An Interactive Introduction to Mathematical Analysis: Instructor’s Manual

This printable form of the instructor’s manual is a companion to the on-screen instructors manual for my book. To install the on-screen instructor’s manual, you need to convert your installation of the book from a student version to an instructor’s version. After installing the book, you should run the executable file

lewin-analysis-book-instructors-manual.exe

which can be found at the address

http://science.kennesaw.edu/~jlewin/analysis/instructors-manual/lewin-analysis-book-instructors-manual.exe

http://science.kennesaw.edu/~jlewin/CUP/lewin-analysis-instructor’s-manual.exe

The running of this executable file requires a password that can be obtained by bona fide instructors at

solutions@cambridge.org

In both the student version and the instructor’s version of the text, each group of exercises comes with a link to a solutions document. In the instructors version, the link takes you to a solutions document that shows the solutions provided to students in blue and shows the solutions provided only to instructors in green. If you make a monochrome print of this printable form of the manual, then the solutions provided to students will appear in a bold italics font and the solutions provided only to instructors will appear in an upright sans serif font.

Jonathan Lewin

2 Mathematical Grammar

Exercises on Use of Quantifiers

Except in Exercise 2, decide whether the the sentence that appears in the exercise is meaningful or meaningless. If the sentence is meaningful, say whether what it says is true or false.

1. a. $\sqrt{x^2} = x$.

   Solution: ***This statement is meaningless because*** x ***is unquantified.***

b. For every real number x we have $\sqrt{x^2} = x$.

Solution: ***This statement is meaningful but false because the equation $\sqrt{x^2} = x$ is false whenever x is negative.***

c. For every positive number x we have $\sqrt{x^2} = x$.

Solution: ***This statement is true.***

2. a. Point at the expression $\sqrt{x^2}$ and click on the Evaluate button .

   b. Point at the expression $\sqrt{x^2}$ and click on the Simplify button .

   c. Point at the equation $\sqrt{x^2} = x$, open the Maple menu and click on Check Equality.

   d. Point at the equation $x = -2$ and click on the button to supply the definition $x = -2$ to *Scientific Notebook*. Then try a Check Equality on the equation $\sqrt{x^2} = x$.

3. For every number x and every number y there is a number z such that $z = x + y$.

   Solution: ***This statement is true.***

4. For every number x there is a number z such that for every number y we have $z = x + y$.

   Solution: ***This statement is false.***

5. For every number x and every number z there is a number y such that $z = x + y$.

   Solution: ***This statement is true.***

6. $\sin^2 x + \cos^2 x = 1$.

   Solution: ***This statement is meaningless because x is unquantified.***

7. For every number x we have $\sin^2 x + \cos^2 x = 1$.

   Solution: ***This statement is true.***

8. For every integer $n > 1$, if $n^2 \leq 3$ then the number 57 is prime.

   Solution: ***Since it is impossible to find an integer $n > 1$ such that $n^2 \leq 3$, the assertion that 57 is prime for every such integer is true. The fact that the number 57 happens not to be prime has no bearing on this exercise.***

Exercises on Order of Appearance of Unknowns

For each of the following pairs of statements, decide whether or not the statements are saying the same thing. Except in the first two exercises, say whether or not the given statements are true.

1. a. Every person in this room has seen a good movie that has started playing this week.

   b. A good movie that has started playing this week has been seen by every person in this room.

   Solution: ***The second statement asserts that there is one particular good movie that everyone in the room has seen. The first statement says less. It asserts that everyone has seen a good movie but leaves open the possibility that different people may have seen different movies.***

2. a. Only men wearing top hats may enter this hall.

b. Only men may enter this hall wearing top hats.

c. Men wearing top hats only may enter this hall.

d. Men wearing only top hats may enter this hall.

e. Men wearing top hats may enter this hall only.

Hint: ***These five statements are all different from one another.*** The first statement is ambiguous. It could mean that the only people who may enter this hall are men who are wearing top hats. However, with a different voice inflection it could mean that top hats are permitted only to men and that women will not be permitted entry if they are wearing top hats, leaving open the question of whether a woman who is not wearing a top hat may enter the hall. In other words, a change in voice inflection could make statements a. and b. say the same thing.

3. a. For every nonzero number x there is a number y such that $xy = 1$.

   Solution: *This statement is true because of x is any nonzero number then we have $x(1/x) = 1$.*

   b. There is a number y for which the equation $xy = 1$ is true for every nonzero number x.

   Solution: ***This statement is false.***

4. a. For every number $x \in [0,1)$ there exists a number $y \in [0,1)$ such that $x < y$.

   Solution: ***True***

   b. There is a number $y \in [0,1)$ satisfying $x < y$ for every number $x \in [0,1)$.

   Solution: ***False***

5. a. For every number $x \in [0,1]$ there exists a number $y \in [0,1]$ such that $x < y$.

   b. There is a number $y \in [0,1]$ satisfying $x < y$ for every number $x \in [0,1]$.

Solution: ***These two statements are not saying the same thing but who cares! Both statements are false.***

6. a. For every number $x \in [0,1)$ there exists a number $y \in [0,1)$ such that $x \le y$.

   This statement is true. Given any $x \in [0,1]$, if we define $y = x$ then we have found a number $y \in [0,1]$ such that $x \le y$.

   b. There is a number $y \in [0,1)$ satisfying $x \le y$ for every number $x \in [0,1)$.

   This statement is false because it asserts that the interval $[0,1)$ has a largest member.

7. a. For every number $x \in [0,1]$ there exists a number $y \in [0,1]$ such that $x \le y$.

   b. There is a number $y \in [0,1]$ satisfying $x \le y$ for every number $x \in [0,1]$.

This time, both of the statements are true.

8. a. For every odd integer m it is possible to find an integer n such that mn is even.

   b. It is possible to find an integer n such that for every odd integer m the number mn is even.

Solution: ***These two statements do not say the same thing but they are both true.***

9. a. For every number x it is possible to find a number y such that $xy = 0$.

   b. It is possible to find a number y such that for every number x we have $xy = 0$.

Solution: ***These two statements do not say the same thing but they are both true.***

10. a. For every number x it is possible to find a number y such that $xy \neq 0$.

b. It is possible to find a number y such that for every number x we have $xy \neq 0$.

Solution: ***These two statements do not say the same thing but they are both false.***

11. a. For every number a and every number b there exists a number c such that $ab = c$.

Solution: ***True***

b. For every number a there exists a number c such that for every number b we have $ab = c$.

Solution: ***False***

12. a. For every number a and every number c there exists a number b such that $ab = c$.

b. For every number a there exists a number b such that for every number c we have $ab = c$.

Solution: ***These two statements do not say the same thing but they are both false.***

13. a. For every nonzero number a and every number c there exists a number b such that $ab = c$.

Solution: ***True***

b. For every nonzero number a there exists a number b such that for every number c we have $ab = c$.

Solution: ***False***

Some Exercises on Negations and the Quantifiers

Write a negation for each of the following statements:

1. All roses are red.

   Solution: ***The negation could say: Not all roses are red. Alternatively it could say: There is at least one rose that isn't red. However it would be wrong to write the negation as: All roses are not red.***

2. In Sam's flower shop there is at least one rose that isn't red.

   Solution: ***All the roses in Sam's flower shop are red.***

3. In every flower shop there is at least one rose that isn't red.

   Solution: ***There is at least one flower shop in which all the roses are red.***

4. I believe that all roses are red.

   Solution: ***I do not believe that all roses are red.***

5. There is at least one person in this room who thinks that all roses are red.

   Solution: ***No one in this room thinks that all roses are red.***

6. Every person in this room believes that all roses are red.

   There is at least one person in this room who does not believe that all roses are red.

7. At least half of the people in this room believe that all roses are red.

   Solution: ***More than half of the people in this room do not believe that all roses are red.***

8. Every man believes that all women believe that all roses are red.

   Solution: ***There are men who do not believe that all women believe that all roses are red.***

9. You were at least an hour late for work every day last week.

Solution: ***There was at least one day last week on which you began work less than an hour late.***

10. It has never rained on a day on which you have remembered to take your umbrella.

Solution: ***There have been days on which it has rained and on which you have remembered to take your umbrella.***

11. You told me that it has never rained on a day on which you have remembered to take your umbrella.

Solution: ***You have not told me that it has never rained on a day on which you have remembered to take your umbrella.***

12. You lied when you told me that it has never rained on a day on which you have remembered to take your umbrella.

Solution: ***You did not lie when you told me that it has never rained on a day on which you have remembered to take your umbrella.***

13. I was joking when I said that you lied when you told me that it has never rained on a day on which you have remembered to take your umbrella.

Solution: ***I was not joking when I said that you lied when you told me that it has never rained on a day on which you have remembered to take your umbrella.***

14. This, Watson, if I mistake not, is our client now.

Solution: ***This statement is meaningless because it refers to itself. We should have been able to expect better logical precision from Sherlock Holmes.***

15. a. For every real number x there exists a real number y such that

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement is true?

Solution: ***Yes this statement is true.***

Negation: ***There exists a real number*** x ***such that for every real number*** y***, the equation***

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}$$

is either false or meaningless.

b. There exists a real number x such that for every real number y we have

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement is true?

Solution: ***This statement is false because the equation shown her can't be true when*** $y = x$***.***

Negation: ***For every real number*** x ***there exists a real number*** y ***for which the equation***

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}$$

is either false or meaningless.

c. For every real number x and every real number $y \neq x$ we have

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement is true?

Solution: ***This statement is true.***

Negation: ***There exists a real number x and there exists a real number $y \neq x$ such that the equation***

$$\frac{2x^2 + xy - y^2}{x^3 - y^3} = \frac{2}{3(x-y)} + \frac{5y + 4x}{3(x^2 + xy + y^2)}$$

is either false or meaningless.

Some Exercises on The Use of Conditionals

In the exercises that follow you should assume that P, Q, R and S are given statements that may be either true or false.

1. Write down the denial, the converse and the contrapositive form of each of the following statements:

   a. *All cats scratch.*

   The converse: ***All things that scratch are cats.***

   The contrapositive: ***All things that do not scratch are not cats.***

   The denial: ***There is at least one cat that does not scratch.***

   b. *If what you said yesterday is correct, then Jim has red hair.*

   The converse: ***If Jim has red hair then what you said yesterday is correct.***

   The contrapositive: ***If Jim does not have red hair then what you said yesterday is false.***

   The denial: ***What you said yesterday is correct and Jim does not have red hair.***

   c. *If a triangle $\triangle ABC$ has a right angle at C then*

   $$(AB)^2 = (AC)^2 + (BC)^2.$$

   The converse: ***Every triangle $\triangle ABC$ satisfying the the condition***

   $$(AB)^2 = (AC)^2 + (BC)^2.$$

   must have a right angle at C.

   The contrapositive: ***Every triangle $\triangle ABC$ that fails to satisfy the the condition***

   $$(AB)^2 = (AC)^2 + (BC)^2.$$

   will not have a right angle at C.

   The denial: ***There is at least one triangle $\triangle ABC$ that has a right angle at C and for which the condition***

   $$(AB)^2 = (AC)^2 + (BC)^2.$$

   does not hold.

   d. *If some cats scratch, then all dogs bite.*

   The converse: ***If all dogs bite then some cats scratch.***

   The contrapositive: ***If there exists a dog that does not bite then no cats scratch.***

   The denial: ***Some cats scratch but not all dogs bite.***

   e. *It is with regret that I inform you that someone in this room is smoking.*

   The converse: ***This statement doesn't have a converse.***

   The contrapositive: ***This statement doesn't have a contrapositive form.***

   The denial: ***It is without regret that I inform you that someone in this room is smoking.***

f. *If a function is differentiable at a given number then it must be continuous at that number.*

The converse: ***If a function is continuous at a given number then it must be differentiable at that number.***

The contrapositive: ***If a function is not continuous at a given number then it cannot be differentiable at that number.***

The denial: ***There exists a function and a real number such that the function is differentiable at the number but fails to be continuous at that number.***

g. *Every boy or girl alive is either a little liberal or else a conservative.*

The converse: ***Every little liberal and every conservative is a living boy or girl***

The contrapositive: ***Any individual who fails to be either a little liberal or a conservative cannot be either a living boy or a living girl.***

The denial: ***There is at least one living boy or girl who is neither a little liberal nor a conservative.***

2. In each of the following exercises, write down a denial of the given statement.

a. *All cats scratch and some dogs bite.*

The denial: ***Either there is a cat that does not scratch or no dogs bite.***

b. *Either some cats scratch or if all dogs bite then some birds sing.*

The denial: ***No cats scratch and all dogs bite and no birds sing.***

c. *He walked into my office this morning, told me a pack of lies and punched me on the nose.*

The denial: ***Either he did not walk into my office this morning or he did not tell me a pack of lies or he did not punch me on the nose.***

d. *No one has ever seen an Englishman who is not carrying an umbrella.*

The denial: ***At least one person has seen an Englishman who is not carrying an umbrella.***

e. *For every number x there exists a number y such that* $y > x$.

The denial: There exists a number x ***such that for every number*** y ***we have*** $y \leq x$.

3. In each of the following exercises we assume that f and g are given functions. Write down a denial of each of the following statements:

a. *Whenever* $x > 50$, *we have* $f(x) = g(x)$.

The denial: ***There exists a number*** $x > 50$ ***such that*** $f(x) \neq g(x)$.

b. *There exists a number w such that* $f(x) = g(x)$ *for all numbers* $x > w$.

The denial: ***For every number*** w ***there exists a number*** $x > w$ ***such that*** $f(x) \neq g(x)$.

c. *For every number x there exists a number* $\delta > 0$ *such that for every number t satisfying the condition* $|x - t| < \delta$, *we have* $|f(x) - f(t)| < 1$.

The denial: ***There exists a number*** x ***such that for every number*** $\delta > 0$ ***there is at least one number*** t ***satisfying the condition*** $|x - t| < \delta$ ***such that*** $|f(x) - f(t)| \geq 1$.

d. *There exists a number* $\delta > 0$ *such that for every pair of numbers x and t satisfying the condition* $|x - t| < \delta$, *we have* $|f(x) - f(t)| < 1$.

The denial: ***For every number*** $\delta > 0$ ***it is possible to find a pair of numbers*** x ***and*** t ***satisfying the condition*** $|x - t| < \delta$ ***and for which*** $|f(x) - f(t)| \geq 1$.

e. *For every number* $\varepsilon > 0$ *and for every number* x*, there exists a number* $\delta > 0$ *such that for every number* t *satisfying* $|x - t| < \delta$*, we have* $|f(t) - f(x)| < \varepsilon$.

The denial: ***There exists a number*** $\varepsilon > 0$ ***and a number*** x ***such that for every number*** $\delta > 0$ ***it is possible to find a number*** t ***such that*** $|x - t| < \delta$ ***and*** $|f(x) - f(t)| \geq \varepsilon$.

f. *For every positive number* ε *there exists a positive number* δ *such that for every pair of numbers* x *and* t *satisfying the condition* $|x - t| < \delta$*, we have* $|f(x) - f(t)| < \varepsilon$.

The denial: ***There exists a number*** $\varepsilon > 0$ ***such that for every number*** $\delta > 0$ ***it is possible to find a pair of numbers*** x ***and*** t ***satisfying the condition*** $|x - t| < \delta$ ***for which*** $|f(x) - f(t)| \geq \varepsilon$.

4. Explain why the statement $\neg(P \Rightarrow Q)$ is equivalent to the statement $P \wedge (\neg Q)$.

   The assertion $P \Rightarrow Q$ says that if P is true then Q must also be true. This assertion says nothing at all about Q in the event that P is false. The only way in which the assertion $P \Rightarrow Q$ can be false is that P is true and Q is not. In other words, the denial of the condition $P \Rightarrow Q$ says that $P \wedge (\neg Q)$.

5. Explain why the statement $\neg(P \Leftrightarrow Q)$ is equivalent to the assertion that either (P is true and Q is false) or (P is false and Q is true).

   The assertion $P \Leftrightarrow Q$ says that P and Q have the same truth value. So the assertion $\neg(P \Leftrightarrow Q)$ says that they don't, which means that one of them is true and the other is false.

6. Explain why the statement $\neg(P \vee Q)$ is equivalent to the statement $(\neg P) \wedge (\neg Q)$.

   The assertion $P \vee Q$ says that at least one of the statements P and Q is true. So the denial of the the assertion $P \vee Q$ says that they are both false.

7. Explain why the statement $\neg(P \Rightarrow (Q \vee R))$ is equivalent to the assertion that P is true and that both of the statements Q and R are false.

   The denial of the assertion $P \Rightarrow (Q \vee R)$ says that P is true but that the assertion $Q \vee R$ is false. In other words, it says that P is true and that both of the statements Q and R are false.

8. Explain why the converse of the statement $P \Rightarrow (Q \vee R)$ is equivalent to the condition $(R \Rightarrow P) \wedge (Q \Rightarrow P)$.

   The converse of the statement $P \Rightarrow (Q \vee R)$ says that $(Q \vee R) \Rightarrow P$ and this says that P must be true if at least one of the statments Q and R are true. In other words, the satement $(Q \vee R) \Rightarrow P$ says that $Q \Rightarrow P$ and also that $R \Rightarrow P$.

9. Write the assertion $P \Rightarrow (Q \vee R)$ as simply as you can in its contrapositive form.

   The contrapositive form of the assertion $P \Rightarrow (Q \vee R)$ says that $(\neg(Q \vee R)) \Rightarrow (\neg P)$ and this says that if both of the statements Q and R are false then P is false. In other words, this contrapositive form says that $((\neg Q) \wedge (\neg R)) \Rightarrow (\neg P)$.

10. Write the assertion $(P \wedge Q) \Rightarrow (R \vee S)$ as simply as you can in its contrapositive form.

    The contrapositive form of the assertion $(P \wedge Q) \Rightarrow (R \vee S)$ says that

    $$\neg(R \vee S) \Rightarrow \neg(P \wedge Q)$$

    which we can write as

    $$(\neg R) \wedge (\neg S) \Rightarrow (\neg P) \vee (\neg Q).$$

    The latter statement says that if both of the statements R and S are false then at least one of the statements P and Q is false.

3 Strategies for Writing Proofs