shares of the two firms?

2. Given your answer to (1), compute the equilibrium profits, consumer surplus, and social welfare.

3. Prove that if c_2 decreases slightly, then social welfare increases if the market share of firm 2 exceeds 1/6, but decreases if the market share of firm 2 is less than 1/6. Give an economic interpretation of this finding.

Solutions to Exercise 9

1. The Nash equilibrium of the game is obtained by solving the following system of equations:

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= (a - q_1 - q_2) - q_1 - c_1 = 0, \\ \frac{\partial \pi_1}{\partial q_2} &= (a - q_1 - q_2) - q_2 - c_2 = 0. \end{aligned}$$

The equilibrium is given by the pair

$$q_1^* = \frac{a - 2c_1 + c_2}{3}, q_2^* = \frac{a - 2c_2 + c_1}{3}.$$

Given q_1^* and q_2^*, the equilibrium market share are:

$$\begin{aligned} \alpha_1 &= \frac{q_1^*}{q_1^* + q_2^*} = \frac{a - 2c_1 + c_2}{2a - c_1 - c_2}, \\ a_2 &= \frac{q_2^*}{q_1^* + q_2^*} = \frac{a - 2c_2 + c_1}{2a - c_1 - c_2}. \end{aligned}$$

2. The equilibrium profits are given by

$$\begin{aligned}\pi_1^* &= (a - q_1^* - q_2^*)q_1^* - c_1 q_1^* = (q_1^*)^2 \\ &= \frac{(a - 2c_1 + c_2)^2}{9}, \\ \pi_2^* &= \frac{(a - 2c_2 + c_1)^2}{9}.\end{aligned}$$

and the consumer surplus is given by

$$\begin{aligned}CS^* &= \int_0^{q_1^*+q_2^*} (a-q)dq - (a - q_1^* - q_2^*)(q_1^* + q_2^*) \\ &= \frac{(q_1^* + q_2^*)^2}{2} = \frac{(2a - c_1 - c_2)^2}{18}.\end{aligned}$$

Adding the three together, social welfare is given by

$$\begin{aligned}W^* &= CS^* + \pi_1^* + \pi_2^* \\ &= \frac{(q_1^* + q_2^*)^2}{2} + (q_1^*)^2 + (q_2^*)^2.\end{aligned}$$

3. Differentiating W^* with respect to c_2 yields

$$\begin{aligned}\frac{\partial W^*}{\partial c_2} &= (q_1^* + q_2^*)\frac{\partial (q_1^* + q_2^*)}{\partial c_2} + 2q_1^* \frac{\partial q_1}{\partial c_2} + 2q_2^* \frac{\partial q_2}{\partial c_2} \\ &= \frac{q_1^* + q_2^* - 6(q_2^*)^2}{3} = 2(q_1^* + q_2^*)(\frac{1}{6} - \alpha_2).\end{aligned}$$

Hence, a small reduction in c_2 increases W^* if $\alpha_2 > 1/6$ and deceases W^* is $\alpha_2 < 1/6$.

Not that the result that a cost reduction may be socially undesirable was first demonstrated in Lahiri and Ono (1988), "Helping Minor Firms Reduces Welfare," Economic Journal.

Exercise 10 *Selling independent products to budget-constrained consumers*[1]

Consider two sellers 1 and 2 and a continuum of buyers. Seller i offers product i at price p_i and incurs zero marginal cost of production. Buyers are identical and derive utility u_i from one unit of each product. Thus their utility is u_i if they buy one unit of product i and zero units of product j, $j \neq i$; it is $u_1 + u_2$ if they buy one unit of each product. Additional units do not give any extra utility. Each buyer has a budget y, which she cannot exceed. Each seller is assumed to prefer not to sell a unit rather than setting a zero price. Suppose that $u_1 > u_2$.

[1] This exercise is inspired by Jeon, D.-S. and D. Menicucci (2006). Bundling Electronic Journals and Competition among Publishers. Journal of the European Economic Association 4: 1038-1083.

1. Derive the demand function of each buyer.
2. Consider the game in which sellers simultaneously set price. Characterize the Nash equilibrium of the game.
3. Determine consumer surplus and total surplus that realize in equilibrium. Is the equilibrium necessarily efficient or are there inefficiencies? Explain.

Solutions to Exercise 10

1. For $p_1 \leq u_1$, $p_2 \leq u_2$ and $p_1 + p_2 \leq y$, $Q_1(p_1, p_2) = 1$ and $Q_2(p_1, p_2) = 1$. For $p_1 \leq u_1$, $p_2 \leq u_2$ and $p_1 + p_2 > y$, $Q_1(p_1, p_2) = 1$ and $Q_2(p_1, p_2) = 0$ if $u_1 - p_1 > u_2 - p_2$ and $Q_1(p_1, p_2) = 0$ and $Q_2(p_1, p_2) = 1$ if the reverse inequality holds. For $p_i \leq \max\{u_i, y\}$ and $p_j > u_j$, $Q_i(p_i, p_j) = 1$ and $Q_j(p_i, p_j) = 0$. For all other prices, $Q_1(p_1, p_2) = 0$ and $Q_2(p_1, p_2) = 0$.
2. For $u_1 - u_2 \geq y$, $p_1^* = y$ and seller 2 does not sell. Thus, the buyer only buys product 1. Consider next the case $u_1 + u_2 > y > u_1 - u_2$. Note that when both products are bought under this inequality, we must have in equilibrium that $p_1 + p_2 = y$. Suppose that prices satisfy that $u_1 - p_1 = u_2 - p_2$. Then, buyers are indifferent between product 1 and product 2 if they were to make a discrete choice. In this case, if firm i slightly increases the price, the buyer can no longer afford to buy both products. However, the buyer than buys product j and, therefore, firm i does not have an incentive to deviate. Whenever $u_1 - p_1 \neq u_2 - p_2$ and $p_1 + p_2 = y$, the firm which offers a larger net surplus has an incentive to increase its price. Hence, in equilibrium, $u_1 - p_1 = u_2 - p_2$. Solving $u_1 - p_1^* = u_2 - p_2^*$ and $p_1^* + p_2^* = y$ gives $p_i^* = (y + u_i - u_j)/2$. Clearly, for $u_1 + u_2 \leq y$, each seller charges $p_1^* = u_1$ and $p_2^* = u_2$.
3. Consumers make strictly positive surplus if $u_1 + u_2 > y > u_1 - u_2$. Here, firms compete for the limited budget of the buyer. For a smaller as well as for a larger budget, consumer surplus is zero. Thus, consumer surplus in non-monotone in y. For $u_1 - u_2 \geq y$, there is a total surplus loss compared to the social optimum because in the social optimum each buyer obtains one unit of each product.

Exercise 11 *Differentiated duopoly with uncertain demand*

1. Consider a monopolist facing an uncertain inverse demand curve
$$p = a - bq + \theta.$$
When setting its price or quantity the monopolist does not know θ but knows that $E[\theta] = 0$ and $E[\theta^2] = \sigma^2$. The cost function of the monopolist is given by
$$C(q) = c_1 q + \frac{c_2 q^2}{2},$$
with $a > c_1 > 0$ and $c_2 > -2b$.
Show that the monopolist prefers to set a quantity if the marginal cost curve is increasing and a price if the marginal cost curve is decreasing. Provide a short intuition for the result.

2. Now consider a differentiated duopoly facing the uncertain inverse demand system
$$p_1 = a - bq_1 - dq_2 + \theta$$
and
$$p_2 = a - bq_2 - dq_1 + \theta,$$
with $0 < d < b$, $E[\theta] = 0$ and $E[\theta^2] = \sigma^2$. Again, the cost functions are similar for both firms and are given by $C(q) = c_1 q + \frac{c_2 q^2}{2}$, with $a > c_1 > 0$ and $c_2 > -\frac{2(b^2-d^2)}{b}$.
Both firms play a one-shot game in which they choose the strategy variable and the value of this variable simultaneously.

   Argue by the same line of reasoning as in (1) that

   (a) if $c_2 > 0$ in the unique Nash equilibrium both firms choose quantities

   (b) if $c_2 < 0$ in the unique Nash equilibrium both firms choose prices

   (c) if $c_2 = 0$ there exist four Nash equilibria in pure strategies.

Solutions to Exercise 11[2]

Price vs. quantity setting

quantity setting:

$$\begin{aligned}
E(\Pi) &= pq - C(q) = (a-bq)q - c_1 q - \frac{c_2 q^2}{2} \\
\frac{\partial E(\Pi)}{\partial q} &= a - 2bq - c_1 - c_2 q = 0 \\
\rightarrow E(q^*) &= \frac{a-c_1}{2b+c_2} \qquad p^* = \frac{ab + ac_2 + bc_1}{2b+c_2} + \theta \\
E(\Pi_q) &= \frac{(a-c_1)^2}{2(2b+c_2)}
\end{aligned}$$

price setting:

$$\begin{aligned}
E(\Pi) &= pq - C(q) = p\left(\frac{a}{b} - \frac{p}{b}\right) - c_1\left(\frac{a}{b} - \frac{p}{b}\right) - \frac{c_2}{2}\left(\frac{a}{b} - \frac{p}{b}\right)^2 \\
\frac{\partial E(\Pi)}{\partial p} &= 0 \\
\rightarrow E(p^*) &= a - \frac{b(a-c_1)}{2b+c_2} \qquad q^* = \frac{(a-c_1)}{2b+c_2} + \frac{\theta}{b} \\
E(\Pi_p) &= \frac{(a-c_1^2)}{2(2b+c_2)} - \frac{c_2\theta^2}{2b^2}
\end{aligned}$$

[2] This exercise is based on Klemperer&Meyer (1986).

$\rightarrow$ compare profits of the two settings (Note that $E(\theta^2) = \sigma^2 > 0$):

$$\Pi_p > \Pi_q \Leftrightarrow -\frac{c_2\theta^2}{2b^2} > 0 \text{ if } c_2 < 0$$

$\rightarrow$ firm chooses price competition if MC are decreasing

$$\Pi_p < \Pi_q \Leftrightarrow -\frac{c_2\Theta^2}{2b^2} < 0 \text{ if } c_2 > 0$$

$\rightarrow$ firm chooses quantity competition if MC are increasing

Calculate best responses given the strategic choice of the other firm (by symmetry we only have to look at firm 1):

- Suppose that firm 2 has decided to set quantity q_2

$\rightarrow$ optimal quantity reaction by firm 1:

$$\begin{aligned} \Pi_1 &= (a - bq_1 - dq_2 + \theta)q_1 - c_1 q_1 - c_2\frac{q_1^2}{2} \\ \frac{\partial \Pi_1}{\partial q_1} &= 0 \\ E(q_1) &= \frac{a - c_1 - dq_2}{2b + c_2} \end{aligned}$$

best response payoff: $E(\Pi_1) = \dfrac{(a - c_1 - dq_2)^2}{2(2b + c_2)}$

$\rightarrow$ optimal price reaction by firm 1:

$$\begin{aligned} \Pi_1 &= \frac{p_1(a - p_1 - dq_2 + \theta)}{b} - c_1\left(\frac{a - p_1 - dq_2 + \theta}{b}\right) - \frac{c_2}{2}\left(\frac{a - p_1 - dq_2 + \theta}{b}\right)^2 \\ \frac{\partial \Pi_1}{\partial p_1} &= 0 \\ E(p_1) &= a - dq_2 - \frac{b(a - dq_2 - c_1)}{2b + c_2} \end{aligned}$$

best response payoff: $E(\Pi_1) = \dfrac{(a - dq_2 - c_1)^2}{2(2b + c_2)} - \dfrac{c_2\Theta^2}{2b^2}$

$\longrightarrow$ optimal reaction to quantity setting by the competitor is to also set quantities if $c_2 > 0$, i.e. if MC are increasing!

- suppose firm 2 has decided to set price p_2

$\longrightarrow$ optimal quantity reaction by firm 1:

$$\begin{aligned}
\text{plug in } q_2 &= \frac{a - p_2 - dq_1 + \theta}{b} \text{ into } p_1 = a - bq_1 - dq_2 + \theta \\
\rightarrow p_1 &= a\frac{(b-d)}{b} - \frac{(b^2-d^2)}{b}q_1 + \frac{d}{b}p_2 + \frac{(b-d)}{b}\theta
\end{aligned}$$

rename:

$$\alpha \equiv a\frac{(b-d)}{b}; \qquad \beta \equiv \frac{(b^2-d^2)}{b}; \qquad \gamma \equiv \frac{d}{b}; \qquad \widehat{\theta} \equiv \frac{(b-d)}{b}\theta$$

$$\begin{aligned}
\Pi_1 &= (\alpha - \beta q_1 + \gamma p_2 + \widehat{\theta})q_1 - c_1 q_1 - \frac{c_2 q_1^2}{2} \\
\frac{\partial \Pi_1}{\partial q_1} &= 0 \\
E(q_1) &= \frac{\alpha - c_1 + \gamma p_2}{2b + c_2} \\
E(\Pi_1) &= \frac{(a - c_1 + \gamma p_2)^2}{2(2\beta + c_2)}
\end{aligned}$$

$\longrightarrow$ optimal price reaction by firm 1:

$$\begin{aligned}
q_1 &= \frac{\alpha - p_1 + \gamma p_2 + \widehat{\theta}}{\beta} \\
\frac{\partial \Pi_1}{\partial q_1} &= 0 \\
E(p_1) &= \alpha + \gamma p_2 - \frac{\beta(\alpha + \gamma p_2 - c_1)}{2\beta + c_2} \\
E(\Pi_1) &= \frac{(\alpha + \gamma p_2 - c_1)^2}{2(2\beta + c_2)} - \frac{c_2 \widehat{\theta}^2}{2b^2}
\end{aligned}$$

$\longrightarrow$ optimal reaction to price setting by firm 2 is to set quantities if $c_2 > 0$, i.e., if MC are increasing

Summary:

- if $c_2 > 0$, then it is a dominant strategy for both firms to set quantities $\rightarrow$ unique Cournot equilibrium
- if $c_2 < 0$, it is a dominant strategy for both firms to set prices $\rightarrow$ unique Bertrand equilibrium
- if $c_2 = 0$, firms are indifferent between choosing prices and quantities $\rightarrow$ there are four NE: {price, quantity}, {quantity, price}, {quantity, quantity}, {price, price}

Exercise 12 *Cournot duopoly and cost information*

Consider a duopoly market for a homogeneous product in which firms set quantity. Inverse demand is $P(q) = 1 - q$ with $q = q_1 + q_2$. Firm 1 has marginal costs equal to 0.7. Firm 2 has marginal cost 0.65 with probability 1/2 and 1 with probability 1/2.

1. Suppose that the cost type is publicly observed by both firms prior to the quantity setting. Characterize the equilibrium outcome of this game.
2. Suppose from now on that firm 2 privately observes its cost type before setting its quantity. Determine the equilibrium of this game. What is the appropriate equilibrium concept? In particular, determine equilibrium quantities and profits.
3. Would firm 2 have an incentive to reveal its cost type to firm 1 if it could do so at zero cost?
4. Would firm 1 have an incentive to find out about firm 2's costs? Would it like to do so privately (assuming that firm 2 does not know the cost of firm 1 has when investigating) or publicly?
5. Are consumers better off if firm 2's cost type remains private information? Discuss.

Solutions to Exercise 12

1. firms share information:

high costs:

firm 1:

$$\Pi_1 = (1 - q_1 - q_2 - 0.7)q_1$$
$$\frac{\partial \Pi_1}{\partial q_1} = 0.3 - q_2 - 2q_1 = 0$$
$$q_1 = \frac{0.3 - q_2}{2}$$

firm 2:

$$\Pi_2 = (1 - q2 - q1 - 1)q2 = (-q2 - q1)q2$$
$$\rightarrow \text{ will not produce !} \rightarrow \Pi_2^H = 0$$
$$q_1 = \frac{0.3}{2} = 0.15$$
$$\Pi_1 = (1 - 0.15 - 0.7)0.15 = (0.15)^2 = 0.0225$$

low costs:

firm 2:

$$\begin{aligned}
\Pi_2 &= (0.35 - q_1 - q_2)q_2 \\
\frac{\partial \Pi_2}{\partial q_2} &= 0.35 - q_1 - 2q_2 = 0 \\
q_2 &= \frac{0.35 - q_1}{2}
\end{aligned}$$

firm 1:

$$\begin{aligned}
q_1 &= \frac{0.3}{2} - \frac{0.35}{4} + \frac{q_1}{4} \\
q_1 &= \frac{4}{3}\frac{1}{16} = \frac{1}{12} \approx 0.008\overline{3} \\
q_2 &= \frac{2}{15} \approx 0.1\overline{3} \\
\Pi_1 &= (0.3 - \frac{1}{12} - \frac{2}{15})\frac{1}{12} = \frac{1}{144} \approx 0.0069 \\
\Pi_2 &= (0.35 - \frac{1}{12} - \frac{2}{15})\frac{2}{15} = \frac{4}{225} \approx 0.018 \\
&\Rightarrow \text{ the lower cost firm makes higher profits}
\end{aligned}$$

2. Firm 2 privately observes its cost type:

$\rightarrow$ Solve for a Bayesian NE since its a static game with imperfect info

firm 2:

$$\begin{aligned}
\Pi_2 &= (1 - q_1 - q_2 - c_2)q_2 \\
\rightarrow \max_q \Pi_2 &\Leftrightarrow q_2(c_2) = \frac{1 - c_2 - q_1}{2} \\
q_2^L &= \frac{0.35 - q_1}{2} \\
q_2^H &= 0
\end{aligned}$$

firm 1:

$$\begin{aligned}
\Pi_1 &= \frac{1}{2}(0.3 - q_1 - q_2^L)q_1 + \frac{1}{2}(0.3 - q_1 - q_2^H)q_1 = (0.3 - q_1)q_1 - \underbrace{(\frac{1}{2}q_2^L + \frac{1}{2}q_2^L)}_{E(q_2)}q_1 \\
q_1 &= \frac{0.3 - E(q_2)}{2} \\
E(q_2) &= \frac{0.35 - q_1}{4} \\
q_1 &= \frac{0.3}{2} - \frac{0.35 - q_1}{8} = \frac{17}{140} \approx 0.121 \\
q_2^L &= \frac{4}{35} \approx 0.114 \\
q_2^H &= 0
\end{aligned}$$

NE strategies: $\{q_1 = \frac{17}{140}; q_2^L = \frac{4}{35}; q_2^H = 0\}$

profits:

$$\begin{aligned}
E(\pi_1) &= \frac{1}{2}(0.3 - \frac{17}{140} - \frac{4}{35})\frac{17}{140} + \frac{1}{2}(0.3 - \frac{17}{140})\frac{17}{140} = \frac{289}{19600} \approx 0.015 \\
\pi_2^L &= (0.35 - \frac{17}{140} - \frac{4}{35})\frac{4}{35} = \frac{4}{35} = \frac{16}{1225} \approx 0.0131 \\
\pi_2^H &= 0
\end{aligned}$$

3. yes, because if he is of high cost, he makes zero profits in both cases, but if he has lower costs, then his profit is higher if costs are revealed!

4.

- If firm 1 finds out costs publicly:

$$E(\pi_1) = \frac{1}{2}(0.15)^2 + \frac{1}{2}\frac{1}{144} = \frac{53}{3600} \approx 0.01472$$

- If costs are not revealed:

$$E(\pi_1) = \frac{289}{19600} \approx 0.01474$$

- If firm 1 finds out the costs privately:
  It knows that firm 2 thinks firm 1 does not know and will therefore play $q_2^L = \frac{4}{35}$ and $q_2^H = 0$. Hence, firm 1 will take this into account when maximizing profits.
  high costs:

$$\begin{aligned}
\pi_1 &= (0.3 - q_1)q_1 \Leftrightarrow q_1 = 0.15 \\
\pi_1 &= (\frac{3}{20})^2 = \frac{9}{400}
\end{aligned}$$

low costs:

$$\begin{aligned}
\pi_1 &= (0.3 - q_1 - \frac{4}{35})q_1 \Leftrightarrow q_1 = \frac{13}{140} \\
\pi_1 &= (\frac{13}{140})^2 = \frac{169}{19600} \\
E(\pi_1) &= \frac{1}{2}\frac{9}{400} + \frac{1}{2}\frac{169}{19600} = \frac{61}{3920} \approx 0.01556
\end{aligned}$$

$\Rightarrow$ firm 1 has an incentive to find out costs privately, whereas it does not want to find out publicly!

5. Are consumers better off if firm 2's costs stay private information?

1. costs are publicly observed:

$$\begin{aligned}
\text{low cost: } q &= \frac{1}{12} + \frac{2}{15} = \frac{13}{60} \approx 0.216 \\
CS &= \frac{1}{2}(\frac{13}{60})^2 = \frac{169}{7200} \approx 0.0235 \\
\\
\text{high cost: } q &= 0.15 \\
CS &= \frac{1}{2}(0.15)^2 = \frac{9}{800} \approx 0.01125 \\
\rightarrow E(CS) &= \frac{5}{288} \approx 0.01736
\end{aligned}$$

2. costs are private info:

$$\begin{aligned}
\text{high cost: } q &= \frac{17}{140} \rightarrow CS = \frac{289}{39200} \approx 0.00737 \\
\text{low cost: } q &= \frac{17}{140} + \frac{4}{35} \rightarrow CS = \frac{1089}{39200} \approx 0.02778 \\
\rightarrow E(CS) &= \frac{689}{39200} \approx 0.01758 \qquad (1)
\end{aligned}$$

3. firm 1 observes costs without firm 2 knowing it:

$$\begin{aligned}
\text{high cost: } q &= 0.15 \rightarrow CS = \frac{9}{800} \\
\text{low cost: } q &= \frac{4}{35} + \frac{13}{140} = \frac{29}{140} \rightarrow CS = \frac{841}{39200} \\
\rightarrow E(CS) &= \frac{641}{39200} \approx 0.01635
\end{aligned}$$

$\rightarrow$ Consumers are better off if costs stay private info. They are worst off when firm 1 can find out the costs privately.

Exercise 13 *Strategic capacity choice*

Consider a market in which firms $1, ..., N$ set simultaneously capacities for a homogeneous product and afterwards a third party, which observes market demand and the capacity choice of each firm sets the market clearing price. Suppose that inverse demand is linear and of the form $P(q) = a - q$, where p is the price, a a positive constant, and q aggregate output, $q = \sum_{i=1}^{N} q_i$; q_i is the quantity sold by firm i. Suppose that firm i has constant marginal costs of production c_i (firms have different marginal costs!). Suppose that $a > \max\{c_1, ..., c_N\}$. Suppose furthermore that the parameters of the model are such that each firm produces positive output. Solve for the Nash equilibrium where capacity is the strategic choice of the firms. Determine aggregate output and price level in equilibrium. Determine the output of each firm i.

Exercise 14 *Price setting in a market with limited capacity [included in the 2nd edition of the book]*

Suppose that two identical firms in a homogeneous-product market compete in prices. The capacity of each firm is 3. The firms have constant marginal cost equal to 0 up to the capacity constraint. The demand in the market is given by $Q(p) = 9 - p$. If the firms set the same price, they split the demand equally. If the firms set a different price, the demand of each one of the firms is calculated according to the Efficient Rationing Rule. Show that $p_1 = p_2 = 3$ can be sustained as an equilibrium. Calculate the equilibrium profits.

Solutions to Exercise 14

At the prices $p_1 = p_2 = 3$, both firms produce at full capacity. We conclude that the firms have no incentives to deviate to a lower price. This would result in the same amount of sales but at a lower price. The efficient rationing rule implies that a firm that deviates upwards faces a demand of $D(p) = 6 - p$. Thus, the profits are $(6 - p)p$ when deviating to some $p > 3$. It is easy to show that $(6 - p)p$ is decreasing in p for all $p > 3$. This, in turn, implies that deviating to a price above 3 is not profitable. Hence, $p_1 = p_2 = 3$ is an equilibrium. The equilibrium profits are: $\pi^{NE} = 3(9 - 3)/2 = 9$.

Exercise 15 *Capacity-constrained imperfect competition*

Suppose two firms in an industry face linear inverse demand curves $P_i(q_i, q_j) = 7 - q_i - q_j$, $i = 1, 2$, $i \neq j$. Firms compete in a two-stage game; first they set capacity and then they set price or output. At the first stage firms set capacities, at this stage the marginal costs of capacity is 6. Suppose that firms have zero marginal costs of production up until installed capacity and that production above capacity is not feasible. In case of rationing, rationing is assumed to be efficient.

1. Suppose each firm has a capacity of 7. Analyze competition at stage 2. Determine the Nash equilibrium if both firms set prices.

2. Consider the same situation as in (1) but suppose that firms choose quantities, not prices at stage 2. Determine the Nash equilibrium.

3. Consider the same situation as in (1) but suppose that consumers do not observe price and incur a cost of 1/2 if, after visiting one firm, they decide to visit the other firm. [You can think of identical consumers with the demand function as given above]. Characterize the equilibrium if both firms set prices. (What is the appropriate equilibrium concept here?). Give an explanation (at most 2 sentences).

4. Suppose that firms have given capacities $\overline{q}_1$ and $\overline{q}_2$, respectively. If firm 1 is the high-price firm, what is its demand function? Determine the Nash equilibrium in prices (provided that $\overline{q}_i \leq 49/24$, $i = 1, 2$). Show that equilibrium prices satisfy $p_1 = p_2 = a - \overline{q}_1 - \overline{q}_2$.

5. Determine the subgame perfect equilibrium of the two-stage game in which firms first set capacities and then prices. Give an explanation (at most 3 sentences).

6. Suppose that firms collude at the stage at which they set capacity. What should they do?

7. Suppose that firms are able to use a less costly technology (e.g., the marginal cost of capacity falls from 6 to 11/2). What are the competitive effects of this reduction in capacity costs? What would happen if those costs fell to zero? Discuss your results.

Solutions to Exercise 15

1. Suppose that capacity is 7
   $\rightarrow$ NE in prices
   $\rightarrow$ products are homogenous $\rightarrow$ prices equal to MC
   $\rightarrow p_i^B = 0$, $q_i^B = 3.5$ and $\Pi_i^B = 0$ $(i = 1, 2)$

2. Suppose that capacity is 7
   $\rightarrow$ NE in quantities:

$$
\begin{aligned}
\Pi_i &= (7 - q_i - q_j)q_i \\
\frac{\partial \Pi_i}{\partial q_i} &= 7 - 2q_i - q_j = 0 \\
q_i &= \frac{7 - q_2}{2} \text{ with } i = 1, 2;\ i \neq j
\end{aligned}
$$

$\rightarrow q_1^C = q_2^C = \frac{7}{3}$, $p_1^C = p_2^C = \frac{7}{3}$ and $\Pi_1^C = \Pi_2^C = \frac{49}{9}$

3. Consumers do not observe prices before consumption
The appropriate equilibrium concept is a Perfect Bayesian Equilibrium (PBE) since consumers' decision which firm to choose depends on their beliefs. There are multiple PBE which can be supported by different belief systems. Focus will be on the PBE that yields highest profits for the firms.
Assume that consumers believe that both firms set identical prices.
$\rightarrow$ Consumers are indifferent between visiting either firm 1 or firm 2 at first. Therefore, they visit firm i ($i = 1, 2$) with probability $1/2$ and will never consider a second offer given that switching cost are larger than zero (they are equal to $1/2$ in this exercise). In consequence, firms will set their prices equal to the monopoly level. (We can therefore specify consumers' beliefs even more precisely stating that consumers believe that firms will both set monopoly prices.)
Do firms have an incentive to deviate from setting monopoly prices given this belief system? No, since the probability of a consumer switching to check the offer of the second firm is zero given the beliefs and consumers choose a firm randomly in the first place.
Hence, consumers believing that firms set monopoly prices and firms actually choosing prices equal to monopoly level is a PBE.

4. Suppose that firms have given capacities $\bar{q}_1$ and $\bar{q}_2$:

- efficient rationing rule implies the following demand for firm 1 (residual demand for those not served by firm 2):

$$\widehat{Q}(p_1) = \begin{cases} 0 & \text{if } \bar{q}_2 \geq Q_1(p_1) \\ Q(p_1) - \bar{q}_2 & \text{if } \bar{q}_2 < Q(p_1) \end{cases}$$

- maximum capacity chosen is such that its revenues minus costs are non-negative independent of its competitor:
$\rightarrow$ maximum revenue in period 2:

$$\begin{aligned} \Pi^M &= p(7-p) \rightarrow \frac{\partial \Pi^M}{\partial p} = 7 - 2p = 0 \Leftrightarrow p^M = \frac{7}{2} = 3.5 \\ \Pi^M &= 3.5^2 \end{aligned}$$

- profit maximizing capacity must satisfy: $\Pi^M > 6\bar{q}_i \Leftrightarrow \bar{q}_i < \frac{49}{24} \rightarrow$ this condition is fulfilled by assumption

- profits:

$$\begin{aligned} \Pi_2 &= (p_2 - 6)\bar{q}_2 \\ \Pi_1 &= p_1\widehat{Q}(p_1) - 6\bar{q}_1 = p_1(Q/p_1) - \bar{q}_2) - 6\bar{q}_1 \end{aligned}$$

Show that $p^N = p_1 = p_2 = 7 - \bar{q}_1 - \bar{q}_2$ is the NE price, i.e., show that firm i has no incentive to deviate by setting a price above or below p^N ! [Note that p^N

implies market clearing: $\widehat{Q}(p_1) = (7 - p_1 - \bar{q_2}) = (7 - 7 + \bar{q_1} + \bar{q_2} - \bar{q_2}) = \bar{q_1}$ and $Q(p_2) = \bar{q_2}$]

- Suppose $p_i < p^*$:
  $\rightarrow$ at p^* firm i sells all its capacity
  $\rightarrow$ lowering p_i would increase demand above capacity
  $\rightarrow$ still sell $\bar{q_i}$, but at lower price $\rightarrow \Pi_i \downarrow$

- Suppose $p_i > p^*$:
  $\rightarrow$ due to capacity limit of firm j, firm i obtains positive demand for $p_i > p^*$
  $\rightarrow$ in order to show that it is not optimal to raise prices, prove that the revenue maximizing price lies to the left of p^*(Note: costs of capacity are already incurred in period 1)
  $\rightarrow$ if $p_i > p^*$, then revenue is given as

$$p_i \widehat{Q}(p_i) = \begin{cases} p_i(7 - p_i - \bar{q_j}) & \text{if } 7 - p_i > \bar{q_j} \\ 0 & \text{else} \end{cases}$$

  $\rightarrow$ revenue maximizing price: $\widehat{p_i} = \frac{7-\bar{q_j}}{2}$
  $\rightarrow$ this price must be smaller than p^*

$$\widehat{p_i} < p^* \Leftrightarrow \frac{7 - \bar{q_j}}{2} < 7 - \bar{q_i} - \bar{q_j} \Leftrightarrow 7 > \bar{q_j} + 2\bar{q_i}$$

  $\rightarrow$ we know that $\bar{q_i} \leq \frac{49}{24}$ by assumption. Hence, $7 > \frac{49}{24} + \frac{49}{12} \Leftrightarrow \frac{168}{24} > \frac{147}{24} \rightarrow$ it is not optimal to set a price $p_i > p^*$

$\rightarrow p^* = 7 - \bar{q_1} - \bar{q_2}$ is the NE price

5. SPNE of the two-stage game
   $\rightarrow$ second stage: see 4)
   $\rightarrow$ first stage: plug in NE prices in the profit function:

$$\begin{aligned} \Pi_1 &= (7 - \bar{q_1} - \bar{q_2})\bar{q_1} - 6\bar{q_1} \Leftrightarrow \text{ FOC: } 1 - 2\bar{q_1} - \bar{q_2} = 0 \Leftrightarrow \bar{q_1} = \frac{1 - \bar{q_2}}{2} \\ \Pi_2 &= (7 - \bar{q_2} - \bar{q_1})\bar{q_2} - 6\bar{q_2} \Leftrightarrow \text{ FOC: } 1 - 2\bar{q_2} - \bar{q_1} = 0 \Leftrightarrow \bar{q_2} = \frac{1 - \bar{q_1}}{2} \end{aligned}$$

$$\Rightarrow \bar{q_1}^* = \bar{q_2}^* = \frac{1}{3} \text{ and } \Pi_1^* = \Pi_2^* = (7 - \frac{2}{3})\frac{1}{3} - 6\frac{1}{3} = \frac{1}{9}$$

   $\rightarrow$ NE: firms set quantities equal to $\bar{q_i}^* = \frac{1}{3}$ and choose prices according to $p^* = 7 - \bar{q_1} - \bar{q_2}$
   $\rightarrow$ chosen capacities are equal to the quantities that would result in a standard Cournot game with MC equal to 6

6. If firms collude at the capacity setting stage, they maximize joint profits:

$$\begin{aligned}\Pi^M &= (7-\bar{q})\bar{q} - 6\bar{q} \rightarrow \frac{\partial \Pi^M}{\partial \bar{q}} = 1 - 2\bar{q} = 0 \Leftrightarrow \bar{q} = \frac{1}{2} \\ &\Rightarrow \bar{q_1}^M = \bar{q_2}^M = \frac{\bar{q}}{2} = \frac{1}{4} \text{ and } \Pi_1^M = \Pi_2^M = \frac{\Pi^M}{2} = \frac{1}{8}\end{aligned}$$

7. Reduction in MC to $c = \frac{11}{2}$ changes the results of the first stage:

$$\Pi_i = (7 - \bar{q_i} - \bar{q_j})\bar{q_i} - \frac{11}{2}\bar{q_i} \Leftrightarrow \bar{q_i} = \frac{3}{4} - \frac{\bar{q_j}}{2} \Rightarrow \bar{q_1}^* = \bar{q_2}^* = \frac{1}{2}$$

$\rightarrow$ As costs of capacity fall, $\bar{q_1}^*$ and $\bar{q_2}^*$ increase. Furthermore, profits increase to $\Pi_1 = \Pi_2 = \frac{1}{4}$

As $c \rightarrow 0$, we arrive at capacities equal to Cournot quantities given zero marginal production costs (see question 1). In general, the first-stage game corresponds to solving a Cournot game with MC equal to the capacity costs and the unique second stage game NE corresponding to $p_1 = p_2 = p^* = 7 - \bar{q_1} - \bar{q_2}$.

Exercise 16 *Competition, installed capacity, and demand uncertainty*

Suppose two firms, firm 1 and firm 2, operate in a homogeneous good market. The supply of firm i, denoted by q_i, is constrained by installed capacity k_i, i.e., $0 \leq q_i \leq k_i$ for $i = 1, 2$. Firms have zero marginal costs of production for quantities weakly below their capacity. They cannot increase production beyond capacity.

There is a group of consumers of size $\widetilde{\mu}$ who all have unit demand with the same willingness-to-pay r. Demand $\widetilde{\mu}$ is uncertain. The size of demand is determined by the value a random variable μ takes. Suppose that this random variable is uniformly distributed on the unit interval (i.e., on the interval $[0, 1]$). Due to regulatory intervention there is a price ceiling $P \leq r$ that firms are allowed to charge (lower prices are admissible).

Consider the following two-stage game. At stage 1, before observing the demand realization $\widetilde{\mu}$ firms make investment decisions simultaneously. The constant cost per unit of capacity is c. After the investment stage, information about capacities become public knowledge. Next, demand is realized and publicly observed. At stage 2, firms compete in prices. Each firm simultaneously and independently sets its price p_i. Firms maximize expected profits.

1. Determine the allocation at stage 2 for given capacity choice.
2. Determine the Nash equilibrium at stage 2 for given capacities. Here you have to distinguish between four different parameter regions. Note that

in two regions pure-strategy equilibria do no exist and you have to solve for mixed-strategy equilibria. Also note that since all consumer have the same willingness-to-pay you can be agnostic about the rationing rule that is applied. If prices are equal, demand is assumed to be equally split between firms.

3. Analyze the 2-stage game and solve for subgame perfect Nash equilibria (HINTS: Do not treat firms as symmetric. Consider capacity as a pure strategy). What is the expected profit at this stage? Draw the best-response functions of the two firms in a diagram. Determine the aggregate equilibrium capacity $K = k_1 + k_2$ in this game. Implicitly characterize equilibrium capacities k_1 and k_2. Can you say anything about equilibrium profits (e.g., whether they are the same for the two firms or which firm enjoys higher profit)?

Exercise 17 *Cournot equilibrium and competitive limit*

Suppose there are N firms in a homogeneous good market which set their output sequentially (firm i in period i). Suppose that firms have identical constant marginal costs of production c. The industry faces an inverse demand $P(q) = a - q$ where $q = \sum_{i=1}^{N} q_i$ is aggregate demand. Suppose that $a > c$.

1. What is the output level of firm i in the subgame perfect equilibrium?
2. What is the aggregate output for N firms?
3. Describe the equilibrium outcome when the number of firms increases without bounds ($N \to \infty$).

Solutions to Exercise 17

N firms, $MC = 0$, homogenous good, output is set sequentially
$\Rightarrow$ Search for SPNE (backward induction starting in period N with firm N)

firm N:

$$\begin{aligned}
\pi_N &= (p-c)q_N = (a-q-c)q_N = (a - \sum_{j=1}^{N-1} q_j - q_N - c) \cdot q_N \\
\rightarrow \frac{\partial \pi_N}{\partial q_N} &= 0 \Leftrightarrow q_N^* = \frac{1}{2}(a - \sum_{j=1}^{N-1} q_j - c)
\end{aligned}$$

firm $N-1$:

$$\begin{aligned}
\pi_{N-1} &= (a-q-c)q_{N-1} = (a - \sum_{j=1}^{N-2} q_j - q_{N-1} - \underbrace{q_N^*}_{\text{expected: } \Rightarrow \text{ plug in } q_N} - c) \cdot q_{N-1} \\
&= \frac{1}{2} \cdot (a - \sum_{j=1}^{N-2} q_j - q_{N-1} - c) q_{N-1} \\
\rightarrow \frac{\partial \pi_{N-1}}{\partial q_{N-1}} &= 0 \Leftrightarrow q_{N-1}^* = \frac{1}{2} \cdot (a - \sum_{j=1}^{N-2} q_j - c)
\end{aligned}$$

...analogously for <u>$N-2$:</u> $\quad q_{N-2}^* = \frac{1}{2} \cdot (a - \sum_{j=1}^{N-3} q_j - c)$

Hence, $q_{N-k}^* = \frac{1}{2} \cdot (a - \sum_{j=1}^{N-k-1} q_j - c)$ or $q_i^* = \frac{1}{2} \cdot (a - \sum_{j=1}^{i-1} q_j - c)$.

$\Rightarrow$ Compare q_i^* with q_{i-1}^*:

$$\begin{aligned}
q_i^* - q_{i-1}^* &= \frac{1}{2}(a - \sum_{j=1}^{i-1} q_j - c) - \frac{1}{2}(a - \sum_{j=1}^{i-2} q_i - c) \\
\Leftrightarrow q_i^* &= \frac{1}{2}\, q_{i-1}^* \qquad \text{(I)}
\end{aligned}$$

$\Rightarrow$ firms always produce less (half) than the firm before them!

What does the first firm $i=1$ do?

$q_{i=1}^* = \frac{1}{2}(a-c-0) = \frac{1}{2}(a-c)$ (II)

$\Rightarrow$ from (I) and (II) we find solutions for questions (1) to (3):

1. $q_i^* = \dfrac{1}{2^i}(a-c)$

2. aggregate output: $q = \sum_{i=1}^N q_i = \sum_{i=1}^N \frac{1}{2^i}(a-c) = (a-c)\sum_{i=1}^N \frac{1}{2^i} = (a-c) \cdot (1 - \frac{1}{2}N)$

3. For $N \rightarrow \infty$: $q \rightarrow (a-c)$

Note that for $q \rightarrow (a-c)$: $P \rightarrow c$, i.e. Bertrand competition (But firms are asymmetric in the sense of q_i^*.)

Exercise 18 *Competing in price-quantity pairs*

Consider a duopoly for a homogeneous product. Firms $i = 1, 2$ set price-quantity pairs (p_i, q_i) simultaneously. If at these pairs some consumers are rationed, rationing is assumed to be efficient. Suppose firms are constrained by capacities $k_i > 0$ and inverse market demand is $P(q) = q^{-1}$. Does a Nash equilibrium in pure strategies exist? Is it unique? Characterize all Nash equilibria of this game

Exercise 19 *Sequential price setting with differentiated products [included in the 2nd edition of the book]*

Consider a market with two horizontally differentiated products and inverse demands given by $P_i(q_i, q_j) = a - bq_i - dq_j$. Set $b = 2/3$ and $d = 1/3$. The system of demands is then given by $Q_i(p_i, p_j) = a - 2p_i + p_j$. Suppose firm 1 has cost $c_1 = 0$ and firm 2 has cost $c_2 = c$ (with $7c < 5a$). The two firms compete in prices. Compute the firms' profits:

1. at the Nash equilibrium of the simultaneous Bertrand game,
2. at the subgame perfect equilibrium of the sequential game
   (a) with firm 1 being the leader, and
   (b) with firm 2 being the leader.
3. Show that firm 2 always has a second-mover advantage, whereas firm 1 has a first-mover advantage if c is large enough.
4. Solve for the Nash equilibria of the endogenous timing game in which firms simultaneously choose whether to play 'early' or to play 'late'. If they both make the same choice (either 'early' or 'late'), the simultaneous Bertrand game follows; if they make different choices, a sequential game follows with the firm having chosen 'early' being the leader.

Solutions to Exercise 19

1. Nash equilibrium of the simultaneous Bertrand game. Firm 1 chooses p_1 to maximize $\pi_1 = (a - 2p_1 + p_2)p_1$. Solving the first-order condition for p_1, one finds firm 1's reaction function as $p_1 = (a + p_2)/4$. Firm 2 chooses p_2 to maximize $\pi_2 = (a - 2p_2 + p_1)(p_2 - c)$. Solving the first-order condition for p_2, one finds firm 2's reaction function as $p_2 = (a + 2c + p_1)/4$. The solution to the system made of the two reaction functions is $p_1 = (5a + 2c)/15$ and $p_2 = (5a + 8c)/15$. We can then compute the equilibrium profits as $\pi_1^B = \frac{2}{225}(5a + 2c)^2$ and $\pi_2^B = \frac{2}{225}(5a - 7c)^2$.
2. Sequential game
   (a) If firm 1 is the leader, it takes firm 2's price reaction into account when maximizing profits; that is, it chooses p_1 to maximize $\pi_1 = (a - 2p_1 + \frac{1}{4}(a + 2c + p_1))p_1$. The optimal price is found as $p_1 = (5a + 2c)/14$. Firm 2 then reacts by setting $p_2 = (19a + 30c)/56$. Plugging the prices into the profit functions, one obtains $\pi_1^L = (5a + 2c)^2/112$ and $\pi_2^F = (19a - 26c)^2/1568$.

(b) If firm 2 is the leader, it takes firm 1's price reaction into account when maximizing profits; that is, it chooses p_2 to maximize $\pi_2 = (a - 2p_2 + \frac{1}{4}(a + p_2))(p_2 - c)$. The optimal price is found as $p_1 = (5a + 7c)/14$. Firm 1 then reacts by setting $p_1 = (19a + 7c)/56$. Plugging the prices into the profit functions, one obtains $\pi_2^L = (5a - 7c)^2/112$ and $\pi_1^F = (19a + 7c)^2/1568$.

3. Firm 2 always has a second-mover advantage: $\pi_2^F = (19a - 26c)^2/1568 > \pi_2^L = (5a - 7c)^2/112$ is equivalent to $(19 - 5\sqrt{14})a + (7\sqrt{14} - 26)c > 0$, which is satisfied as $19 - 5\sqrt{14} = 0.292 > 0$ and $7\sqrt{14} - 26 = 0.192 > 0$. Firm 1 has a first-mover advantage if $\pi_1^L = (5a+2c)^2/112 > \pi_1^F = (19a+7c)^2/1568$, which is equivalent to $7c^2 + 14ac - 11a^2 > 0$ or $c > 0.604a$. (Note: $0.604 < 5/7$, i.e., there is a positive interval of c where the above condition is fulfilled.)

Exercise 20 *Timing game [included in the 2nd edition of the book]*

Use the results of the previous exercise to solve for the Nash equilibria of the endogenous timing game in which firms simultaneously choose whether to play 'early' or to play 'late'. If they both make the same choice (either 'early' or 'late'), the simultaneous Bertrand game follows; if they make different choices, a sequential game follows with the firm having chosen 'early' being the leader. Discuss the economic intuition behind your result.

Solutions to Exercise 20 The following matrix represents the normal form of the game:

Firm 1 / Firm 2	Early	Late	
Early	π_1^B, π_2^B	π_1^L, π_2^F	α
Late	π_1^F, π_2^L	π_1^B, π_2^B	$1-\alpha$
	β	$1-\beta$	

As far as firm 2 is concerned, we already know from Exercise 4.1 that $\pi_2^F > \pi_2^L$. It is also easy to see that $\pi_2^L = 2(5a - 7c)^2/224$ is larger than $\pi_2^B = 2(5a - 7c)^2/225$. We have thus $\pi_2^F > \pi_2^L > \pi_2^B$. As for firm 1, it is easy to see that $\pi_1^L = 2(5a+2c)^2/224 > \pi_1^B = 2(5a + 2c)^2/225$. We can also show that $\pi_1^F > \pi_1^B$:

$$\begin{aligned} \pi_1^F - \pi_1^B &= \tfrac{1}{1568}(19a + 7c)^2 - \tfrac{2}{225}(5a + 2c)^2 \\ &= \tfrac{1}{352\,800}(5a - 7c)(565a + 217c), \end{aligned}$$

which is positive as we assume that $7c < 5a$. Hence, there are two pure-strategy NE: (Early, Late) and (Late, Early). There also exist a mixed-strategy NE. To characterize it, let firm 1 choose α, the probability with which it plays (Early), such that firm 2 is indifferent between (Early) and (Late); that is, $\alpha\pi_2^B + (1-\alpha)\pi_2^L = \alpha\pi_2^F + (1-\alpha)\pi_2^B$, which is equivalent to $\alpha = (\pi_2^L - \pi_2^B)/(\pi_2^F + \pi_2^L - 2\pi_2^B)$. We proceed in the same way for firm 2, which chooses β, the probability with which it chooses (Early), to make firm 1 indifferent between (Early) and (Late); that is, $\beta\pi_1^B + (1-\beta)\pi_1^L = \beta\pi_1^F + (1-\beta)\pi_1^B$;

which implies that $\beta = \left(\pi_1^L - \pi_1^B\right) / (\pi_1^F + \pi_1^L - 2\pi_1^B)$. Replacing the various profit level by their respective value, one finds

$$\alpha = \frac{14(5a-7c)^2}{3175a^2-3760ac-878c^2} \text{ and } \beta = \frac{14(5a+2c)^2}{3175a^2-2590ac-1463c^2}.$$

Exercise 21 *Information sharing in Cournot duopoly [included in the 2nd edition of the book]*

Consider the Cournot duopoly with linear demand $P(q) = 1 - q$ with $q = q_1 + q_2$ and constant marginal cost. Firm one has marginal cost of zero. This is commonly known. The marginal cost of firm 2 is privately known to firm 2; firm 1 only knows that they are prohibitively high or zero and that both events are equally likely (this is commonly known). High marginal costs are assumed to be prohibitively high such that firm 2 does not produce. Consider the three-stage game in which, at stage 1, firm 2 draws its marginal costs, then, at stage 2, it decides whether to share its information with its competitor and, at stage 3, in which both firms compete in quantity.

1. Characterize the equilibrium of this game. Does firm 2 have an incentive to share its private information?
2. Would firm 1 be better off under information sharing?

Solutions to Exercise 21

Suppose first that firm 2 shares its information. In this case firm 1 learns the cost type of firm 2. If firm 2's costs are high, firm 1 knows that firm 2 will produce zero; thus, firm 1 will produce the monopoly quantity $q_1^m = 1/2$; the monopoly profit is $\pi = 1/4$. If the costs of firm 2 are zero, we have a symmetric Cournot duopoly.

Firms will produce duopoly quantities. $q_1^d = q_2^d = 1/3$; and profits are $\pi_i^d = 1/9$.

Now suppose that firm 2 does not share its information. Then firm 1 produces a quantity between q_1^d and q_1^m. Profits of firm 1 and the low-cost firm 2 are

$$\begin{aligned} \pi_1 &= \frac{1}{2}(1 - q_1 - q_2)q_1 + \frac{1}{2}(1 - q_1)q_1 \\ \pi_2 &= (1 - q_1 - q_2)q_2 \end{aligned}$$

First-order conditions of profit maximization at stage 2 can be written as:

$$\begin{aligned} q_2 &= \frac{1}{2} - \frac{1}{2}q_1, \\ q_1 &= \frac{1}{2} - \frac{1}{4}q_2. \end{aligned}$$

Solving this system we obtain $q_2 = 2/7$ and $q_1 = 3/7$. We note that $q_1^m = 1/2 > 3/7 > 1/3 = q_1^d$. Equilibrium at stage 2 under information sharing are

$$\pi_1^s = \frac{1}{2}\left(1-\frac{3}{7}-\frac{2}{7}\right)\frac{3}{7}+\frac{1}{2}\left(1-\frac{3}{7}\right)\frac{3}{7}=\frac{9}{49}$$
$$\pi_2^{sL} = \left(1-\frac{3}{7}-\frac{2}{7}\right)\frac{2}{7}=\frac{4}{49}$$

Because of strategic substitutes firm 1's profit is larger than firm 2's profit.

We now turn to the decision of firm 2 at stage 1. If the costs of firm 2 are high, it obtains zero profit independent of whether it share information. If the costs of firm 2 are low, then its profit will be higher if it shares information ($1/9$ rather than $4/49$), because then firm 1 will produce less. Thus, firm 2 wants to share its information.

Note this is certainly true ex ante (i.e., if firm 2 has to decide before knowing its cost). In the example, it is also true interim (after learning its cost), because the high cost type is indifferent whether to share information.

What about firm 1's profit? If firm 2 shares information, firm 1's expected profit is: $(1/2)\pi_1^m+(1/2)\pi_1^d=(1/2)(1/4)+(1/2)(1/9)=13/72$. If firm 2 does not share information, its profit has been calculated to be $9/49$. Since $9/49>13/72$, firm 1 would be better off if firm 2 did share information.

Exercise 22 *Price competition and information sharing*

Consider the same setting except that firms face a different demand function and that firms set prices at stage 3. Let demand be $Q_i=1-p_i-dp_j$ with $d>0$ so that products are substitutes and $d<1$. Characterize the equilibrium of this game. Does firm 2 have an incentive to share its private information?

Solutions to Exercise 22

Suppose that firm 2 decides to share its information at stage 2. If its costs are high, firm 1 knows that firm 2 will produce zero. Thus, firm 1 will set the monopoly price $p_1^m=\arg\max_{p_1}(1-p_1)p_1=1/2$. If the costs are symmetric, we have a symmetric price competition model with linear demand. Each firm i maximizes $(1-p_i-dp_j)p_i$ subject to p_i. The first-order condition of profit maximization can be written as $p_i=1/2-(1/2)dp_j$. Using symmetry, we obtain that $p_i^d=1/(2+d)$.

Now suppose that firm 2 does not share information. Then firm 1 sets an intermediate price that is between $p_1^d$and p_1^m, as we will show next. Profits are

$$\pi_1 = \frac{1}{2}(1-p_1)p_1+\frac{1}{2}(1-p_1-dp_2)p_1,$$
$$\pi_2 = (1-p_2-dp_1)p_2.$$

First-order conditions of profit maximization can be written as

$$p_1 = \frac{1}{2}-\frac{1}{4}ap_2,$$
$$p_2 = \frac{1}{2}-\frac{1}{2}ap_1.$$

With information sharing, the price of firm 1 is

- higher if the cost of firm 2 are high (monopoly vs. intermediate price)
- lower if the cost of firm 2 are high (duopoly vs. intermediate price)

Let's compare the profits of firm 2: Suppose that the costs of firm 2 are high. Then it obtains zero profits anyhow.

Suppose that the costs of firm 2 are low. Then its profits are lower if it has shared information, because in this situation firm 1 will produce less. Thus, firm 2 does not want to share information! Moral: it depends on Cournot vs. Betrand.