Chapter 2

Methods of Proof

1. If m, n are odd then, by definition, $m = 2k + 1$ and $n = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. Multiplying, we obtain:

$$m \cdot n = (2k + 1) \cdot (2\ell + 1) = 2(2k\ell + k + \ell) + 1,$$

which is odd by definition.

2. If n is even then, by definition, $n = 2\ell$ for some $\ell \in \mathbb{Z}$. Multiplying, we obtain:

$$m \cdot n = m(2\ell) = 2(m\ell),$$

which is even by definition.

4. Writing things out, we obtain

$$\begin{aligned} 2 + 4 + \cdots + (2k-2) + 2k &= 2\,(1 + 2 + \cdots + (k-1) + k) \\ &= 2\,\frac{k(k+1)}{2} \qquad \text{by Prop. 2.4.2} \\ &= k^2 + k. \end{aligned}$$

5. Writing things out, we obtain

$$\begin{aligned}
&1 + 3 + \cdots + (2k-3) + (2k-1) \\
&\qquad = (2-1) + (4-1) + \cdots + ((2k-2)-1) + (2k-1) \\
&\qquad = (2 + 4 \cdots + (2k-2) + 2k) - k \\
&\qquad = (k^2 + k) - k \qquad \text{by Exercise 2.4} \\
&\qquad = k^2.
\end{aligned}$$

7. We can write $m = 3^k$ and $n = 3^\ell$ for some $k, \ell \in \mathbb{N}$. Let us assume, without loss of generality, that $k \leq \ell$. Seeking a contradiction, assume that $m + n = 3^r$, for some $r \in \mathbb{N}$. Then we have:

$$m + n = 3^\ell + 3^k = 3^k(1 + 3^{\ell-k}) = 3^r, \text{ with } \ell - k \geq 0.$$

Now note that since $3^k = m < m + n = 3^r$, it must be that $k < r$. Hence,

$$1 + 3^{\ell-k} = 3^{r-k}, \text{ with } r - k \geq 0.$$

The righthand side of the last equality is either 1 or divisible by 3, whereas the lefthand side is bigger than or equal to 2 and definitely not divisible by three. Thus, the equality must be false, which means that our original hypothesis $m + n = 3^r$, for some $r \in \mathbb{N}$) was false.

11. Following the given scheme, we have $2q^2 = p^2$. If q has, say, r prime factors, then q^2 has $2r$ prime factors. Thus $2q^2$ has $2r + 1$ prime factors. On the other hand, p^2 must have an even number of prime factors, and we arrive at a contradiction. Hence our original assumption $\sqrt{2} = p/q$ must be false.

12. Write $n = k^2$, and suppose that $n + 1 = \ell^2$ for some $l \in \mathbb{N}$. Then $1 = (n+1) - n = \ell^2 - k^2 = (\ell + k)(\ell - k)$. Then $\ell + k = 1$, and that is impossible for natural numbers ℓ and k.

13. We proved in Exercise 2.1 that the product of two odd numbers is odd. Therefore, if the product of two numbers is even, at least one of them must be even.

15. If n is even, then $n - 1$ is odd. Now, $n = (n-1) + 1$, so n is the sum of two odds. If n is odd, then $n = n$, so n is the sum of one odd integer.

17. False: $1^2 + 2^2 = 5$, which is not a perfect square.

18. True: Suppose, seeking a contradiction, that there are no perfect squares in that list. This means that all the numbers in the list fall between two consecutive squares, i.e., there is a k such that $k^2 < n < n+1 < \ldots < 2n+2 < (k+1)^2$. We then have:

$$2n + 2 < (k+1)^2 = k^2 + 2k + 1 < n + 2k + 1,$$

or

$$n + 1 < 2k.$$

Squaring both sides, we obtain:

$$n^2 + 2n + 1 < 4k^2 < 4n,$$

or

$$(n-1)^2 = n^2 - 2n + 1 < 0,$$

which is impossible.

19. True: 6=1+2+3, or 28=1+2+4+7+14. In fact, these numbers have a name: Perfect numbers. Not much is known about perfect numbers. It is conjectured that there are no odd perfect numbers, mainly because nobody ever found an odd perfect number, but it has never been proved. It has also not been proved that there are infinitely many perfect numbers.

21. False: $2^2 + 1^2 = 5$, which is a prime.

23. False: Take $n = 2, a_1 = 1, a_2 = 4$. Then the inequality would read:

$$\frac{5}{2} = \frac{1+4}{2} \leq (1 \cdot 4)^{1/2} = 2,$$

which is clearly false.

25. True: Write the rationals as fractions $p_1/q, p_2/q$ with the same denominator q so that $p_1 + 2 \leq p_2$ (one can always achieve this by taking q big enough). Then either $\sqrt{{p_1}^2 + 1}$ or $\sqrt{{p_1}^2 + 2}$ is an irrational number that lies between p_1 and p_2 (cf. Exercise 2.12). Divide this irrational by q to obtain another irrational that lies between the two rationals.

27. We will use the alternative form of the principle of complete mathematical induction given in the text (see also Exercise 2.31). The property is clearly true for 2 (since 2 is prime). Assume that it is true for any $k < n$. We have to prove that it is true for $k = n$. If the only divisors of n are n itself and 1, we are done, since that would imply that n is itself a prime. Otherwise n has a divisor $d < n$. By the mathematical induction hypothesis, d must have some prime factor p. Now, since p divides d which divides n, p must also divide n. But then p will be a factor of n. Hence the property is also true for n.

29. The property is true for $k = 3$, since $2^3 = 8 > 7 = 1 + 2 \cdot 3$. Assume that the property is true for $k = n - 1$. We want to show that it is true for $k = n$. In other words, we want to prove that

$$2^n > 1 + 2n.$$

Observing that $2^n = 2 \cdot 2^{n-1}$, we find an obvious place to apply the mathematical induction hypothesis:

$$\begin{aligned} 2^n &= 2 \cdot 2^{n-1} \\ &> 2 \cdot (1 + 2(n-1)) \quad \text{from the mathematical induction hypothesis} \\ &= (1 + 2n) + (2n - 3) \\ &> 1 + 2n \quad \text{since } 2n - 3 > 0 \text{ for } n \geq 2. \end{aligned}$$

31. If one starts the mathematical induction process from a number $n_0 + 1,\ n_0 \geq 1$, then $P(1)$ might not be true (it might not even be defined), and we would not be able to use mathematical induction according to the statement in the text. But we can modify this statement using the following trick:

Define a property P' as follows:

$$P'(k) \text{ is true } \quad if \quad P(n_0 + k) \text{ is true .}$$

Then $P'(1)$ is true since $P(n_0 + 1)$ is true, and $P'(n-1) \Rightarrow P'(n)$ because $P(n_0 + n - 1) \Rightarrow P(n_0 + n)$ by hypothesis. But now we can apply mathematical induction (as stated in the text) to the property P', so that $P'(n)$ holds for any natural number n. This implies that $P(n_0 + n)$ holds for all $n \in \mathbb{N}$, or equivalently, $P(m)$ holds for all $n \geq n_0 + 1$.

34. It is true for $n = 5$:

$$2^5 = 32 > 26 = 5^2 + 1.$$

Assume that it is true for $n - 1 \geq 5$. We must prove it for n. We can write:

$$\begin{aligned} 2^n &= 2 \cdot 2^{n-1} \\ &> 2((n-1)^2 + 1) \quad \text{by the mathematical induction hypothesis} \\ &= n^2 + 1 + (n^2 - 4n + 3) \\ &= n^2 + 1 + (n^2 - 4n + 4) - 1 \\ &= n^2 + 1 + (n - 2)^2 - 1 \\ &> n^2 + 1 \quad \text{since } (n-2)^2 - 1 > 0 \text{ if } n > 4. \end{aligned}$$

35. Let $P(n)$ be the statement, "If $n + 1$ letters are placed into n mailboxes, then some mailbox must contain two letters. When $n = 1$, the claim is that if we put two letters into one mailbox, then some mailbox must contain two letters. Obvious. Now suppose that $P(n - 1)$ has been proved. We have n mailboxes, and we place $n + 1$ letters into n mailboxes. If the last mailbox contains two letters, then we are done. If not, then the last box contains one or two letters. But then the first $n-1$ mailboxes contain at least n letters. So the inductive hypothesis applies, and one of them must contain two letters. That completes the inductive step, and the proof.

36. Assume that we have n mailboxes. Let $\ell(j)$ be the number of letters in box j. Now

$$\ell(1) + \ell(2) + \cdots + \ell(n) = n + 1\,,$$

since all the letters taken together total $n + 1$ letters. Dividing by n gives

$$\frac{\ell(1) + \ell(2) + \cdots + \ell(n)}{n} = \frac{n+1}{n} > 1\,.$$

So the average number of letters per box exceeds 1. This can only be true if some box contains more than 1 letter. Thus some box contains two letters.

38. Consider the set of S all ordered pairs (ℓ, p) where ℓ is a line passing through (at least) two of the given points and p is a point not on that line (certainly p exists because the points are not all colinear). Define a function f on S by

$$f(\ell, p) = \text{distance of } \ell \text{ to } p.$$

Then f is a function with a finite domain, so there is a particular ordered pair (ℓ_0, p_0) that minimizes the function. Then ℓ_0 is the line that we seek. We invite the reader to check cases to verify this assertion.

40. Let

$B(x) = x$ is a boy under the age of 10.

and

$P(x) = x$ practices all pieces in his/her piano book every day.

Then our statement is

$$\forall x, B(x) \Rightarrow P(x)\,.$$

We can rewrite this as

$$\sim \exists x, \sim (B(x) \Rightarrow P(x))\,.$$

41. The assertion is true for $n = 1$ by inspection.

Assume now that the assertion is verified for $n = j$. Then we have

$$\begin{aligned}
\begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix}^{j+1} &= \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix}^{j} \\
&= \begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} a^j & 2ja^{j-1} \\ 0 & a^j \end{pmatrix} \\
&= \begin{pmatrix} a^{j+1} & 2(j+1)a^j \\ 0 & a^{j+1} \end{pmatrix}.
\end{aligned}$$

That completes the inductive step.

42. The assertion is clear for $n = 1$. Now assume that it is true for $n = j$. We write

$$\begin{aligned}
(j+1)^3 - (j+1) &= \left(j^3 + 3j^2 + 3j + 1\right) - (j+1) \\
&= j^3 + 3j^2 + 2j \\
&= (j^3 - j) + (3j^2 + 3j)\,.
\end{aligned}$$

Now, by the inductive hypothesis, $j^3 - j$ is divisible by 6. Also

$$3j^2 + 3j = 3j(j+1)\,.$$

Since either j or $j+1$ is divisible by 2, this last expression is also divisible by 6. Hence $(j+1)^3 - (j+1)$ is divisible by 6, and the mathematical induction is complete.

44. We will prove that, for any positive integer n,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}\,.$$

The claim is plainly true for $n = 1$. Now assume that it has been established for $n = j$. Then we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{j}} + \frac{1}{\sqrt{j+1}} \geq \sqrt{j} + \frac{1}{\sqrt{j+1}}\,.$$

So we need to show that

$$\sqrt{j} + \frac{1}{\sqrt{j+1}} \geq \sqrt{j+1}\,.$$

Multiplying both sides by $\sqrt{j+1}$, we see that this is the same as

$$\sqrt{j(j+1)} + 1 \geq j + 1$$

or

$$\sqrt{j(j+1)} \geq j\,.$$

Now squaring both sides gives the result. The mathematical induction is complete, and the result proved.