Chapter 2

2.1 Take as sample space the set of all ordered pairs (i, j), where the outcome (i, j) represents the two numbers shown on the dice. Each of the 36 possible outcomes is equally likely. Let A be the event that the sum of the two dice is 8 and B be the event that the two numbers shown on the dice are different. There are 30 outcomes (i, j) with $i \neq j$. In four of those outcomes i and j sum to 8. Therefore $P(AB) = \frac{4}{36}$ and $P(B) = \frac{30}{36}$. The sought probability $P(A \mid B)$ is $\frac{4/36}{30/36} = \frac{2}{15}$.

2.2 The ordered sample space consists of the eight equally likely elements (H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, T, T), (T, T, H), (T, H, T), and (T, H, H), where the first component refers to the nickel, the second to the dime and the third to the quarter. Let A the event that the quarter shows up heads and B be the event that the coins showing up heads represent an amount of at least 15 cents. To find $P(A \mid B) = P(AB)/P(B)$, note that the set AB consists of four elements (H, H, H), ,(H,T,H),(T,T,H) and (T, H, H), while the set B consists of the 5 elements (H, H, H), (H, H, T), (H, T, H), (T, T, H), and (T, H, H). This gives $P(AB) = \frac{4}{8}$ and $P(B) = \frac{5}{8}$. Hence the desired probability $P(A \mid B)$ is $\frac{4}{5}$.

2.3 Take as sample space the set of the ordered pairs (G, G), (G, F), (F, G), and (F, F), where G stands for a "correct prediction" and F stands for a "false prediction," and the first and second components of each outcome refer to the predictions of weather station 1 and weather station 2. The probabilities $0.9 \times 0.8 = 0.72$, $0.9 \times 0.2 = 0.18$, $0.1 \times 0.8 = 0.08$, and $0.1 \times 0.2 = 0.02$ are assigned to these elements. Let the event $A = \{(G, F)\}$ and the event $B = \{(G, F), (F, G)\}$. The sought probability is $P(A \mid B) = \frac{0.18}{0.26} = \frac{9}{13}$.

2.4 Let A be the event that a randomly chosen student passes the first test and B be the event that this student also passes the second test. Then $P(B \mid A) = \frac{0.50}{0.80} = 0.625$. The answer is 62.5%.

2.5 Let A be the event that a randomly chosen household has a cat and B be the event that the household has a dog. Then, $P(A) = 0.3$, $P(B) = 0.25$, and $P(B \mid A) = 0.2$. The sought probability $P(A \mid B)$ satisfies

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B \mid A)}{P(B)}$$

and thus is equal to $0.3 \times 0.2/0.25 = 0.24$.

2.6 The ordered sample space is the set $\{(H,1),\ldots,(H,6),(T,1),\ldots,(T,6)\}$. Each outcome is equally likely to occur. Let A be the event that the coin lands heads and B the event that the die lands six. The set A consists of six elements, the set AB consists of a single element and the set $A \cup B$ consists of seven elements. Hence the desired probabilities are given by

$$P(AB \mid A \cup B) = \frac{P(AB)}{P(A \cup B)} = \frac{1}{7} \text{ and } P(A \mid A \cup B) = \frac{P(A)}{P(A \cup B)} = \frac{6}{7}.$$

2.7 Label the two red balls as R_1 and R_2, the blue ball as B and the green ball as G. Take as unordered sample space the set consisting of the six equally likely combinations $\{R_1, R_2\}$, $\{R_1, B\}$, $\{R_2, B\}$, $\{R_1, G\}$, $\{R_2, G\}$, and $\{B, G\}$ of two balls. Let C be the event that two non-red balls have been grabbed, D be the event that at least one non-red ball has been grabbed, and E be the event that the green ball has been grabbed. Then, $P(CD) = \frac{1}{6}$, $P(D) = \frac{5}{6}$, $P(CE) = \frac{1}{6}$ and $P(E) = \frac{3}{6}$. The sought probabilities are $P(C \mid D) = \frac{1}{5}$ and $P(C \mid E) = \frac{1}{3}$. In the second situation you have more information.

2.8 The ordered sample space is the set $\{(i,j) : i,j = 1,2,\ldots,6\}$. Each element is equally likely to occur. Let A be the event that both dice show up an even number and let B the event that at least one of the two dice shows up an even number. The set AB is equal to the set A consisting of 9 elements and the set B consists of 27 elements. The probability $P(A \mid B)$ of your winning of the bet is equal to $\frac{9/36}{27/36} = \frac{1}{3}$. The bet is not fair to you.

2.9 Take as unordered sample space the set of all possible combinations of 13 distinct cards. Let A be the event that the hand contains exactly one ace, B be the event that the hand contains at least one ace, and C be the event that the hand contains the ace of hearts. Then

$$P(A \mid B) = \frac{\binom{4}{1}\binom{48}{12}/\binom{52}{13}}{1 - \binom{48}{13}/\binom{52}{13}} = 0.6304 \text{ and } P(A \mid C) = \frac{\binom{1}{1}\binom{48}{12}}{\binom{1}{1}\binom{51}{12}} = 0.4388.$$

The desired probabilities are 0.3696 and 0.5612. The second case involves more information.

2.10 The probability that the number of tens in the hand is the same as the number of aces in the hand is given by

$$\sum_{k=0}^{4} \binom{4}{k}\binom{4}{k}\binom{44}{13-2k} \Big/ \binom{52}{13} = 0.3162.$$

Hence, using a symmetry argument, the probability that the hand contains more aces than tens is $\frac{1}{2}(1-0.3162) = 0.3419$. Letting A be the event that the hand contains more aces than tens and B the event that the hand contains at least one ace, then $P(A \mid B) = P(AB)/P(B) = P(A)/P(B)$. Therefore

$$P(A \mid B) = \frac{0.3419}{\sum_{k=1}^{4}\binom{4}{k}\binom{48}{13-k}/\binom{52}{13}} = 0.4911.$$

2.11 Let A be the event that each number rolled is higher than all those that were rolled earlier and B be the event that the three different numbers are rolled. Then $P(A) = P(AB)$ and so $P(A) = P(B)P(A \mid B)$. We have $P(B) = \frac{6\times5\times4}{6^3} = \frac{5}{9}$ and $P(A \mid B) = \frac{1}{3!}$. Thus

$$P(A) = \frac{20}{36} \times \frac{1}{3!} = \frac{5}{54}.$$

2.12 (**a**) Since $P(A \mid B) > P(B \mid A)$ is the same as $\frac{P(AB)}{P(B)} > \frac{P(BA)}{P(A)}$, it follows that $P(A) > P(B)$.
(**b**) Since $P(B \mid A) = P(AB)/P(A) = P(A \mid B)P(B)/P(A)$, we get $P(B \mid A) > P(B)$. Also, by $P(B^c \mid A) + P(B \mid A) = [P(B^cA) + P(BA)]/P(A) = P(A)/P(A) = 1$, we get $P(B^c \mid A) = 1 - P(B \mid A) \leq 1 - P(B) = P(B^c)$.
(**c**) If A and B are disjoint, then $P(AB) = 0$ and so $P(A \mid B) = 0$. If B is a subset of A, then $P(AB) = P(B)$ and so $P(A \mid B) = 1$.

2.13 Let A be the event that a randomly chosen student takes Spanish and B be the event that the student takes French. Then, $P(A) = 0.35$, $P(B) = 0.15$, and $P(A \cup B) = 0.40$. Thus $P(AB) = 0.35 + 0.15 - 0.40 = 0.10$ and so $P(B \mid A) = \frac{0.10}{0.35} = \frac{2}{7}$.

2.14 Let A be the event that a randomly chosen child is enrolled in swimming and B be the event that the child is enrolled in tennis. The sought probability $P(A \mid B)$ follows from $P(A \mid B) = P(AB)/P(B) = P(A)P(B \mid A)/P(B)$ and is equal to $(1/3) \times 0.48/0.40 = 0.64$.

2.15 Let A be the event that a randomly chosen voter is a Democrat, B be the event that the voter is a Republican, and C be the event that the voter is in favor of the election issue.
(**a**) Since $P(A) = 0.45$, $P(B) = 0.55$, $P(C \mid A) = 0.7$ and $P(C \mid B) = 0.5$, it follows from $P(AC) = P(C \mid A)P(A)$ and $P(BC) = P(C \mid B)P(B)$ that $P(AC) = 0.7 \times 0.45 = 0.315$ and $P(BC) = 0.5 \times 0.55 = 0.275$.
(**b**) Since $P(C) = P(AC) + P(BC)$, we get $P(C) = 0.59$.
(**c**) $P(A \mid C) = \frac{0.315}{0.59} = 0.5339$.

2.16 Let A be the event that a randomly selected household subscribes to the morning newspaper and B be the event that the household subscribes to the afternoon newspaper. To find the sought probability $P(A^c \mid B)$, use the relation $P(A) = P(AB) + P(A^cB)$. Thus

$$P(A^c \mid B = \frac{P(A) - P(AB)}{P(B)} = \frac{0.50 - 0.40}{0.70} = \frac{1}{7}.$$

2.17 Let A_1 (A_2) be the event that the first (second) card picked belongs to one of the three business partners. Then $P(A_1A_2) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$.

2.18 Let A_i be the event that the ith card you receive is a picture card that you have not received before. Then, by $P(A_1A_2A_3A_4) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1A_2)P(A_4 \mid A_1A_2A_3)$, the sought probability can be computed as

$$P(A_1A_2A_3A_4) = \frac{16}{52} \times \frac{12}{51} \times \frac{8}{50} \times \frac{4}{49} = 9.46 \times 10^{-4}.$$

2.19 Let A be the event that one or more sixes are rolled and B the event that no one is rolled. Then, by $P(AB) = P(B)P(A \mid B)$, we have that the sought probability is

$$P(AB) = \left(\frac{5}{6}\right)^6 \left(1 - \left(\frac{4}{5}\right)^6\right) = 0.2471.$$

2.20 It is no restriction to assume that the drawing of lots begins with the Spanish teams. Let A_0 be the event that the two Spanish team are paired and A_i be the event that the ith Spanish team is not paired to the other Spanish team or to a German team. The events A_0 and A_1A_2 are disjoint. The sought probability is

$$P(A_0) + P(A_1A_2) = \frac{1}{7} + P(A_1)P(A_2 \mid A_1) = \frac{1}{7} + \frac{4}{7} \times \frac{3}{5} = \frac{17}{35}.$$

2.21 Let A_i be the event that you get a white ball on the ith pick. The probability that you need three picks is $P(A_1A_2A_3) = \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} = \frac{6}{125}$. Five picks require that one black ball is taken in the first three picks. By the chain rule, the probability that five picks are needed is $\frac{2}{5} \times \frac{4}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} + \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} + \frac{3}{5} \times \frac{2}{5} \times \frac{4}{5} \times \frac{2}{5} \times \frac{1}{5} = \frac{6}{125}$.

2.22 (**a**) The sought probability is the same as the probability of getting two red balls when two balls are drawn at random from a bowl with three red bed balls and three blue balls. Let A_i be the event that the ith ball drawn is red. The sought probability is $P(A_1A_2)$. This probability is evaluated as $P(A_1)P(A_2 \mid A_1) = \frac{3}{6} \times \frac{2}{5} = \frac{1}{5}$.
(**b**) Let A_i be the event that the ith number drawn is not 10 and E_i be the event that the ith number drawn is more than 10. The first probability is

$$1 - P(A_1 \cdots A_6) = 1 - \frac{41}{42} \times \frac{40}{41} \cdots \times \frac{36}{37} = \frac{6}{42}.$$

The second probability is $P(E_1 \cdots E_6) = \frac{23}{42} \times \frac{22}{41} \cdots \times \frac{18}{37} = 0.0192$.
(**c**) Suppose that first the two cups of coffee are put on the table. Let A_i be the event that the ith cup of coffee is given to a person who ordered coffee. The sought probability is

$$P(A_1A_2) = P(A_1)P(A_2 \mid A_1) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}.$$

(**d**) Suppose that the two socks are chosen one by one. Let A_i be the event that the ith sock chosen is black for $i = 1, 2$. The sought probability is $2P(A_1A_2)$. We have $P(A_1A_2) = P(A_1)P(A_2 \mid A_1) = \frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$. Hence the sought probability is $2 \times \frac{1}{20} = \frac{1}{10}$.
(**e**) Imagine that two apartments become vacant one after the other. Let A_1 be the event that the first vacant apartment is not on the top floor and A_2 be the event that the second vacant apartment is not on the top floor. The sought probability is $1 - P(A_1A_2)$. The probability $P(A_1A_2)$ is evaluated as $P(A_1)P(A_2 \mid A_1) = \frac{48}{56} \times \frac{47}{55} = 0.7325$.

2.23 Let A_i be the event that the ith person in line is the first person matching a birthday with one of the persons in front of him. Then $P(A_2) = \frac{1}{365}$ and $P(A_i) = \frac{364}{365} \times \cdots \times \frac{364-i+3}{365} \times \frac{i-1}{365}$ for $i \geq 3$. The probability $P(A_i)$ is maximal for $i = 20$ and has then the value 0.0323.

2.24 Let A_1 be the event that the luggage is not lost in Amsterdam, A_2 the event that the luggage is not lost in Dubai and A_3 the event that the luggage is not lost in Singapore. Then,

$$\begin{aligned} P(\text{the luggage is lost}) &= 1 - P(A_1A_2A_3) \\ &= 1 - P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1A_2) \\ &= 1 - 0.95 \times 0.97 \times 0.98 = 0.09693. \end{aligned}$$

Letting A_i^c be the complementary event of the event A_i, we have

$$\begin{aligned} &P(\text{the luggage is lost in Dubai} \mid \text{the luggage is lost}) \\ &= \frac{P(A_1A_2^c)}{P(\text{the luggage is lost})} = \frac{P(A_1)P(A_2^c \mid A_1)}{P(\text{the luggage is lost})} \\ &= \frac{0.95 \times 0.03}{0.09693} = 0.2940. \end{aligned}$$

2.25 Let A_i be the event that the ith leaving person has not to squeeze past a still seated person. The sought probability is the same as $P(A_1A_2A_3A_4A_5) = \frac{2}{7} \times \frac{2}{6} \times \frac{2}{5} \times \frac{2}{4} \times \frac{2}{3} = 0.0127$.

2.26 Let A_k be the event that the first ace appears at the kth card, and let $p_k = P(A_k)$. Then, by $P(A_1A_2 \cdots A_n) = P(A_1)P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \ldots A_{n-1})$, it follows that $p_1 = \frac{4}{52}$, $p_2 = \frac{48}{52} \times \frac{4}{51}$, and

$$p_k = \frac{48}{52} \times \frac{47}{51} \times \cdots \times \frac{48-k+2}{52-k+2} \times \frac{4}{52-k+1}, \qquad k = 3, \ldots, 49.$$

The three players do not have the same chance to become the dealer. For $P = A$, B, and C, let r_P be the probability that player P becomes the dealer. Then $r_A > r_B > r_C$, because the probability p_k is decreasing in k. The probabilities can be calculated as $r_A = \sum_{n=0}^{16} p_{1+3n} = 0.3600$, $r_B = \sum_{n=0}^{15} p_{2+3n} = 0.3328$, and $r_C = \sum_{n=0}^{15} p_{3+3n} = 0.3072$.

2.27 Under the condition that the events $A_1, \ldots, A_{i-1}$ have occurred, the ith couple can match the birthdays of at most one of the couples 1 to $i-1$. Thus $P(A_i^c \mid A_1 \cdots A_{i-1}) = \frac{i-1}{365^2}$ and so $P(A_i \mid A_1 \cdots A_{i-1}) = 1 - \frac{i-1}{365^2}$. The sought probability is $1 - P(A_2A_3 \cdots A_n)$ and equals $1 - \prod_{i=2}^{n} \left(1 - \frac{i-1}{365^2}\right)$, by the chain rule.

2.28 The desired probability is $1 - P(A_1A_2 \cdots A_{m-2})$. We have

$$P(A_1) = \frac{\binom{39}{5}\left[\binom{39}{5} - 1\right]\left[\binom{39}{5} - 2\right]}{\binom{39}{5}^3}, \ P(A_i \mid A_1 \cdots A_{i-1}) = \frac{\binom{39}{5} - 2}{\binom{39}{5}}$$

for $i \geq 2$. The desired probability now follows by applying the chain rule $P(A_1 A_2 \cdots A_{m-2}) = P(A_1)P(A_2 \mid A_1) \cdots P(A_{m-2} \mid A_1 \cdots A_{m-3})$ and is equal to

$$1 - \frac{\left[\binom{39}{5} - 1\right]\left[\binom{39}{5} - 2\right]^{m-2}}{\binom{39}{5}^{m-1}}.$$

2.29 Using the chain rule for conditional probabilities, the sought probability is $\frac{r}{r+b}$ for $k = 1$, $\frac{b}{r+b} \times \frac{r}{r+b-1}$ for $k = 2$ and $\frac{b}{r+b} \times \frac{b-1}{r+b-1} \times \cdots \times \frac{b-(k-2)}{r+b-(k-2)} \times \frac{r}{r+b-(k-1)}$ for $3 \leq k \leq b+1$. The sought probability can be written as

$$\frac{\binom{b}{k-1}}{\binom{r+b}{k-1}} \times \frac{r}{r+b-(k-1)} = \frac{\binom{r+b-k}{r-1}}{\binom{r+b}{r}}.$$

This representation can be explained as the probability that the first $k-1$ picks are blue balls multiplied with the conditional probability that the kth pick is a red ball given that the first $k-1$ picks are blue balls. The answer to the last question is $\frac{b}{r+b}$, as can be directly seen by a symmetry argument. The probability that the last ball picked is blue is the same as the probability that the first ball picked is blue.

2.30 The probability that the rumor will not be repeated to any one person once more is $P(A_1 A_2 \cdots A_{10})$, where A_i is the event that the rumor reaches only different persons during the first i times that the rumor is told. Noting that $P(A_1) = 1$, $P(A_2 \mid A_1) = 1$ and $P(A_i \mid A_1 \cdots A_{i-1}) = \frac{25-i}{23}$ for $i \geq 3$, it follows that the probability that the rumor will not be repeated to any one person once more is

$$\frac{22}{23} \times \frac{21}{23} \times \cdots \times \frac{15}{23} = 0.1646.$$

The probability that the rumor will not return to the originator is $(\frac{22}{23})^8 = 0.7007$.

2.31 Since $P(A) = \frac{18}{36}$, $P(B) = \frac{18}{36}$, and $P(AB) = \frac{9}{36}$, we get $P(AB) = P(A)P(B)$. This shows that the events A and B are independent.

2.32 The number is randomly chosen from the matrix and so $P(A) = \frac{30}{50}$, $P(B) = \frac{25}{50}$ and $P(AB) = \frac{15}{50}$. Since $P(AB) = P(A)P(B)$, the events A and B are independent. This result can also be explained by noting that you obtain a random number from the matrix by choosing first a row at random and choosing next a column at random.

2.33 Since A is the union of the disjoint sets AB and AB^c, we have $P(A) = P(AB)+P(AB^c)$. This gives $P(AB^c) = P(A)-P(A)P(B) = P(A)[1-P(B)]$ and so $P(AB^c) = P(A)P(B^c)$, showing that A and B^c are independent events. Applying this result with A replaced by B^c and B by A, we next get that B^c and A^c are independent events.

2.34 Since $A = AB \cup AB^c$ and the events AB and AB^c are disjoint, it follows that $P(A) = P(AB) + P(AB^c) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$. This gives $P(A) = P(A \mid B)P(B)+P(A \mid B)P(B^c) = P(A \mid B)$. Thus $P(A) = \frac{P(AB)}{P(B)}$ and so $P(AB) = P(A)P(B)$.

2.35 The result follows directly from $P(A_1 \cup \cdots \cup A_n) = 1 - P(A_1^c \cdots A_n^c)$ and the independence of the A_i^c, using $P(A_1^c \cdots A_n^c) = P(A_1^c) \cdots P(A_n^c)$ and $P(A_i^c) = 1 - P(A_i)$.

2.36 Using Problem 2.35, the probability is $1 - \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{3}{4}$.

2.37 The set A can be represented as $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Since the sequence of sets $\bigcup_{k=n}^{\infty} A_k$ is nonincreasing, we have $P(A) = \lim_{n\to\infty} P\big(\bigcup_{k=n}^{\infty} A_k\big)$, by the continuity property of probability. Next use the fact that $P\big(\bigcup_{k=n}^{\infty} A_k\big) = 1-P\big(\bigcap_{k=n}^{\infty} A_k^c\big)$. Using the independence of the events A_n and the continuity property of probability measure, it is readily verified that $P\big(\bigcap_{k=n}^{\infty} A_k^c\big) = \prod_{k=n}^{\infty} P(A_k^c)$. By $P(A_k^c) = 1 - P(A_k)$ and the inequality $1 - x \leq e^{-x}$, we get

$$P\Big(\bigcap_{k=n}^{\infty} A_k^c\Big) \leq \prod_{k=n}^{\infty} e^{-P(A_k)} = e^{-\sum_{k=n}^{\infty} P(A_k)} = 0 \text{ for } n \geq 1,$$

where the last equality uses the assumption $\sum_{n=1}^{\infty} P(A_n) = \infty$. This verifies that $P\big(\bigcup_{k=n}^{\infty} A_k\big) = 1$ for all $n \geq 1$ and so $P(A) = 1$.

2.38 Let A be the event that you have picked the ball with number 7 written on it and B_i the event that you have chosen box i for $i = 1, 2$. By the law of conditional probability, $P(A) = P(A \mid B_1)P(B_1) + P(A_2 \mid B_2)P(B_2)$. Therefore

$$P(A) = \frac{1}{10} \times \frac{1}{2} + \frac{1}{25} \times \frac{1}{2} = 0.07.$$

2.39 Let A be the event that HAPPY HOUR appears again, B_1 be the event that either the two letters H or the two letters P have been removed, and B_2 be the event that two different letters have been removed. Then

$P(B_1) = \frac{2}{9} \times \frac{1}{8} + \frac{2}{9} \times \frac{1}{8}$ and $P(B_2) = 1 - P(B_1)$. Obviously, $P(A \mid B_1) = 1$ and $P(A \mid B_2) = \frac{1}{2}$. By the law of conditional probability,

$$P(A) = \sum_{i=1}^{2} P(A \mid B_i)P(B_i) = 1 \times \frac{1}{18} + \frac{1}{2} \times \frac{17}{18} = \frac{19}{36}.$$

2.40 Let A be the event that the cases with \$1,000,000 and \$750,000 are still in the game when you have opened 20 cases. Also, let B_0 be the event that your chosen case does not contain either of the amounts \$1,000,000 and \$750,000 and B_1 be the complementary event of B_0. Then $P(A) = P(A \mid B_0)P(B_0) + P(A \mid B_1)P(B_1)$, which gives

$$P(A) = \left[\binom{23}{20}\Big/\binom{25}{20}\right] \times \frac{24}{26} + \left[\binom{24}{20}\Big/\binom{25}{20}\right] \times \frac{2}{26} = \frac{3}{65}.$$

2.41 Let A be the event that you ever win the jackpot when buying a single ticket only once. Also, let B be the event that you match the six numbers drawn and C be the event that you match exactly two of these numbers. It follows from $P(A) = P(A \mid B)P(B) + P(A \mid C)P(C)$ that $P(A) = P(B) + P(A)P(C)$. Since $P(B) = 1/\binom{59}{6}$ and $P(C) = \binom{6}{2}\binom{53}{4}/\binom{59}{6}$, we get $P(A) = 1/40{,}665{,}099$.

2.42 Let A be the event that Joe's dinner is burnt, B_0 be the event that he did not arrive home on time, and B_1 be the event that he arrived home on time. The probability $P(A) = P(A \mid B_0)P(B_0) + P(A \mid B_1)P(B_1)$ is equal to $0.5 \times 0.2 + 0.15 \times 0.8 = 0.22$. The inverse probability $P(B_1 \mid A)$ is given by

$$\frac{P(B_1A)}{P(A)} = \frac{P(B_1)P(A \mid B_1)}{P(A)} = \frac{0.8 \times 0.15}{0.22} = \frac{6}{11}.$$

2.43 Let A be the event of reaching your goal, B_1 be the event of winning the first bet and B_2 be the event of losing the first bet. Then, by $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)$, we get $P(A) = 1 \times \frac{12}{37} + \frac{9}{37} \times \frac{25}{37}$. Thus the probability of reaching your goal is 0.4887. *Note*: This probability is slightly more than the probability 0.4865 of reaching your goal when you use bold play and stake the whole \$10,000 on a 18-numbers bet.

2.44 Let A be the event that the player wins and B_i be the conditioning event that the first roll of the two dice gives a dice sum of i points for $i = 2, \ldots, 12$. Then, $P(A) = \sum_{k=2}^{12} P(A \mid B_k)P(B_k)$. We have $P(A \mid B_i) = 1$ for $i = 7, 11$, and $P(A \mid B_i) = 0$ for $i = 2, 3, 12$. Put for abbreviation $p_k = P(B_k)$, then $p_k = \frac{k-1}{36}$ for $k = 2, \ldots, 7$ and $p_k = p_{14-k}$ for $k = 8, \ldots, 12$. The other conditional probabilities $P(A \mid B_i)$ can be given in terms of the p_k. For example, the conditional probability $P(A \mid B_4)$ is no other than the unconditional probability that the total of 4 will appear before the total of 7 does in the (compound) experiment of repetitive dice rolling. The total of 4 will appear before the total of 7 if and only if one of the events $E_1, E_2, \ldots$ occurs, where E_k is the event that the first consecutive $k-1$ rolls give neither the total of 4 nor the total of 7 and the kth consecutive roll gives a total of 4. The events $E_1, E_2, \ldots$ are mutually exclusive and so

$$P(4 \text{ before } 7) = P\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} P(E_i).$$

Any event E_k is generated by physically independent subexperiments and thus the probabilities of the individual outcomes in the subexperiments are multiplied by each other in order to obtain $P\left(E_k\right) = \big(1 - p_4 - p_7\big)^{k-1} p_4$ for any $k \geq 1$. This leads to the formula

$$P(4 \text{ before } 7) = \sum_{k=1}^{\infty} \big(1 - p_4 - p_7\big)^{k-1} p_4 = \frac{p_4}{p_4 + p_7},$$

In this way, we find that $P(A \mid B_i) = \frac{p_i}{p_i + p_7}$ for $i = 4, 5, 6, 8, 9, 10$. Putting all the pieces together, we get

$$P(A) = \sum_{k=2}^{12} P(A \mid B_k) p_k = 0.4929.$$

2.45 Apply the gambler's ruin formula with $p = 0.3$, $a = 3$ and $b = 7$. The sought probability is 0.0025.

2.46 For fixed integer r, let A_r be the event that there are exactly r winning tickets among the fifty thousand tickets sold. Let B_k be the event that there exactly k winning tickets among the one hundred thousand tickets printed. Then, by the law of conditional probability,

$$P(A_r) = \sum_{k=0}^{\infty} P(A_r \mid B_k) P(B_k).$$

Obviously, $P(A_r \mid B_k) = 0$ for $k < r$. For all practical purposes the so-called Poisson probability $e^{-1}/k!$ can be taken for the probability of the event B_k for $k = 0, 1, \ldots$, see Example 1.19. This gives

$$P(A_r) = \sum_{k=r}^{\infty} \binom{k}{r} \left(\frac{1}{2}\right)^r \frac{e^{-1}}{k!} = e^{-1} \frac{(1/2)^r}{r!} \sum_{j=0}^{\infty} \frac{1}{j!} = e^{-\frac{1}{2}} \frac{(1/2)^r}{r!}.$$

Hence the probability of exactly r winning tickets among the fifty thousand tickets sold is given by the Poisson probability $e^{-\frac{1}{2}} \frac{(1/2)^r}{r!}$ for $r = 0, 1, \ldots$.

2.47 It is no restriction to assume that the starting point is 1 and the first transition is from point 1 to point 2 (otherwise, renumber the points). Some reflection shows that the probability of visiting all points before returning to the starting point is nothing else than the probability $\frac{1}{1+10} = \frac{1}{11}$ from the gambler's ruin model.

2.48 Let A be the event that the card picked is a red card, B_1 be the event that the removed top card is red and B_2 be the event that the removed top card is black. The sought probability $P(A)$ is given by $P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)$. Therefore

$$P(A) = \frac{r-1}{r+b-1} \times \frac{r}{r+b} + \frac{r}{r+b-1} \times \frac{b}{r+b} = \frac{r}{r+b}.$$

2.49 Let A be the event that John needs more tosses than Pete and B_j be the event that Pete needs j tosses to obtain three heads. Then $P(B_j) = \binom{j-1}{2}(\frac{1}{2})^j$ and $P(A \mid B_j) = \binom{j}{0}(\frac{1}{2})^j + \binom{j}{1}(\frac{1}{2})^j$. By the law of conditional probability, the sought probability $P(A)$ is

$$P(A) = \sum_{j=3}^{\infty} P(A \mid B_j)P(B_j) = 0.1852.$$

2.50 Take any of the twenty balls and mark this ball. Let A be the event that this ball is the last ball picked for the situation that three balls were overlooked and were added to the bin at the end. If we can show that $P(A) = \frac{1}{20}$, the raffle is still fair. Let B_1 (B_2) be the event that the marked ball is (is not) one of the three balls that were unintentionally overlooked. Then, by the law of conditional probabilities,

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) = \frac{1}{3} \times \frac{3}{20} + 0 \times \frac{17}{20}.$$

Hence $P(A) = \frac{1}{20}$, the same win probability as for the case in which no balls would have been overlooked.

2.51 Let state i mean that player A's bankroll is i. Also, let E be the event of reaching state k without having reached state $a+b$ when starting in state a and F be the event of reaching state $a+b$ without having reached state $k-1$ when starting in state k. Then the unconditional probability of player A winning and having k as the lowest value of its bankroll during the game is given by $P(EF) = P(E)P(F \mid E)$. Using the gambler's ruin formula, $P(E) = \frac{b}{a+b-k}$ and $P(F \mid E) = \frac{1}{a+b-k+1}$. Thus the sought conditional probability is

$$\frac{b(a+b)}{a(a+b-k)(a+b-k+1)} \qquad \text{for } k = 1, \ldots a.$$

This probability has the values 0.1563, 0.2009, 0.2679, and 0.3750 for $k = 1$, 2, 3, and 4 when $a = 4$ and $b = 5$.

2.52 Let A be the event that two or more participating cyclists will have birthdays on the same day during the tournament and B_i be the event that exactly i participating cyclists have their birthdays during the tournament. The conditional probability $P(A \mid B_i)$ is easy to calculate. It is the standard birthday problem. We have

$$P(A \mid B_i) = 1 - \frac{23 \times 22 \times \cdots \times (23 - i + 1)}{23^i} \qquad \text{for } 2 \leq i \leq 23.$$

Further, $P(A \mid B_i) = 1$ for $i \geq 24$. Also, $P(A \mid B_0) = P(A \mid B_1) = 0$. Therefore the probability $P(B_i)$ is given by

$$P(B_i) = \binom{180}{i} \left(\frac{23}{365}\right)^i \left(1 - \frac{23}{365}\right)^{180-i} \qquad \text{for } 0 \leq i \leq 180.$$

Putting the pieces together and using $P(A) = \sum_{i=2}^{180} P(A \mid B_i)P(B_i)$, we get

$$P(A) = 1 - P(B_0) - P(B_1) - \sum_{i=2}^{23} \frac{23 \times 22 \times \cdots \times (23 - i + 1)}{(23)^i} P(B_i).$$

This yields the value 0.8841 for the probability $P(A)$.

2.53 Let $p_n(i)$ be the probability of reaching his home no later than midnight without having reached first the police station given that he is

i steps away from his home and he has still time to make n steps before it is midnight. The sought probability is $p_{180}(10)$. By the law of conditional probability, the $p_n(i)$ satisfy the recursion

$$p_n(i) = \frac{1}{2}p_{n-1}(i+1) + \frac{1}{2}p_{n-1}(i-1).$$

The boundary conditions are $p_k(30) = 0$ and $p_k(0) = 1$ for $k \geq 0$, and $p_0(i) = 0$ for $i \geq 1$. Applying the recursion, we find $p_{180}(10) = 0.4572$. In the same way, the value 0.1341 can be calculated for the probability of reaching the police station before midnight. *Note*: As a sanity check, we verified that $p_n(10)$ tends to $\frac{2}{3}$ as n gets large, in agreement with the gambler's ruin formula. The probability $p_n(10)$ has the values 0.5905, 0.6659, and 0.6665 for $n = 360$, 1,200 and 1,440.

2.54 Let A be the event that John and Pete meet each other in the semi-finals. To find $P(A)$, let B_1 be the event that John and Pete are allocated to either group 1 or group 2 but not to the same group and B_2 be the event that John and Pete are allocated to either group 3 or group 4 but not to the same group. Then $P(B_1) = P(B_2) = \frac{1}{2} \times \frac{2}{7} = \frac{1}{7}$. By the law of conditional probability,

$$\begin{aligned} P(A) &= P(A \mid B_1) \times \frac{1}{7} + P(A \mid B_2) \times \frac{1}{7} \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{7} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{7} = \frac{1}{14}. \end{aligned}$$

Let C be the event that John and Pete meet each other in the final. To find $P(C)$, let D_1 be the event that John is allocated to either group 1 or group 2 and Pete to either group 3 or group 4 and D_2 be the event that John is allocated to either group 3 or group 4 and Pete to either group 1 or group 2. Then $P(D_1) = P(D_2) = \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$. By the law of conditional probability,

$$\begin{aligned} P(C) &= P(C \mid D_1) \times \frac{2}{7} + P(C \mid D_2) \times \frac{2}{7} \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{2}{7} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{2}{7} = \frac{1}{28}. \end{aligned}$$

The latter result can also be directly seen by a symmetry argument. The probability that any one pair contests the final is the same as that for any other pair. There are $\binom{8}{2}$ different pairs and so the probability that John and Pete meet each other in the final is $1/\binom{8}{2} = \frac{1}{28}$.

2.55 Let A be the event that you have chosen the bag with one red ball and B be the event that you have the other bag. Also, let E be the event that the first ball picked is red. The sought probability that the second ball picked is red is

$$\frac{1}{4}P(A \mid E) + \frac{3}{4}P(B \mid E),$$

by the law of conditional probability. We have

$$P(A \mid E) = \frac{P(AE)}{P(E)} = \frac{P(A)P(E \mid A)}{P(E)}.$$

Further, $P(B \mid E) = 1 - P(A \mid E)$. Since $P(A) = P(B) = \frac{1}{2}$, $P(E) = P(A) \times \frac{1}{4} + P(B) \times \frac{3}{4} = \frac{1}{2}$ and $P(E \mid A) = \frac{1}{4}$, we get $P(A \mid E) = \frac{1}{4}$ and $P(B \mid E) = \frac{3}{4}$. Thus the sought probability is $\frac{1}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{3}{4} = \frac{5}{8}$.

2.56 The key idea for the solution approach is to parameterize the starting state. Define D_s as the event that Dave wins the game when the game begins with Dave rolling the dice and Dave has to roll more than s points in his first roll. Similarly, the event E_s is defined for Eric. The goal is to find $P(D_1)$. This probability can be found from a recursion scheme for the $P(D_s)$. The recursion scheme follows by conditioning on the events B_j, where B_j is the event that a roll of the two dice results in a sum of j points. The probabilities $p_j = P(B_j)$ are given by $p_j = \frac{j-1}{36}$ for $2 \leq j \leq 7$ and $p_j = p_{14-j}$ for $8 \leq j \leq 12$. By the law of conditional probability,

$$P(D_s) = \sum_{j=s+1}^{12} P(D_s \mid B_j)p_j \quad \text{for } s = 1, 2, \ldots, 11.$$

Obviously, $P(D_{12}) = 0$. Since $P(D_s \mid B_j) = 1 - P(E_j)$ for $j > s$ and $P(E_k) = P(D_k)$ for all k, we get the recursion scheme

$$P(D_s) = \sum_{j=s+1}^{12} [1 - P(D_j)]p_j \quad \text{for } s = 1, 2, \ldots, 11.$$

Starting with $P(D_{12}) = 0$, we recursively compute $P(D_{11}), \ldots, P(D_1)$. This gives the value $P(D_1) = 0.6541$ for the probability of Dave winning the game.

2.57 Fix j. Label the $c = \binom{7}{j}$ possible combinations of j stops as $l = 1, \ldots, c$. Let A be the event that there will be exactly j stops at which nobody gets off and B_l be the event that nobody gets off at the j stops from combination l. Then, $P(A) = \sum_{l=1}^{c} P(A \mid B_l)P(B_l)$. We have that $P(B_l) = (7-j)^{25}/7^{25}$ for all l and $P(A \mid B_l)$ is the unconditional probability that at least one person gets off at each stop when there are $7-j$ stops and 25 persons. Thus $P(A \mid B_l) = 1-\sum_{k=1}^{7-j}(-1)^{k+1}\binom{7-j}{k}(7-j-k)^{25}/(7-j)^{25}$, using the result of Example 1.18. Next we get after some algebra the desired result

$$P(A) = \sum_{k=0}^{7-j}(-1)^k \binom{j+k}{j}\binom{7}{j+k}\frac{(7-j-k)^{25}}{7^{25}}.$$

Note: More generally, the probability of exactly j empty bins when $m \geq b$ balls are sequentially placed at random into b bins is given by

$$\sum_{k=0}^{b-j}(-1)^k \binom{j+k}{j}\binom{b}{j+k}\frac{(b-j-k)^m}{b^m}.$$

2.58 Let A be the event that all of the balls drawn are blue and B_i be the event that the number of points shown by the die is i for $i = 1, \ldots, 6$. By the law of conditional probability, the probability that all of the balls drawn are blue is given by

$$P(A) = \sum_{i=1}^{6} P(A \mid B_i)P(B_i) = \sum_{i=1}^{5} \frac{\binom{5}{i}}{\binom{10}{i}} \times \frac{1}{6} = \frac{5}{36}.$$

The probability that the number of points shown by the die is r given that all of the balls drawn are blue is equal to

$$P(B_r \mid A) = \frac{P(B_r A)}{P(A)} = \frac{(1/6)\binom{5}{r}/\binom{10}{r}}{5/36}.$$

This probability has the values $\frac{3}{5}, \frac{4}{15}, \frac{1}{10}, \frac{1}{35}, \frac{1}{210}$ and 0 for $r = 1, \ldots, 6$.

2.59 Let A be the event that the both rolls of the two dice show the same combination of two numbers. Also, let B_1 be the event that the first roll of the two dice shows two equal numbers and B_2 be the event that the first roll shows two different numbers. Then $P(B_1) = \frac{6}{36}$ and

$P(B_2) = \frac{30}{36}$. Further, $P(A \mid B_1) = \frac{1}{36}$ and $P(A \mid B_2) = \frac{2}{36}$. By the law of conditional probability,

$$P(A) = \sum_{i=1}^{2} P(A \mid B_i)P(B_i) = \frac{1}{36} \times \frac{6}{36} + \frac{2}{36} \times \frac{30}{36} = \frac{11}{216}.$$

2.60 Let A_j be the event that the team placed jth in the competition wins the first place in the draft and B_j be the event that this team wins the second place in the draft for $7 \leq j \leq 14$. Obviously,

$$P(A_j) = \frac{15-j}{36} \quad \text{for } j = 7, \ldots, 14.$$

By the law of conditional probability, $P(B_j) = \sum_{k \neq j} P(B_j \mid A_k)P(A_k)$. We have $P(B_j \mid A_k) = \frac{15-j}{36-15+k}$ for $k \neq j$. Therefore

$$P(B_j) = \sum_{k \neq j} \frac{15-j}{36-15+k} \times \frac{15-k}{36} \quad \text{for } j = 7, \ldots, 14.$$

The probability $P(B_j)$ has the numerical values 0.2013, 0.1848, 0.1653, 0.1431, 0.1185, 0.0917, 0.0629, and 0.0323 for $j = 7, 8, \ldots, 14$.

2.61 This problem can be seen as a random walk on the integers, where the random walk starts at zero. In the first step the random walk moves from to 1 with probability p and to -1 with probability $q = 1 - p$. Take $p < \frac{1}{2}$. Starting from 1, the random walk will ever return to zero with probability $1 - \lim_{b \to \infty}[1-(q/p)^a]/[1-(q/p)^{a+b}]$ with $a = 1$. This probability is 1. Starting from -1, the random walk will ever return to zero with probability $1 - \lim_{b \to \infty}[1-(p/q)^a]/[1-(p/q)^{a+b}]$ with $a = 1$. This probability is $\frac{p}{q}$. The sought probability is $p \times 1 + (1-p) \times \frac{p}{1-p} = 2p$ (this result is also valid for $p = \frac{1}{2}$).

2.62 Let A be the event that the drunkard will ever visit the point which is one unit distance south from his starting point. Let B_1 be the event that the first step is one unit distance to the south and B_2 be the event that the first step is two units distance to the north. By the law of conditional probability, $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)$. Obviously, $P(B_1) = P(B_2) = \frac{1}{2}$ and $P(A \mid B_1) = 1$. Noting that the probability of ever going three units distance to the south from any starting point is $P(A) \times P(A) \times P(A)$, it follows that

$$P(A) = 0.5 + 0.5\big[P(A)\big]^3.$$

The cubic equation $x^3 - 2x + 1 = 0$ has the root $x = 1$ and so the equation can be factorized as $(x-1)(x^2+x-1) = 0$. The only positive root of $x^2+x-1 = 0$ is $x = \frac{1}{2}(\sqrt{5}-1)$. This is the desired value for the sought probability $P(A)$. Next some reflection shows that $\frac{1}{2}(\sqrt{5}-1)$ gives also the probability of the number of heads ever exceeding twice the number of tails if a fair coin is tossed over and over.

2.63 It does not matter what question you ask. To see this, let A be the event that your guess is correct, B_1 be the event that the answer of your friend is yes and B_2 be the event that the answer is no. For the question whether the card is red, we have $P(A) = \frac{1}{26} \times \frac{1}{2} + \frac{1}{26} \times \frac{1}{2} = \frac{1}{26}$, by the law of conditional probability. For the other question, $P(A) = 1 \times \frac{1}{52} + \frac{1}{51} \times \frac{51}{52} = \frac{1}{26}$. The same probability.

2.64 Let A be the event that player 1 wins the game. We have $P(A) = 0.5$, regardless of the value of m. The simplest way to see this is to define E_1 as the event that player 1 has more heads than player 2 after m tosses, E_2 as the event that player 1 has fewer heads than player 2 after m tosses, and E_3 as the event that player 1 has the same number of heads as player 2 after m tosses. Then $P(A) = \sum_{i=1}^{3} P(A \mid E_i)P(E_i)$, by the law of conditional probability. To evaluate this, it is not necessary to know the $P(E_i)$. Since $P(E_2) = P(E_1)$ and $P(E_3) = 1 - 2P(E_1)$, it follows that

$$\begin{aligned} P(A) &= 1 \times P(E_1) + 0 \times P(E_2) + \frac{1}{2} \times P(E_3) \\ &= P(E_1) + \frac{1}{2} \times \big(1 - 2P(E_1)\big) = 0.5. \end{aligned}$$

2.65 Let A be the event that you roll two *consecutive* totals of 7 before a total of 12. Let B_1 be the event that each of the first two rolls results in a total of 7, B_2 be the event that the first roll gives a total of 7 and the second roll a total different from 7 and 12, B_3 be the event that the first roll gives a total different from 7 and 12, B_4 be the event that the first roll gives a total of 7 and the second roll a total of 12, and B_5 be the event that the first roll gives a total of 12. Then,

$$P(A) = 1 \times \frac{6}{36} \times \frac{6}{36} + P(A) \times \frac{6}{36} \times \frac{29}{36} + P(A) \times \frac{29}{36} + 0 \times \frac{6}{36} \times \frac{1}{36} + 0 \times \frac{1}{36}$$

and so $P(A) = \frac{6}{13}$.

2.66 A minor modification of the analysis of Example 2.11 shows that the optimal stopping level for player A remains the same, but the win probability of player A changes to 0.458.

2.67 The recursion is

$$p(i,t) = \frac{1}{6-i+1} \sum_{j=0}^{6-i} p(i+1, t-j),$$

as follows by conditioning upon the number of tokens you lose at the ith cup. This leads to $p(1,6) = \frac{169}{720}$.

2.68 For fixed n, Let A be the event that the total score ever reaches the value n. To find $p_n = P(A)$, condition on the outcome of the first roll of the die. Let B_j be the event that the outcome of this roll is j. Then, $P(A \mid B_j) = p_{n-j}$ and so, by the law of conditional probability,

$$p_n = \sum_{k=1}^{6} p_{n-k} \times \frac{1}{6} \quad \text{for all } n \geq 1$$

with the convention $p_j = 0$ for $j \leq 0$. The result that p_n tends to $\frac{1}{3.5}$ as n gets large can be intuitively explained from the fact that after each roll of the die the expected increase in the total score is equal to $\frac{1}{6}(1+2+\cdots+6) = 3.5$.

2.69 (**a**) Define r_n as the probability of getting a run of either r successes or r failures in n trials.Also, define s_n as probability of getting a run of either r successes or r failures in n trials given that the first trial results in a success, and f_n as probability of getting a run of either r successes or r failures in n trials given that the first trial results in a failure. Then $r_n = ps_n + (1-p)f_n$. The s_n and f_n satisfy the recursive schemes

$$s_n = p^{r-1} + \sum_{k=1}^{r-1} p^{k-1}(1-p)f_{n-k}$$

$$f_n = (1-p)^{r-1} + \sum_{k=1}^{r-1} (1-p)^{k-1} p s_{n-k}$$

for $n \geq r$, where $s_j = f_j = 0$ for $j < r-1$.
(**b**) Parameterize the starting state and let $p(r, b, L)$ be the probability

that the longest run of red balls will be L or more when the bowl initially contains r red and b blue balls. Fix $r > L$ and $b \geq 1$. Let A be the event that a run of L red balls will occur. To find $P(A) = p(r, b, L)$, let B_L be the conditioning event that the first L balls picked are red and B_{j-1} be the conditioning event that each of the first $j-1$ balls picked is red but the jth ball picked is blue, where $1 \leq j \leq L$. Then

$$P(B_L) = \frac{r}{r+b} \times \cdots \times \frac{r-(L-1)}{r+b-(L-1)}$$
$$P(B_{j-1}) = \frac{r}{r+b} \times \cdots \times \frac{r-(j-2)}{r+b-(j-2)} \times \frac{b}{r+b-(j-1)}, 1 \leq j \leq L.$$

Note that $P(A \mid B_L) = 1$ and $P(A \mid B_{j-1}) = p\big(r-(j-1), b-1, L\big)$ for $1 \leq j \leq L$. Then

$$P(A) = P(B_L) + \sum_{j=1}^{L} P(A \mid B_{j-1}) P(B_{j-1})$$

gives a recursion scheme for the calculation of the probability $p(r, b, L)$.

2.70 For the case of n dwarfs, $p(k, n)$ is defined as the probability that the kth dwarf will *not* sleep in his own bed when the first dwarf chooses randomly one of the n beds (the dwarfs $1, 2, \ldots, n$ go to bed in this order and dwarf j has bed j). Let us first note that the dwarfs $2, \ldots, j-1$ sleep in their own beds if the first dwarf chooses bed j. The first dwarf chooses each of the n beds with the same probability $\frac{1}{n}$. Fix $k \geq 2$. Under the condition that the first dwarf chooses bed j with $2 \leq j \leq k$, the conditional probability that the kth dwarf will not sleep in his own bed is equal to 1 for $j = k$ and is equal to the unconditional probability $p\big(k-(j-1), n-(j-1)\big)$ for $2 \leq j \leq k-1$ (when dwarf j goes to bed, we face the situation of $n-(j-1)$ dwarfs where bed 1 is now the bed of dwarf j and dwarf k is in the $k-(j-1)$-th position). Hence, by the law of conditional probability, we find the recursion

$$p(k, n) = \frac{1}{n} + \sum_{j=2}^{k-1} p\big(k-(j-1),\, n-(j-1)\big)\frac{1}{n}$$

for $k = 2, \ldots, n$ and all $n \geq 2$. Noting that $p(1, n) = \frac{n-1}{n}$ for all $n \geq 1$, we get $p(2, n) = \frac{1}{n}$ and $p(3, n) = \frac{1}{n-1}$. Next, by induction, we obtain

$$p(k, n) = \frac{1}{n-k+2} \qquad \text{for } 2 \leq k \leq n.$$

Hence the probability that the kth dwarf can sleep in his own bed is equal to $1 - \frac{n-1}{n} = \frac{1}{n}$ for $k = 1$ and $1 - \frac{1}{n-k+2} = \frac{n-k+1}{n-k+2}$ for $2 \leq k \leq n$. A remarkable result is that $p(n,n) = \frac{1}{2}$ for all $n \geq 2$. A simple intuitive explanation can be given for the result that the last dwarf will sleep in his own bed with probability $\frac{1}{2}$, regardless of the number of dwarfs. The key observation is that the last free bed is either the bed of the youngest dwarf or the bed of the oldest dwarf. This is an immediate consequence of the fact that any of the other dwarfs always chooses his own bed when it is free. Each time a dwarf finds his bed occupied, the dwarf chooses at random a free bed and then the probability of the youngest dwarf's bed being chosen is equal to the probability of the oldest dwarf's bed being chosen. Thus the last free bed is equally likely to be the bed of the youngest dwarf or the bed of the oldest dwarf.

Note: Consider the following variant of the problem with seven dwarfs. The jolly youngest dwarf decides not to choose his own bed but rather to choose at random one of the other six beds. Then, the probability that the oldest dwarf can sleep in his own bed is $\frac{5}{6} \times \frac{1}{2} = \frac{5}{12}$, as can be seen by using the intuitive reasoning above.

2.71 Let's assume that the numbers $1, 2, \ldots, R$ are on the wheel. It is obvious that the optimal strategy of the second player B is to stop after the first spin if and only if the score is larger than the final score of player A and is larger than a_2, where a_2 is the optimal switching point for the first player in the two-player game ($a_2 = 53$ for $R = 100$). Denote by $S_3(a)$ [$C_3(a)$] the probability that the first player A will beat both player B and player C if player A obtains a score of a points in the first spin and stops [continues] after the first spin. Let the switching point a_3 be the largest value of a for which $C_3(a)$ is still larger than $S_3(a)$. Then, in the three-player game it is optimal for player A to stop after the first spin if the score of this spin is more than a_3 points. Denote by $P_3(A)$ the overall win probability of player A. Then, by the law of conditional probability,

$$P_3(A) = \frac{1}{R}\sum_{a=1}^{a_3} C_3(a) + \frac{1}{R}\sum_{a=a_3+1}^{R} S_3(a).$$

To obtain $S_3(a)$, we first determine the conditional probability that player A will beat player B when player A stops after the first spin with a points. Denote this conditional probability by P_a. To find, we first note that the probability $S(a) = \frac{a^2}{R^2}$ from the two-player game represents the probability that the second player in this game scores no more than a points in the first spin and has not beaten the first player after the second spin. Thus, taking into account the form of the optimal strategy of player B, we find for $a \geq a_2$,

$$\begin{aligned} P_a &= P(B \text{ gets no more than } a \text{ in the first spin and } A \text{ beats } B) \\ &= S(a) = \frac{a^2}{R^2} \end{aligned}$$

and for $1 \leq a < a_2$,

$$\begin{aligned} P_a = {} & P(B \text{ gets no more than } a \text{ in the first spin and } A \text{ beats } B) \\ & + P(B \text{ gets between } a \text{ and } a_2 + 1 \text{ in the first spin and } A \text{ beats } B) \\ = {} & S(a) + \frac{1}{R} \sum_{j=a+1}^{a_2} \Big(1 - \frac{R-j}{R}\Big) = \frac{a^2}{R^2} + \frac{1}{2R^2}\big[a_2(a_2+1) - a(a+1)\big], \end{aligned}$$

where $1 - (R-j)/R$ denotes the probability that B's total score after the second spin exceeds R when B's score in the first spin is j. Obviously, for the case that player A stops after the first spin with a points, the conditional probability of player A beating player C given that player A has already beaten player B is equal to $\frac{a^2}{R^2}$. Thus, the function $S_3(a)$ is given by

$$S_3(a) = \begin{cases} \frac{a^2}{R^2} \times \frac{a^2}{R^2}, & a_2 \leq a \leq R \\ \Big(\frac{a^2}{R^2} + \frac{1}{2R^2}[a_2(a_2+1) - a(a+1)]\Big) \times \frac{a^2}{R^2}, & 1 \leq a < a_2. \end{cases}$$

Further,

$$C_3(a) = \frac{1}{R} \sum_{k=1}^{R-a} S_3(a+k) \quad \text{ for } 1 \leq a \leq R.$$

Noting that $S_3(a) = \frac{a^4}{R^4}$ and $C_3(a) = \frac{1}{R}\sum_{k=1}^{R-a} \frac{(a+k)^4}{R^4}$ for $a \geq a_2$ and taking for granted that $a_3 \geq a_2$, the switching point a_3 is nothing else than the largest integer $a \geq a_2$ for which

$$\frac{1}{R} \sum_{k=1}^{R-a} \frac{(a+k)^4}{R^4} > \frac{a^4}{R^4}.$$

The probability $P_3(B)$ of the second player B being the overall winner can be calculated as follows. For the situation of optimal play by the players, let $p_3(a)$ denote the probability that the final score of player A will be a points and, for $b > a$, let $p_3(b \mid a)$ denote the probability that the final score of player B will be b points given that players's A final score is a. Then,

$$P_3(B) = \sum_{a=0}^{R-1} p_3(a) \sum_{b=a+1}^{R} p_3(b \mid a) \frac{b^2}{R^2}.$$

It easily follows that $p_3(0) = \sum_{k=1}^{a_3}(1/R) \times (k/R) = \frac{1}{2}a_3(a_3+1)/R^2$, $p_3(a) = \sum_{k=1}^{a-1}(1/R) \times (1/R) = (a-1)/R^2$ for $1 \leq a \leq a_3$, and $p_3(a) = 1/R + a_3/R^2$ for $a_3 < a \leq R$. Then, for $0 \leq a < a_2$ and $b > a$,

$$p_3(b \mid a) = \frac{b-1}{R^2} \text{ for } b \leq a_2, \quad p_3(b \mid a) = \frac{1}{R} + \frac{a_2}{R^2} \text{ for } b > a_2,$$
$$p_3(b \mid a) = 1/R + a/R^2 \text{ for } a_2 \leq a < R \text{ and } b > a.$$

Numerical results: For $R = 20$, we find $a_3 = 13$ ($a_2 = 10$), $P_3(A) = 0.3414$, $P_3(B) = 0.3307$, and $P_3(C) = 0.3279$ with $P_3(C) = 1 - P_3(A) - P_3(B)$. For $R = 100$, the results are $a_3 = 65$ ($a_2 = 53$), $P_3(A) = 0.3123$, $P_3(B) = 0.3300$, and $P_3(C) = 0.3577$, while for $R = 1{,}000$ the results are $a_3 = 648$ ($a_2 = 532$), $P_3(A) = 0.3059$, $P_3(B) = 0.3296$, and $P_3(C) = 0.3645$ (see also the solution of Problem 7.35).

Note: The following result can be given for the s-player game with $s > 3$. Denoting by a_s the optimal switching point for the first player A in the s-player game, the value of a_s can be calculated as the largest integer $a \geq a_{s-1}$ for which

$$\frac{1}{R} \sum_{k=1}^{R-a} \left(\frac{a+k}{R}\right)^{2(s-1)} > \left(\frac{a}{R}\right)^{2(s-1)}.$$

For $R = 20$, a_s has the values 14, 15, 16, and 17 for s=4, 5, 7, and 10. These values are 71, 75, 80, and 85 when $R = 100$ and are 711, 752, 803, and 847 when $R = 1{,}000$.

2.72 Let the hypothesis H be the event that a 1 is sent and the evidence E be the event that a 1 is received. The posterior odds are

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{P(H)}{P(\overline{H})} \times \frac{P(E \mid H)}{P(E \mid \overline{H})} = \frac{0.8}{0.2} \times \frac{0.95}{0.01} = 380.$$

Hence the posterior probability $P(H \mid E)$ that a 1 has been sent is $\frac{380}{1+380} = 0.9974$.

2.73 Let the hypothesis H be the event that oil is present and the evidence E be the event that the test is positive. Then $P(H) = 0.4$, $P(\overline{H}) = 0.6$, $P(E \mid H) = 0.9$, and $P(E \mid \overline{H}) = 0.15$. Thus the posterior odds are

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{P(H)}{P(\overline{H})} \times \frac{P(E \mid H)}{P(E \mid \overline{H})} = \frac{0.4}{0.6} \times \frac{0.90}{0.15} = 4$$

The posterior probability $P(H \mid E) = \frac{4}{1+4} = 0.8$.

2.74 Let the hypothesis H be the event that it rains tomorrow and E be the event that rain is predicted for tomorrow. The prior odds of the event H are $P(H)/P(\overline{H}) = 0.1/0.9$. The likelihood ratio is given by $P(E \mid H)/P(E \mid \overline{H}) = 0.85/0.25$. Then, by Bayes' rule in odds form, the posterior odds are

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{P(H)}{P(\overline{H})} \times \frac{P(E \mid H)}{P(E \mid \overline{H})} = \frac{0.1}{0.9} \times \frac{0.85}{0.25} = \frac{17}{45}.$$

It next follows that the posterior probability $P(E \mid H)$ of rainfall tomorrow given the information that rain is predicted for tomorrow is equal to $\frac{17/45}{1+17/45} = 0.2742$.

2.75 Let the hypothesis H be the event that the blue coin is unfair and the evidence E be the event that all three tosses of the blue coin show a head. The posterior odds are $\frac{0.2}{0.8} \times \frac{(0.75)^3}{(0.5)^3} = \frac{27}{32}$. The posterior probability $P(H \mid E) = \frac{27}{59} = 0.4576$.

2.76 Let the hypothesis H be the event that Dennis Nightmare played the final and the evidence E be the event that the Dutch team won the final Then, $P(H) = 0.75$, $P(\overline{H}) = 0.25$, $P(E \mid H) = 0.5$, and $P(E \mid \overline{H}) = 0.3$. Therefore the posterior odds are

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{0.75}{0.25} \times \frac{0.5}{0.3} = 5.$$

Thus the sought posterior probability $P(H \mid E) = \frac{5}{6}$.

2.77 Let the hypothesis H be the event that both children are boys.
(**a**) If the evidence E is the event that at least one child is a boy, then the posterior odds are

$$\frac{1/4}{3/4} \times \frac{1}{2/3} = \frac{1}{2}.$$

The posterior probability $P(H \mid E) = \frac{1}{3}$.
(**b**) If the evidence E is the event that at least one child is a boy born on a Tuesday, then the posterior odds are

$$\frac{1/4}{3/4} \times \Big[1 - \Big(\frac{6}{7}\Big)^2\Big] \Big/ \Big[\frac{1}{3} \times \frac{1}{7} + \frac{1}{3} \times \frac{1}{7} + \frac{1}{3} \times 0\Big] = \frac{13}{14}.$$

The posterior probability $P(H \mid E) = \frac{13}{27}$.
(**c**) If the evidence E is the event that at least one child is a boy born on one of the first k days of the week, then the posterior odds are

$$\frac{1/4}{3/4} \times \Big[1 - \Big(1 - \frac{k}{7}\Big)^2\Big] \Big/ \Big[\frac{1}{3} \times \frac{k}{7} + \frac{1}{3} \times \frac{k}{7} + \frac{1}{3} \times 0\Big] = \frac{14-k}{14}.$$

The posterior probability $P(H \mid E) = \frac{14-k}{28-k}$ for $k = 1, 2, \ldots, 7$.

2.78 Let the hypothesis H be the event that the inhabitant you overheard spoke truthfully and the evidence E be the event that the other inhabitant says that the inhabitant you overheard spoke the truth. The posterior odds are

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{1/3}{2/3} \times \frac{1/3}{2/3} = \frac{1}{4}.$$

Hence the posterior probability $P(H \mid E)$ that the inhabitant you overheard spoke the truth is $\frac{1/4}{1+1/4} = \frac{1}{5}$.

2.79 Let the hypothesis H be the event that the suspect is guilty and the evidence E be the event that the suspect makes a confession. To verify that $P(H \mid E) > P(H)$ if and only if $P(E \mid H) > P(E \mid \overline{H})$, we use the fact that $\frac{a}{1-a} > \frac{b}{1-b}$ for $0 < a,\, b < 1$ if and only if $a > b$. Bayes' rule in odds form states that

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{P(H)}{P(\overline{H})} \times \frac{P(E \mid H)}{P(E \mid \overline{H})}$$

If $P(E \mid H) > P(E \mid \overline{H})$, then it follows from Bayes' rule in odds form that $\frac{P(H|E)}{1-P(H|E)} > \frac{P(H)}{1-P(H)}$ and so $P(H \mid E) > P(H)$. Next suppose that $P(H \mid E) > P(H)$. Then $\frac{P(H|E)}{1-P(H|E)} > \frac{P(H)}{1-P(H)}$ and thus, by Bayes' rule in odds form, $P(E \mid H) > P(E \mid \overline{H})$.

2.80 Let the hypothesis H be the event that the bowl originally contained a red ball and the evidence E be the event that a red ball is picked from the bowl after a red ball was added. Then, $P(H) = 0.5$, $P(\overline{H}) = 0.5$, $P(E \mid H) = 1$, and $P(E \mid \overline{H}) = 0.5$. Therefore $\frac{P(H|E)}{P(\overline{H}|E)} = \frac{1/2}{1/2} \times \frac{1}{1/2} = 2$. Thus the posterior probability $P(H \mid E) = \frac{2}{3}$.

2.81 Let the hypothesis H be the event that the woman has breast cancer and the evidence E be the event that the test result is positive. Since $P(H) = 0.01$, $P(\overline{H}) = 0.99$, $P(E \mid H) = 0.9$, and $P(E \mid \overline{H}) = 0.1$, the posterior odds are $\frac{0.01}{0.99} \times \frac{0.9}{0.1} = \frac{1}{11}$. Therefore the posterior probability $P(H \mid E) = \frac{1}{12}$.
Note: As a sanity check, the posterior probability can also be obtained by a heuristic but insightful approach. This approach presents the relevant information in terms of frequencies instead of probabilities. Imagine 10,000 (say) women who undergo the test. On average, there are 90 positive tests for the 100 women having the malicious disease, whereas there are 990 false positives for the 9,900 healthy women. Thus, based on the information presented in this way, we find that the sought probability is $90/(90 + 990) = \frac{1}{12}$.

2.82 Let the hypothesis H be the event that Elvis was an identical twin and the evidence E be the event that Elvis's twin was male. Then $P(H) = \frac{300}{425} = \frac{5}{17}$, $P(\overline{H}) = \frac{12}{17}$, $P(E \mid H) = 1$, and $P(E \mid \overline{H}) = 0.5$. Then, by Bayes in odds form, $\frac{P(H|E)}{P(\overline{H}|E)} = \frac{5}{6}$. This gives $P(H \mid E) = \frac{5}{11}$.
Note: A heuristic way to get the answer is as follows. In 3000 births (say), we would expect $3000/300 = 10$ sets of identical twins. Roughly half of those we would expect to be boys. That's 5 sets of boy-boy identical twins. In 3000 births, we would expect $3000/125 = 24$ sets of fraternal twins. One fourth would be boy-boy, one-fourth would be girl-girl, one fourth would be boy-girl, and one fourth girl-boy. Therefore six sets would be boy-boy. So, out of 3000 births, five out of eleven sets of boy-boy twins would be identical. Therefore the chances that Elvis was an identical twin is about 5/11.

2.83 Let the hypothesis H be the event that you have chosen the two-headed coin and the evidence E be the event that all n tosses result in heads. The posterior odds are

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{1/10{,}000}{9{,}999/10{,}000} \times \frac{1}{0.5^n}.$$

This gives $P(H \mid E) = \frac{2^n}{2^n + 9{,}999}$. The probability $P(H \mid E)$ has the values 0.0929, 0.7662, and 0.9997 for $n = 10$, 15, and 25.

2.84 Let the random variable Θ represent the unknown probability that a single toss of the die results in the outcome 6. The prior distribution of Θ is given by $p_0(\theta) = 0.25$ for $\theta = 0.1$, 0.2, 0.3 and 0.4. The posterior probability $p(\theta \mid \text{data}) = P(\Theta = \theta \mid \text{data})$ is proportional to $L(\text{data} \mid \theta) p_0(\theta)$, where $L(\text{data} \mid \theta) = \binom{300}{75}\theta^{75}(1-\theta)^{225}$. Hence the posterior probability $p(\theta \mid \text{data})$ is given by

$$\begin{aligned} p(\theta \mid \text{data}) &= \frac{L(\text{data} \mid \theta) p_0(\theta)}{\sum_{k=1}^{4} L(\text{data} \mid k/10) p_0(k/10)} \\ &= \frac{\theta^{75}(1-\theta)^{225}}{\sum_{k=1}^{4} (k/10)^{75}(1-k/10)^{225}}, \quad \theta = 0.1, 0.2, 0.3, 0.4. \end{aligned}$$

The posterior probability $p(\theta \mid \text{data})$ has the values 3.5×10^{-12}, 0.4097, 0.5903, and 3.5×10^{-12} for $\theta = 0.1$, 0.2, 0.3, and 0.4.

2.85 Let the random variable Θ represent the unknown win probability of Alassi. The prior of Θ is $p_0(0.4) = p_0(0.5) = p_0(0.6) = \frac{1}{3}$. Let E be the event that Alassi wins the best-of-five contest. The likelihood function $L(E \mid \theta)$ is $\theta^3 + \binom{3}{2}\theta^2(1-\theta)\theta + \binom{4}{2}\theta^2(1-\theta)^2\theta$. The posterior probability $p(\theta \mid E)$ is proportional to $p_0(\theta)L(E \mid \theta)$ and has the values 0.2116, 0.3333, and 0.4550 for $\theta = 0.4$, 0.5, and 0.6.

2.86 The prior density of the unknown success probability is

$$p_0(\theta) = \frac{1}{101} \quad \text{for } \theta = 0, 0.01, \ldots, 0.99, 1.$$

For a single observation, the prior is updated with the likelihood factor θ if the observation corresponds to a success of the new treatment and with $1 - \theta$ otherwise. The first observation S leads to an update that is proportional to $\theta p_0(\theta)$, the second observation S to an update that is proportional to $\theta^2 p_0(\theta)$, the third observation F to an update that is proportional to $\theta^2(1-\theta)p_0(\theta)$, and so on, the tenth observation F to an update that is proportional to $\theta^2(1-\theta)\theta^2(1-\theta)\theta^3(1-\theta)p_0(\theta) = \theta^7(1-\theta)^3 p_0(\theta)$. The same posterior as we found in Example 2.17, where we simultaneously used all observations.

2.87 Let the random variable Θ be 1 if the student is unprepared for the exam, 2 if the student is half prepared, and 3 if the student is well prepared. The prior of Θ is $p_0(1) = 0.2$, $p_0(2) = 0.3$, and $p_0(3) = 0.5$. Let E be the event that the student has answered correctly 26 out of 50 questions. The likelihood function $L(E \mid \theta)$ is $\binom{50}{26} a_\theta^{26}(1 - a_\theta)^{24}$, where $a_1 = \frac{1}{3}$, $a_2 = 0.45$ and $a_3 = 0.8$. The posterior probability $p(\theta \mid E)$ is proportional to $p_0(\theta)L(E \mid \theta)$ and has the values 0.0268, 0.9730, and 0.0001 for $\theta = 1$, 2, and 3.

2.88 Let the random variable Θ represent the unknown probability that a free throw of your friend will be successful. The prior probabilities are $p_0(\theta) = P(\Theta = \theta)$ has the values 0.2, 0.6, and 0.2 for $\theta = 0.25$, 0.50, and 0.75. The posterior probability $p(\theta \mid \text{data}) = P(\Theta = \theta \mid \text{data})$ is proportional to $L(\text{data} \mid \theta)p_0(\theta)$, where $L(\text{data} \mid \theta) = \binom{10}{7}\theta^7(1 - \theta)^3$. Hence the posterior probability $p(\theta \mid \text{data})$ is given by

$$\frac{\theta^7(1-\theta)^3 p_0(\theta)}{0.25^7 \times 0.75^3 \times 0.2 + 0.50^7 \times 0.50^3 \times 0.6 + 0.75^7 \times 0.25^3 \times 0.2}$$

for $\theta = 0.25$, 0.50, and 0.75. The possible values 0.25, 0.50 and 0.75 for the success probability of the free throws of your friend have the posterior probabilities 0.0051, 0.5812 and 0.4137.