Chapter 2

Methods of Proof

2.1 What is a Proof?

2.2 Direct Proof

1. If m, n are odd then, by definition, $m = 2k+1$ and $n = 2\ell+1$ for some $k, \ell \in \mathbf{Z}$. Multiplying, we obtain:

$$m \cdot n = (2k+1) \cdot (2\ell+1) = 2(2k\ell + k + \ell) + 1,$$

which is odd by definition.

2. If n is even then, by definition, $n = 2\ell$ for some $\ell \in \mathbf{Z}$. Multiplying, we obtain:

$$m \cdot n = m(2\ell) = 2(m\ell),$$

which is even by definition.

3. If n is even, then $n-1$ is odd. Now, $n = (n-1)+1$, so n is the sum of two odds. If n is odd, then $n = n$, so n is the sum of one odd integer.

4. Let $\alpha < \beta$ be distinct real numbers. Let $\epsilon = \beta - \alpha > 0$. Choose a positive integer N so large that $1/N < \epsilon$. Look at the numbers $\{j/N\}$ as j runs through the integers. These numbers, in sequence, are closer together than ϵ. So one of them must lie between α and β. The solution of Exercise 5 is similar.

6. The property is true for $k = 3$, since $2^3 = 8 > 7 = 1 + 2 \cdot 3$. Assume that the property is true for $k = n - 1$. We want to show that it is true for $k = n$. In other words, we want to prove that

$$2^n > 1 + 2n.$$

Observing that $2^n = 2 \cdot 2^{n-1}$, we find an obvious place to apply the mathematical induction hypothesis:

$$\begin{aligned} 2^n &= 2 \cdot 2^{n-1} \\ &> 2 \cdot (1 + 2(n-1)) \quad \text{from the mathematical induction hypothesis} \\ &= (1 + 2n) + (2n - 3) \\ &> 1 + 2n \quad \text{since } 2n - 3 > 0 \text{ for } n \geq 2. \end{aligned}$$

7. It is true for $n = 5$:

$$2^5 = 32 > 26 = 5^2 + 1.$$

Assume that it is true for $n - 1 \geq 5$. We must prove it for n. We can write:

$$\begin{aligned} 2^n &= 2 \cdot 2^{n-1} \\ &> 2((n-1)^2 + 1) \quad \text{by the mathematical induction hypothesis} \\ &= n^2 + 1 + (n^2 - 4n + 3) \\ &= n^2 + 1 + (n^2 - 4n + 4) - 1 \\ &= n^2 + 1 + (n-2)^2 - 1 \\ &> n^2 + 1 \quad \text{since } (n-2)^2 - 1 > 0 \text{ if } n > 4. \end{aligned}$$

8. Assume that we have n mailboxes. Let $\ell(j)$ be the number of letters in box j. Now

$$\ell(1) + \ell(2) + \cdots + \ell(n) = n + 1\,,$$

since all the letters taken together total $n + 1$ letters. Dividing by n gives

$$\frac{\ell(1) + \ell(2) + \cdots + \ell(n)}{n} = \frac{n+1}{n} > 1\,.$$

So the average number of letters per box exceeds 1. This can only be true if some box contains more than 1 letter. Thus some box contains two letters.

9. It cannot be that just one letter is in the wrong envelope. If letter a went into envelope B, then at least letter b went into envelope A (of course it could be more complicated than that).

10. If $k = 2j + 1$ is odd and $\ell = 2m + 1$ is odd, then

$$k + \ell = (2j + 1) + (2m + 1) = 2j + 2m + 2 = 2(j + m + 1)\,,$$

which is even.

2.3 Proof by Contradiction

1. By the usual pigeonhole principle, some mailbox will contain at least two letters. Clearly, of these two letters, either both are red or both are blue or there is one of each color.

2. We can write $m = 3^k$ and $n = 3^\ell$ for some $k, \ell \in \mathbf{N}$. Let us assume, without loss of generality, that $k \leq \ell$. Seeking a contradiction, assume that $m + n = 3^r$, for some $r \in \mathbf{N}$. Then we have:

$$m + n = 3^\ell + 3^k = 3^k(1 + 3^{\ell-k}) = 3^r, \quad \text{with} \quad \ell - k \geq 0.$$

Now note that since $3^k = m < m + n = 3^r$, it must be that $k < r$. Hence,

$$1 + 3^{\ell-k} = 3^{r-k}, \quad \text{with} \quad r - k \geq 0.$$

The righthand side of the last equality is either 1 or divisible by 3, whereas the lefthand side is bigger than or equal to 2 and definitely not divisible by three. Thus, the equality must be false, which means that our original hypothesis $m + n = 3^r$, for some $r \in \mathbf{N}$) was false.

6. Following the given scheme, we have $2q^2 = p^2$. If q has, say, r prime factors, then q^2 has $2r$ prime factors. Thus $2q^2$ has $2r+1$ prime factors. On the other hand, p^2 must have an even number of prime factors, and we arrive at a contradiction. Hence our original assumption $\sqrt{2} = p/q$ must be false.

7. Write $n = k^2$, and suppose that $n + 1 = \ell^2$ for some $l \in \mathbf{N}$. Then $1 = (n + 1) - n = \ell^2 - k^2 = (\ell + k)(\ell - k)$. Then $\ell + k = 1$, and that is impossible for natural numbers ℓ and k.

8. Suppose not. If neither integer is even, then they are both odd. So one is $k = 2m + 1$ and the other is $\ell = 2j + 1$. But then

$$k \cdot \ell = (2m+1)(2j+1) = 4mj + 2j + 2m + 1 = 2 \cdot (2mj + j + m) + 1\,,$$

which is odd. That is a contradiction.

9. Refer to the solution of Exercise 8.

10. False. $1^2 + 2^2 = 5$, which is not a perfect square.

11. True: Suppose, seeking a contradiction, that there are no perfect squares in that list. This means that all the numbers in the list fall between two consecutive squares, i.e., there is a k such that $k^2 < n < n + 1 < \ldots < 2n + 2 < (k + 1)^2$. We then have:

$$2n + 2 < (k + 1)^2 = k^2 + 2k + 1 < n + 2k + 1,$$

or

$$n + 1 < 2k.$$

Squaring both sides, we obtain:

$$n^2 + 2n + 1 < 4k^2 < 4n,$$

or

$$(n - 1)^2 = n^2 - 2n + 1 < 0,$$

which is impossible.

12. Certainly $6 = 3+2+1$. There are many other examples. Such integers are called perfect numbers.

13. Notice that

$$m^2 - n^2 = (m + n) \cdot (m - n)\,,$$

which is composite.

14. We see that $1^2 + 2^2 = 5$, which is prime.

15. Certainly

$$(1 + x)^2 = 1 + 2x + x^2 > 1 + x^2\,.$$

16. False: Take $n = 2, a_1 = 1, a_2 = 4$. Then the inequality would read:

$$\frac{5}{2} = \frac{1+4}{2} \leq (1 \cdot 4)^{1/2} = 2,$$

which is clearly false.

18. Let $p < q$ be distinct rational numbers. Let $\epsilon = q - p$. Choose an integer N so large, N not a perfect square, so that $1/N < \epsilon^2$. Then $1/\sqrt{N} < \epsilon$, and $1/\sqrt{N}$ is irrational. Examine the sequence $\{j/\sqrt{N}\}$. These numbers, in sequence, are closer together than ϵ. So one of them must lie between p and q.

2.4 Proof by Induction

1. Now $P(n)$ is the statement

$$1^2 + 2^2 + \cdots n^2 = \frac{2n^3 + 3n^2 + n}{6}\,. \tag{$*$}$$

The statement $P(1)$ is obvious becaise $(2+3+1)/6 = 1$. Assume now that we know $P(n)$, and we shall use this hypothesis to prove $P(n+1)$. We add $(n+1)^2$ to both sides of $(*)$ to obtain

$$1^2 + 2^2 + \cdots n^2 + (n+1)^2 = \frac{2n^3 + 3n^2 + n}{6} + (n+1)^2$$

or

$$\begin{aligned} 1^2 + 2^2 + \cdots n^2 + (n+1)^2 &= \frac{2n^3 + 3n^2 + n + 6(n^2 + 2n + 1)}{6} \\ &= \frac{2(n+1)^3 + 3(n+1)^2 + (n+1)}{6}\,. \end{aligned}$$

This is the statement $P(n+1)$. The induction is complete.

2. The inductive statement is of course that

$$2 + 4 + \cdots 2k = k^2 + k\,.$$

The statement is obvious for $k = 1$.

Now assume that $P(k)$ is known. So

$$2 + 4 + \cdots + 2k = k^2 + k\,.$$

We add $2k + 2$ to both sides to obtain

$$2 + 4 + \cdots + 2k + (2k + 2) = k^2 + k + (2k + 2)$$

or

$$2 + 4 + \cdots + 2k + (2k + 2) = (k + 1)^2 + (k + 1)\,.$$

But this is just the statement $P(k + 1)$. The induction is complete.

3. The flaw in the reasoning is that we never enunciated what $P(n)$ was.

4. Of course the case $k = 3$ is a triangle, and we certainly know that the sum of the angles in a triangle is $180°$. Now assume that the result is known for a convex polygon with k sides.

Let $\mathcal{P}$ be a convex polygon with $(k+1)$ sides. Then $\mathcal{P}$ may be thought of as a convex polygon with k sides with a triangle added on. And that adds $180°$ to the interior angles. That completes the induction.

5. That ordinary induction implies complete induction is obvious.

Now suppose that complete induction is valid. Assume that we have a statement $P(n)$ and that $P(1)$ and $P(j)$ have been established.

6. If one starts the mathematical induction process from a number $n_0 + 1$, $n_0 \geq 1$, then $P(1)$ might not be true (it might not even be defined), and we would not be able to use mathematical induction according to the statement in the text. But we can modify this statement using the following trick:

Define a property P' as follows:

$$P'(k) \text{ is true } \quad if \quad P(n_0 + k) \text{ is true }.$$

Then $P'(1)$ is true since $P(n_0 + 1)$ is true, and $P'(n - 1) \Rightarrow P'(n)$ because $P(n_0 + n - 1) \Rightarrow P(n_0 + n)$ by hypothesis. But now we can apply mathematical induction (as stated in the text) to the property P', so that $P'(n)$ holds for any natural number n. This implies that $P(n_0 + n)$ holds for all $n \in \mathbf{N}$, or equivalently, $P(m)$ holds for all $n \geq n_0 + 1$.

9. Let $P(n)$ be the statement, "If $n+1$ letters are placed into n mailboxes, then some mailbox must contain two letters. When $n=1$, the claim is that if we put two letters into one mailbox, then some mailbox must contain two letters. Obvious. Now suppose that $P(n-1)$ has been proved. We have n mailboxes, and we place $n+1$ letters into n mailboxes. If the last mailbox contains two letters, then we are done. If not, then the last box contains one or two letters. But then the first $n-1$ mailboxes contain at least n letters. So the inductive hypothesis applies, and one of them must contain two letters. That completes the inductive step, and the proof.

10. The assertion is true for $n=1$ by inspection.

Assume now that the assertion is verified for $n=j$. Then we have

$$\begin{aligned}\begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix}^{j+1} &= \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix}^{j} \\ &= \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a^j & 2ja^{j-1} \\ 0 & a^j \end{pmatrix} \\ &= \begin{pmatrix} a^{j+1} & 2(j+1)a^j \\ 0 & a^{j+1} \end{pmatrix}.\end{aligned}$$

That completes the inductive step.

11. The assertion is clear for $n=1$. Now assume that it is true for $n=j$. We write

$$\begin{aligned}(j+1)^3-(j+1) &= \big(j^3+3j^2+3j+1\big)-(j+1) \\ &= j^3+3j^2+2j \\ &= (j^3-j)+(3j^2+3j)\,.\end{aligned}$$

Now, by the inductive hypothesis, j^3-j is divisible by 6. Also

$$3j^2+3j=3j(j+1)\,.$$

Since either j or $j+1$ is divisible by 2, this last expression is also divisible by 6. Hence $(j+1)^3-(j+1)$ is divisible by 6, and the mathematical induction is complete.

13. We will prove that, for any positive integer n,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}\,.$$

The claim is plainly true for $n = 1$. Now assume that it has been established for $n = j$. Then we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{j}} + \frac{1}{\sqrt{j+1}} \geq \sqrt{j} + \frac{1}{\sqrt{j+1}}\,.$$

So we need to show that

$$\sqrt{j} + \frac{1}{\sqrt{j+1}} \geq \sqrt{j+1}\,.$$

Multiplying both sides by $\sqrt{j+1}$, we see that this is the same as

$$\sqrt{j(j+1)} + 1 \geq j + 1$$

or

$$\sqrt{j(j+1)} \geq j\,.$$

Now squaring both sides gives the result. The mathematical induction is complete, and the result proved.