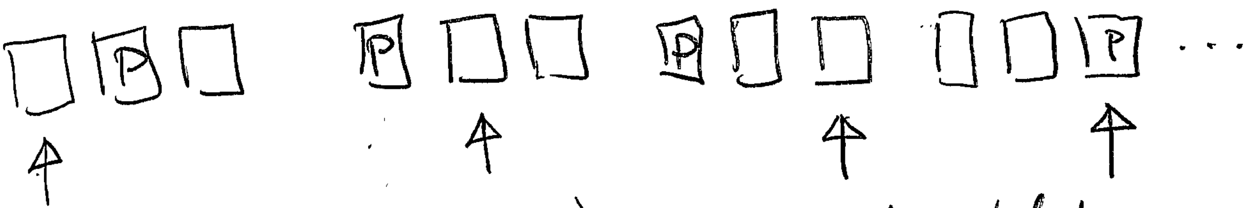
Problem 2.1

Monty Hall. Start with 10^6 sets of three boxes. Prize "P" is randomly & uniformly found in any of the three



Your random (initial) choice indicated by arrows)

The Key: Your initial choice will be right $1/3$ of the time and wrong $2/3$ of the time. That is, there is a 2/3 chance the prize is in a box you did not choose — i.e. ~ 666,666 of 10^6.

What if someone gave you the choice to switch to the other two boxes?

Of course you would, and this is the choice Monty offers — only he tells you which of the remaining is empty.

Thus, we expect that switchers will win $666,666/10^6$ times. Be a switcher!

Problem 2.2

Six faces are $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$

twice as often as others

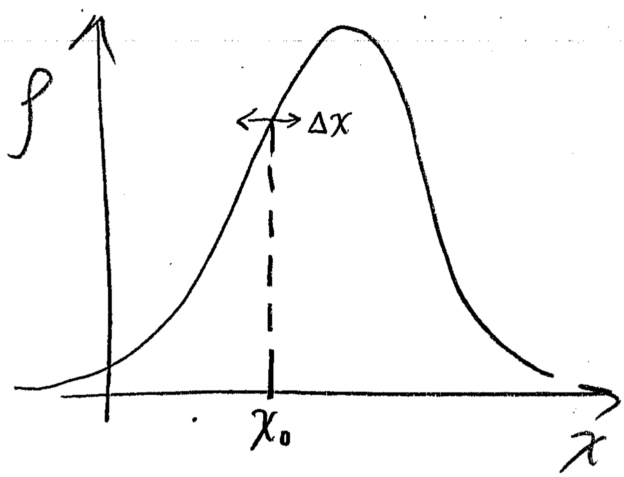
Weights are 1 1 1 2 1 1

$\Rightarrow$

Roll	1	2	3	4	5	6
Prob	1/7	1/7	1/7	2/7	1/7	1/7

The probs sum to one and $p(4) = 2\,p(j \neq 4)$

Problem 2.3

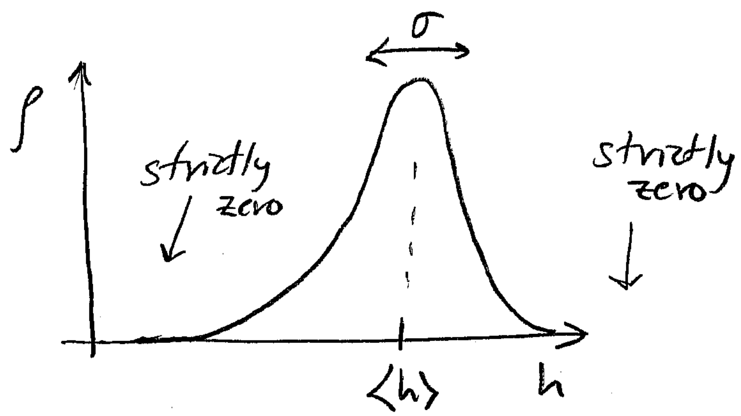


We want the function ρ to change 'minimally' over the interval Δx. In a Gaussian, this will be true if $\Delta x \ll \sigma$, since σ is the lengthscale over which ρ changes significantly. More generally, one can that ρ should change minimally compared to its maximum:

$$\Delta x \cdot \left.\frac{d\rho}{dx}\right|_{x_0} \ll \max(\rho)$$

Problem 2.4

If $\sigma << \langle h \rangle$, The distribution can be approximately Gaussian near $\langle h \rangle$.



Problem 2.5

(a) $w(x) = 5 = \text{const}$ for $0 < x < 10$

$$\rho(x) = \frac{w(x)}{\int dx\, w(x)} = \frac{5}{\int_0^{10} dx \cdot 5} = \frac{5}{50} = \frac{1}{10}$$

Should make sense: $10 \cdot \frac{1}{10} = 1$

(b) $w(x) = x^{-4}$ $(1, \infty)$

$$\int_1^\infty dx\, x^{-4} = \frac{-1}{3} x^{-3} \Big|_1^\infty = \frac{1}{3} \Rightarrow \rho(x) = 3x^{-4}$$

(c) $w(x) = e^{-5x}$ $(0, \infty)$

$$\int_0^\infty e^{-5x} = -\frac{1}{5} e^{-5x} \Big|_0^\infty = \frac{1}{5} \Rightarrow \rho(x) = 5e^{-5x}$$

Problem 2.6

The pdf tells us the probability in a small region dx around x_0 is $\mathrm{pdf}(x_0)\,dx$. The cdf gives us the same info, by def'n, from $\mathrm{cdf}(x_0 + dx/2) - \mathrm{cdf}(x_0 - dx/2)$. Also basic calculus and the def'n (2.6) indicate that

$$\mathrm{pdf}(x) = \frac{d}{dx}\,\mathrm{cdf}(x)$$

Problem 2.7

It is the error function, which does not have an analytic form - there's no equation for it without an integral.

Problem 2.8

The def'n of $\mathrm{cdf}(x)$ tells us to sum probability below x, which is still true if pdf is discrete. Example:

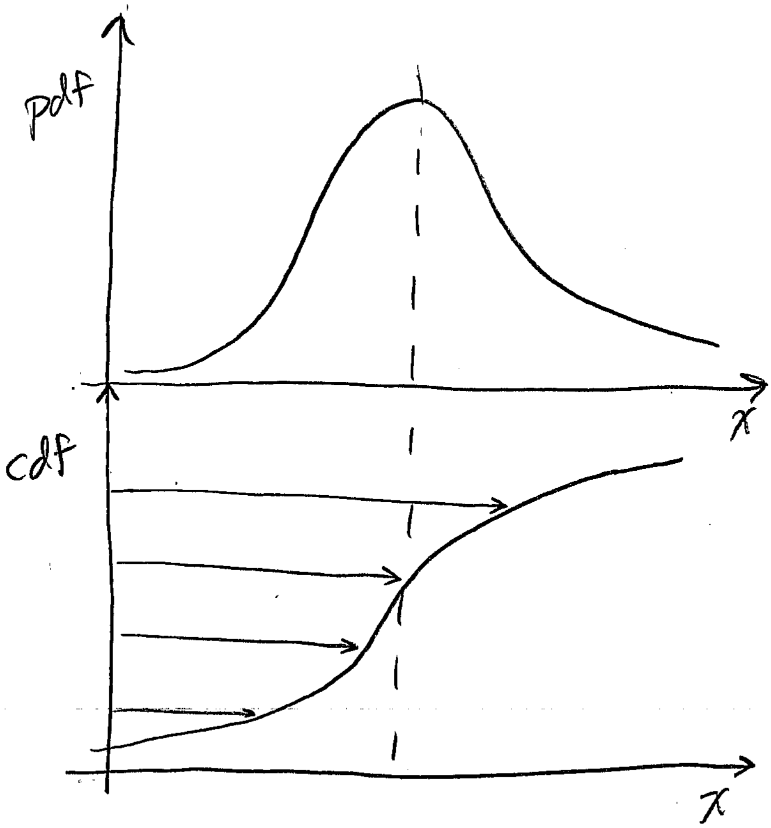
$\int_0^x dx\,\frac{1}{6}$

$$\mathrm{pdf} = \begin{cases} 1/6 & 0 < x < 2 \\ 1/3 & 2 < x < 3 \\ 1/6 & 3 < x < 5 \end{cases} \quad \Rightarrow \quad \mathrm{cdf} = \begin{cases} x/6 & 0 < x < 2 \\ 1/3 + (x-2)/3 & 2 < x < 3 \\ 2/3 + (x-3)/3 & 3 < x < 5 \end{cases}$$

zero elsewhere

CHECK IT YOURSELF!

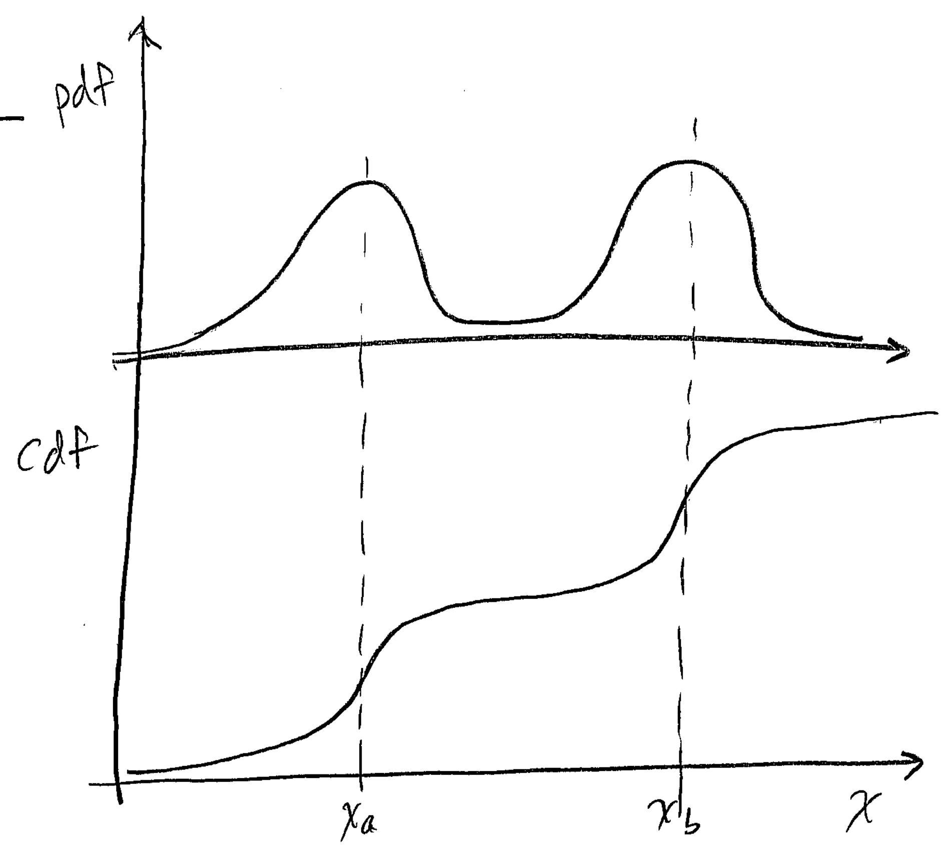
Problem 2.9



Recall
$$\frac{d}{dx}\,\mathrm{cdf}(x) = \mathrm{pdf}(x)$$

Uniformly distributed arrows (at left) tend to hit the steeper part of the cdf more, which is the peak of the pdf.

Problem 2.10



Problem 2.11

$$\rho(x) = \frac{w(x)}{\int dx\, w(x)}$$ a const.

$$\langle f \rangle = \int dx\, f(x)\, \rho(x) = \frac{\int dx\, f(x)\, w(x)}{\int dx\, w(x)}$$

Problem 2.12

By defn, sampling means generating 'configurations' (e.g., x values) in proportion to ρ. Because this only calls for relative/proportional sampling, w is as good as ρ: $\dfrac{\rho(x_a)}{\rho(x_b)} = \dfrac{w(x_a)}{w(x_b)}$

As long as we get the right ratio of x_a and x_b values, we are sampling correctly.

Problem 2.13

(computer programming)

Problem 2.14

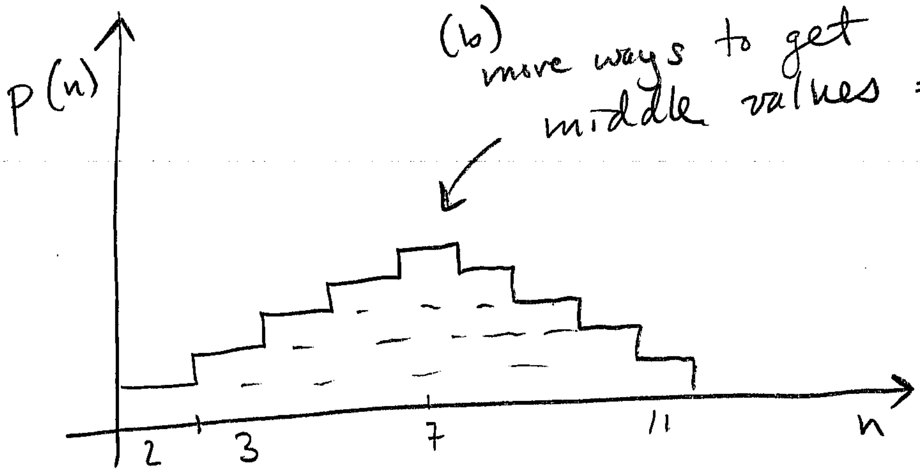
(a)

1 — 1	1 ╳ 1
2 2	2 2
3 3	3 3
4 4	4 4
5 5	5 5
6 6	6 6
1 way to roll 2	2 ways to roll 3

Continuing ⟹

Roll	# ways
2, 12	1
3, 11	2
4, 10	3
5, 9	4
6, 8	5
7	6

(b) more ways to get middle values ⟹ peak



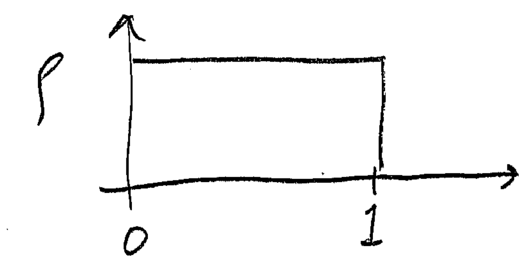
Problem 2.15

$$\rho_u(u) = \int_{-\infty}^{\infty} dx_1\, \rho(x_1)\, \rho(u-x_1)$$

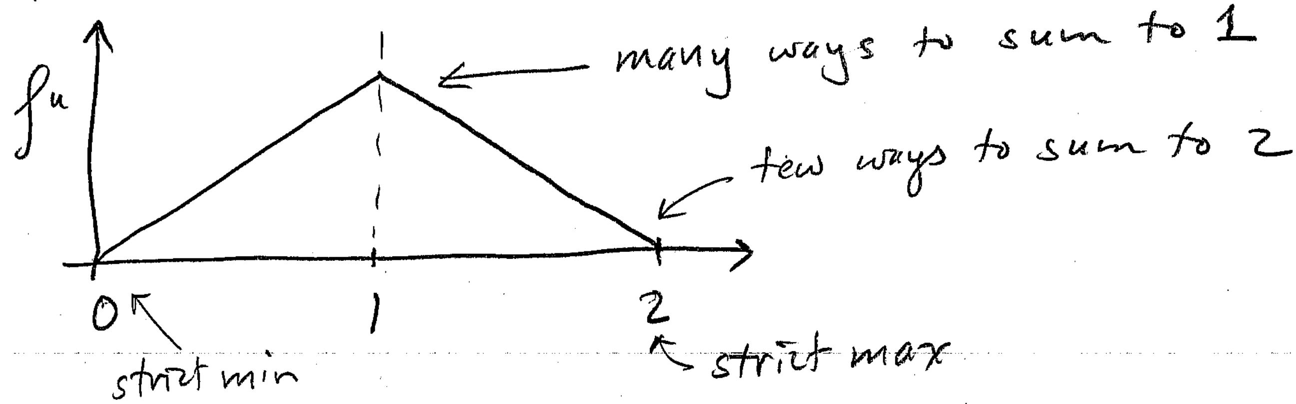
$$\int_{-\infty}^{\infty} du\, \rho_u(u) = \int_{-\infty}^{\infty} dx_1\, \rho(x_1) \underbrace{\int_{-\infty}^{\infty} du\, \rho(u-x_1)}_{\text{integrates to 1 regardless of } x_1 \text{ since all } u \text{ values included}}$$

$$= \int_{-\infty}^{\infty} dx_1\, \rho(x_1) = 1$$

Problem 2.16



Just as in the case of two dice, if we draw uniformly twice in the interval $0 < x < 1$, there are more ways to sum to the middle values near $x = 1$



$$f_u(u) = \int_0^1 dx_1 \; 1 \cdot \underbrace{\rho(u - x_1)}$$

is zero except for a range of x_1 values that depend on u

The math is tricky

Examples

$$f_u\left(\tfrac{1}{2}\right) = \int_0^1 dx_1 \, \rho\left(\tfrac{1}{2} - x_1\right) = \tfrac{1}{2} \quad \text{since } \rho = 0 \text{ for } x_1 > \tfrac{1}{2}$$

$$f_u(1) = \int_0^1 dx_1 \, \rho(1 - x_1) = 1 \quad \rho = 1 \text{ in whole range}$$

$$f_u(3/2) = \int_0^1 dx_1 \, \rho(3/2 - x_1) = \tfrac{1}{2} \quad \rho = 0 \text{ for } x_1 < \tfrac{1}{2}$$

Problem 2.17

Since $x_1 + x_2 = u$ and $x_1, x_2 > 0$
we know $0 < x_1 < u$

$$f_u(u) = \int_0^u dx_1\, e^{-x_1}\, e^{-(u-x_1)} = e^{-u}\int_0^u dx_1 = u e^{-u}$$

This is x_2

Now, for conv. of $u + v = w$, with $\rho(v) = e^{-v}$, $v > 0$

$$f_w(w) = \int_0^w du\, u e^{-u}\, e^{-(w-u)} = e^{-w}\int_0^w du\, u$$

$$= \frac{1}{2} w^2 e^{-w}.$$

Problem 2.18

(computer sim.)

Problem 2.19

Really we used this in Prob 2.17. Using the δ function idea

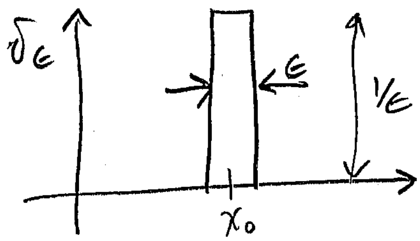
$$\rho(u) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, \rho_1(x_1)\, \rho_2(x_2)\, \delta(u - (x_1 + x_2))$$

$$= \int_{-\infty}^{\infty} dx_1\, \rho_1(x_1)\, \rho_2(u - x_1)$$

Problem 2.20

Rectangular

Let $$\delta_\epsilon(x-x_0) = \begin{cases} 0 & x < x_0 - \epsilon/2 \\ 1/\epsilon & x_0 - \epsilon/2 < x < x_0 + \epsilon/2 \\ 0 & x > x_0 + \epsilon/2 \end{cases}$$



We want, in the limit $\epsilon \to 0$ to evaluate

$$I_\epsilon = \int_{-\infty}^{\infty} f(x)\, \delta_\epsilon(x-x_0)\, dx = \frac{1}{\epsilon} \int_{x_0-\epsilon/2}^{x_0+\epsilon/2} dx\, f(x)$$

If the function is well-behaved in the interval, then the limit can be evaluated as

$$\lim_{\epsilon \to 0} I_\epsilon \simeq \frac{1}{\epsilon}\, f(x_0)\, \Big|_{x_0-\epsilon/2}^{x_0+\epsilon/2} = f(x_0)$$

function doesn't change in tiny interval

Problem 2.21

We want $\rho(x)\,dx = \rho(u)\,du$

If $u = x^2$, then $du = 2x\,dx$

Thus if $\rho(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$,

then $\rho(u) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-u/2\sigma^2} \underbrace{\frac{1}{2\sqrt{u}}}_{\text{cancels in } du}$

If we only use variable u, we don't know if $x > 0$ or $x < 0$ since $x^2 = (-x)^2 = u$

Problem 2.22

We note that $\langle x \rangle$ is just a constant for a given distribution. Hence $\langle \langle x \rangle f(x) \rangle = \langle x \rangle \langle f \rangle$

$$\text{var}(x) = \left\langle (x - \langle x \rangle)^2 \right\rangle = \langle x^2 - 2x \underset{\uparrow \atop \text{const}}{\langle x \rangle} + \underset{\uparrow \atop \text{const}}{\langle x \rangle^2} \rangle$$

$$= \langle x^2 \rangle - 2\langle x \rangle \langle x \rangle + \langle x \rangle^2$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

Problem 2.23

$$I_1 = \int_1^\infty dx \; x\, x^{-5/2} = \int_1^\infty dx\, x^{-3/2} = -2x^{-\frac{1}{2}}\Big|_1^\infty$$

$$= 0 - 2 = 2$$

$$I_0 = \int_1^\infty dx\, x^{-5/2} = -\frac{2}{3} x^{-3/2}\Big|_1^\infty = \frac{2}{3}$$

$$\langle x \rangle = I_1 / I_0 = 3 \quad \text{FINITE}$$

$$I_2 = \int_1^\infty dx\, x^2 x^{-5/2} = \int_1^\infty dx\, x^{-\frac{1}{2}} = 2x^{1/2}\Big|_1^\infty$$

DIVERGES

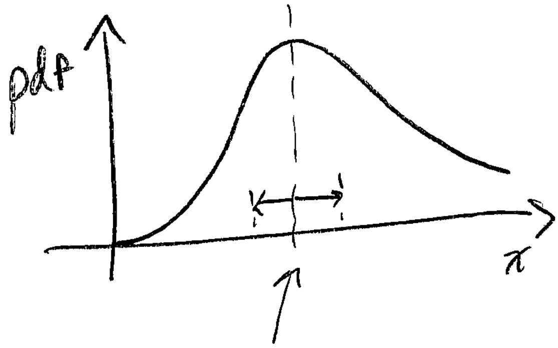
$$\mathrm{var}(x) = \frac{I_2}{I_0} - \left(\frac{I_1}{I_0}\right)^2 \quad \text{DIVERGES}$$

Problem 2.24

Draw once from each dist. and sum. This is the first sum. Repeat draws to get second sum. Repeat many times to get sums, which are treated as individual variables in the variance:

$$\mathrm{var(sum)} = \langle (\mathrm{sum})^2 \rangle - \langle \mathrm{sum} \rangle^2$$

Problem 2.25



more area under curve to right of peak than left

$\Downarrow$

mean is to right of peak

If mean were at peak, then pdf of values to left and right would have to balance (on avg.)

Problem 2.26

Symmetry about x_0 is defined by same pdf equal distances to left and right:

$$\text{pdf}(x_0 - x) = \text{pdf}(x_0 + x) \quad \text{for all } x$$

$$\mu_3 = \left\langle \left(x - \langle x \rangle\right)^3 \right\rangle$$

First note: $\langle x \rangle = x_0$ due to symmetry.

Let $y = x - x_0$
$dy = dx$

$$\mu_3 = \int_{-\infty}^{\infty} dx\, \text{pdf}(x)\,(x - x_0)^3$$

$$= \int_{-\infty}^{\infty} dy\, \text{pdf}(y + x_0)\, y^3 = \underbrace{\int_{-\infty}^{0} dy\, \underbrace{\text{pdf}(y+x_0)}_{\text{pdf}(x_0 - y)} y^3}_{-\int_0^{\infty} dy\, \text{pdf}(y+x_0)\, y^3} + \int_0^{\infty} dy\, \text{pdf}(y + x_0)\, y^3$$

Sign reversals for dy, y^3, and $0 \leftrightarrow \infty$ switch $\rightarrow$ $-\int_0^{\infty} dy\, \text{pdf}(y+x_0)\, y^3$ $\Rightarrow \mu_3 = 0$

Problem 2.27

$$\mathrm{var}\left(\frac{S}{N}\right) = \left\langle \left(\frac{S}{N}\right)^2 - \left\langle\frac{S}{N}\right\rangle^2 \right\rangle = \left\langle \left(\frac{S}{N} - \left\langle\frac{S}{N}\right\rangle\right)^2 \right\rangle$$

$$= \left\langle \frac{1}{N^2}\left(S - \langle S\rangle\right)^2 \right\rangle = \frac{1}{N^2} \underbrace{\mathrm{var}(S)}_{\substack{N \cdot \mathrm{var}(X) \\ \text{from Eq (2.19)}}}$$

$$= \frac{\mathrm{var}(x)}{N}$$

Problem 2.28

Say we take a sample of 10,000 whales. The scale of the uncertainty is $1\,\mathrm{m}/\sqrt{N} = 1\,\mathrm{cm}$. This is the expected scale of variation for repeated sets of measurements of 10^4 whales. We can get our scale of precision much less than the natural scale of variation for <u>one</u> whale (i.e., $\sigma = 1\,\mathrm{m}$) by averaging over many whales

Problem 2.29

Error $\sim \frac{1}{\sqrt{N/100}} = \frac{10}{\sqrt{N}}$ still proportional to $N^{-1/2}$, just a larger prefactor

Problem 2.30

Generalizing the 1D Gaussian, we have

$$\rho_G(x,y) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^2 e^{-(x^2+y^2)/2\sigma^2}$$

$$= \left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}\right]\left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2\sigma^2}\right]$$

Since the distribution factorizes and each factor is normalized, the whole distribution will be normalized

$$\int dx\, dy\, \rho_G(x,y) = \int dx\, \rho_G(x) \int dy\, \rho_G(y)$$

$$= 1 \cdot 1 = 1$$

Problem 2.31

$$\int dx\, \rho_x(x) = \int dx \int dy\, \rho(x,y) = 1$$

Problem 2.32

See sol'n to 2.30

Problem 2.33

Stat indep: $\rho(x,y) = \rho_x(x)\,\rho_y(y)$

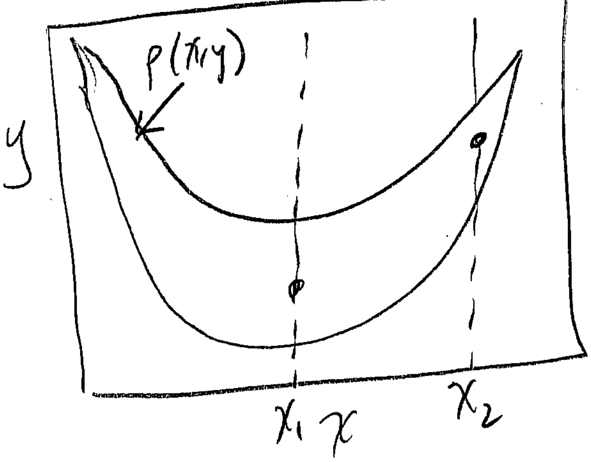
(a) $\rho_x(x) = \int dy\, \rho(x,y) \qquad \rho_y(y) = \int dx\, \rho(x,y)$

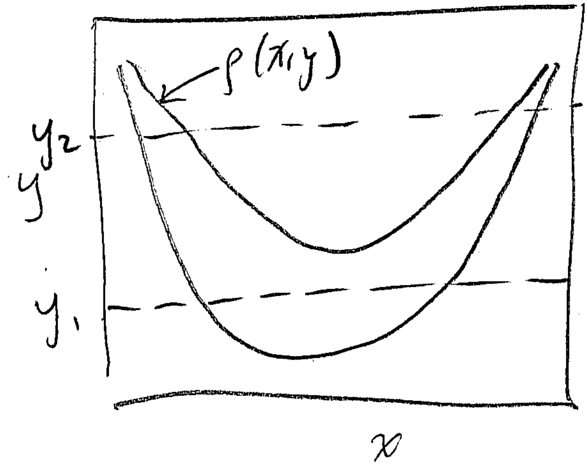
(b) If the variables are indep., the projection will give the single-variable dist., but the key point is that just because one derives a projected ρ_x does <u>not</u> imply anything about independence.

Problem 2.34

We are trying to study intrinsic correlations – i.e., a property that does not depend on the number of samples. The standard dev. is an intrinsic scale, independent of samples.

Problem 2.35





$\rho(y|x_1) \neq \rho(y|x_2)$

just by looking at means

$\rho(x|y_1) \neq \rho(x|y_2)$

unimodal bimodal

Problem 2.36

$$I_{x,y} = \int dx\,dy\, \rho(x,y) \ln \frac{\rho(x,y)}{\rho_x(x)\rho_y(y)} = \text{weighted avg of log term}$$

(a) Independ $\Rightarrow \rho(x,y) = \rho_x(x)\rho_y(y) \Rightarrow \ln 1 = 0 = I_{x,y}$

If not independent $I_{x,y} \neq 0$ (due to properties of log)

So $I = 0$ for indep and $I \neq 0$ when not indep. ✓

(b) Other functions could yield this same qualitative information, so long as they were constructed so that $f(u=1) = 0$, where $u = \rho(x,y)/\rho_x(x)\rho_y(y)$.
For example, a power of $(u-1)$ could work, or the log of a power of u.

Problem 2.37

So long as your hand and foot are both attached to your body, they will always be less than 3m from one another – their locations are not independent. Correlations between hand + elbow will be stronger, however.

Prob 2.38

A professor's style . . .

shirt color / shoe color	white	blue	pink
black	0.1	0.15	0.05
brown	0.2	0.25	0.25

$$\text{prob}(\text{pink shirt} \mid \text{black shoes}) = \frac{0.05}{0.1 + 0.15 + 0.05}$$

$$= \frac{0.05}{0.3} = \frac{1}{6}$$

Prob 2.39

num. is function of y

$$\int dy\, \rho(y|C) = \int dy \frac{\int_C dx\, \rho(x,y)}{\int\int_C dx\,dy\, \rho(x,y)}$$

denom. is a constant

$$= \frac{\int dy \int_C dx\, \rho(x,y)}{\int\int_C dx\,dy\, \rho(x,y)} = 1$$

since C is a condition on x.

Both integrals mean: "integrate over all x + y values given condition on x."

Problem 2.40

The distance between atoms 1 + 3 in protein 1 should be time correlated, but the distance between atom 1 in protein 1 and atom 3 in protein 2 should not

~~Problem 2.41~~

Problem 2.41

The location of a soccer ball during a game should be strongly (auto) correlated on the second timescale, but not on the minute time scale.

Problems 2.42 - 48

Depend on trajectory