Introduction to Radiation Physics

1. $E = N_{\text{telescopes}} P_{\text{detected}} \Delta t$; and by Equation 2.2, $P_{\text{detected}} = F_\nu A_{\text{eff}} \Delta\nu$. Therefore,

   $E \approx 100\,(0.1 \text{ Jy} \times 10^{-26} \text{ W m}^{-2} \text{ Hz}^{-1} \text{ Jy}^{-1})\, 500 \text{ m}^2\, (100 \times 10^6 \text{ Hz})\, (0.8 \times 45 \text{ yr})\, (3.16 \times 10^7 \text{ s yr}^{-1})$

   $= 5.69 \times 10^{-6}$ J

   This equals the kinetic energy gained by a mass, m, falling from a height of 1 m on the surface of the Earth, where $mgh = m$ (9.8 m s^{-2})(1 m) $= 5.69 \times 10^{-6}$ J, or $m = 5.80 \times 10^{-7}$ kg, or 0.58 mg!

2. a. Using Equation 2.2,

$$F_\nu = \frac{P}{A\Delta\nu} = \frac{4.00 \times 10^{-17} \text{ J}/100 \text{ s}}{\pi(1 \text{ m})^2(500 \times 10^3 \text{Hz})}$$

$$= 2.55 \times 10^{-25} \text{ W m}^{-2} \text{ Hz}^{-1} = 25.5 \text{ Jy}$$

   b. Using Equation 2.1,

$$L_\nu = F_\nu 4\pi d^2 = (2.55 \times 10^{-25} \text{ W m}^{-2} \text{ Hz}^{-1}) 4\pi (9.45 \times 10^{16} \text{ m})^2$$

$$= 2.87 \times 10^{10} \text{ W Hz}^{-1}$$

3. By Equation 2.4,

$$I_\nu = \frac{F_\nu}{\Omega} = \frac{5.00 \text{ Jy} \times 10^{-26} \text{ W m}^{-2} \text{ Hz}^{-1}}{(\pi/4)(0.100^\circ \times \ \pi \text{ radians}/180^\circ)^2} = 2.09 \times 10^{-20} \text{ W m}^{-2} \text{ Hz}^{-1}$$

4. a. Using Equation 2.1,

$$F = \frac{L}{4\pi d^2} = \frac{3.90 \times 10^{26} \text{ W}}{4\pi(1.50 \times 10^{11} \text{ m})^2} = 1380 \text{ W m}^{-2}$$

   b.

$$F_\lambda = \frac{F \text{ in visible}}{\Delta\lambda_{\text{visible}}} \approx \frac{0.37(1380 \text{ W m}^{-2})}{300 \times 10^{-9} \text{ m}} = 1.70 \times 10^9 \text{ W m}^{-2} \text{ m}^{-1}$$

   c.

$$F_\nu = \frac{F \text{ in visible}}{\Delta\nu_{\text{visible}}}$$

and

$$\Delta\nu = \frac{c}{\lambda^2}\Delta\lambda \approx \frac{3.00 \times 10^8 \text{ m s}^{-1}}{(550 \times 10^{-9} \text{ m})^2} 300 \times 10^{-9} \text{ m} \approx 2.98 \times 10^{14} \text{ Hz}$$

(see discussion in Section 2.2 leading up to Equation 2.11). Therefore,

$$F_\nu \approx \frac{0.37(1380 \text{ W m}^{-2})}{2.98 \times 10^{14} \text{ Hz}} = 1.71 \times 10^{-12} \text{ W m}^{-2} \text{ Hz}^{-1}$$

d. Using Equation 2.4,

$$I_\lambda = \frac{F_\lambda}{\Omega} \approx \frac{1.70 \times 10^9 \text{ W m}^{-2} \text{ m}^{-1}}{(\pi/4)(0.533^\circ \times \ \pi \text{ radians}/180^\circ)^2}$$

$$= 2.50 \times 10^{13} \text{ W m}^{-2} \text{ m}^{-1} \text{ sr}^{-1}$$

e.

$$I_\nu = \frac{F_\nu}{\Omega} \approx \frac{1.71 \times 10^{-12} \text{ W m}^{-2} \text{ Hz}^{-1}}{(\pi/4)(0.533^\circ \times \ \pi \text{ radians}/180^\circ)^2}$$

$$= 2.52 \times 10^{-8} \text{ W m}^{-2} \text{ Hz}^{-1} \text{ sr}^{-1}$$

5. Equation 2.5:

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1}$$

For the frequency, we use the middle of the visible window, which is about 550 nm.

$$\nu = c/\lambda = 3.00 \times 10^8 \text{ m s}^{-1}/(550 \times 10^{-9} \text{ m}) = 5.45 \times 10^{14} \text{ Hz}$$

Substituting the frequency and temperature into Equation 2.5, we have

$$B_\nu(5800 \text{ K}) = \frac{2(6.626 \times 10^{-34} \text{ J s})(5.45 \times 10^{14} \text{ Hz})^3}{(3.00 \times 10^8 \text{ m s}^{-1})^2}$$

$$\times \frac{1}{\exp[(6.626 \times 10^{-34} \text{ J s})(5.45 \times 10^{14} \text{ Hz})/(1.38 \times 10^{-23} \text{ J K}^{-1})(5800 \text{ K})] - 1}$$

$$= 2.65 \times 10^{-8} \text{ W m}^{-2} \text{ Hz}^{-1} \text{ sr}^{-1}$$

This is 1.06 times larger than the answer to 2.4(e). In 2.4(e) we calculated the total visible flux divided by the bandwidth, which gives, essentially, an average intensity over the visible. Here we calculated the intensity at a specific point near the middle of the visible window. We showed in Section 2.2, that the Planck function at the temperature of the Sun's surface peaks in the infrared, and so $B_\nu(5800 \text{ K})$ decreases across the visible window, although relatively slowly since the peak is not that far from the visible. Therefore, it is not surprising that our calculation of $B_\nu(5800 \text{ K})$ in the middle of the visible window is very close to the average value.

6. a. Equation 2.7: $F = \sigma\, T^4$;
and Equation 2.8: $< E_{\text{ph}} >= (3.73 \times 10^{-23} \text{ J K}^{-1})\ T$.
The number of photons, N, emitted per second, then, is

$$N_{\text{per sec}} = \frac{F}{E_{\text{ph}}} = \frac{\sigma\, T^4}{(3.73 \times 10^{-23} \text{ J K}^{-1})\ T} = \frac{5.67 \times 10^{-8} \text{ J m}^{-2} \text{ s}^{-1} \text{ K}^{-4}}{3.73 \times 10^{-23} \text{ J K}^{-1}}\, T^3$$

$$N_{\text{per sec}} = [1.52 \times 10^{15} \text{ photons m}^{-2} \text{ s}^{-1} \text{ K}^{-3}]\, T^3$$

b. $N \propto T^3$ and $< E_{\text{ph}} > \propto T$.
The number of photons emitted per second increases much more rapidly with an increase in temperature.

7. a. At frequencies low enough that the Rayleigh-Jeans approximation applies, then $I_\nu \propto T$ and so $I_\nu(\text{A}) = 2I_\nu(\text{B})$

   b. By Equation 2.7, $F \propto T^4$, and so $F(\text{A}) \propto 16\ F(\text{B})$

   c. By Equation 2.8, $< E_{\text{ph}} > \ \propto T$ and so $< E_{\text{ph}} > (\text{A}) = 2 < E_{\text{ph}} > (\text{B})$

8. A jansky is a unit of F_ν so we first convert the given F_λ to F_ν. We use Equation 2.11: $F_\lambda = \frac{c}{\lambda^2}\ F_\nu$, or $F_\nu = \frac{\lambda^2}{c}\ F_\lambda$.

$$F_\nu = \frac{(12.0 \times 10^{-6} \text{ m})^2}{3.00 \times 10^8 \text{ m s}^{-1}} 6.00 \times 10^{-7} \text{ W m}^{-3}$$

$$= 2.88 \times 10^{-25} \text{ W m}^{-2} \text{ Hz}^{-1} = 28.8 \text{ Jy}$$

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9. a. Taylor Series Expansion of $f(x)$ for x near 0:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + ...$$

For $f(x) = \exp(x)$,

$$\exp(x) = e^0 + e^0 x + ...$$

When $x \ll 1$ we can keep only the first two terms, and so

$$\exp(x) \approx 1 + x$$

Applying this to B_ν where $x = \frac{h\nu}{kT}$ we have

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2} \frac{1}{1 + (h\nu/kT) - 1} \approx \frac{2h\nu^3}{c^2} \frac{kT}{h\nu} \approx \frac{2k\nu^2 T}{c^2}$$

Substituting in $\frac{1}{\lambda^2}$ for $\frac{\nu^2}{c^2}$ this becomes

$$B_\nu(T) \approx \frac{2kT}{\lambda^2} \quad (\text{for } \frac{h\nu}{kT} \ll 1)$$

b. With $\lambda = 1$ mm, the frequency is

$$\nu = 3.00 \times 10^8 \text{ m s}^{-1} / 10^{-3} \text{ m} = 3.00 \times 10^{11} \text{ Hz.}$$

Starting with $h\nu/kT \approx 0.1$, and solving for T we find

$$T = \frac{(6.626 \times 10^{-34} \text{ J s})(3.00 \times 10^{11} \text{ Hz})}{0.1(1.38 \times 10^{-23} \text{ J K}^{-1})} = 144 \text{ K}$$

Using the Planck function (Equation 2.5), first, we have

$$B_\nu(144\ \mathrm{K}) = \frac{2(6.626\times10^{-34}\ \mathrm{J\ s})(3.00\times10^{11}\ \mathrm{Hz})^3}{(3.00\times10^{8}\ \mathrm{m\ s^{-1}})^2}$$

$$\times\frac{1}{\exp[(6.626\times10^{-34}\ \mathrm{J\ s})(3.00\times10^{11}\ \mathrm{Hz})/1.38\times10^{-23}\ \mathrm{J\ K^{-1}}144\ \mathrm{K}]-1}$$

$$= 3.78\times10^{-15}\ \mathrm{W\ m^{-2}\ Hz^{-1}\ sr^{-1}}$$

Using the Rayleigh-Jeans approximation (Equation 2.13), we have

$$B_\nu(144\ \mathrm{K}) \approx \frac{2(1.38\times10^{-23}\ \mathrm{J\ K^{-1}})144\ \mathrm{K}}{(10^{-3}\ \mathrm{m})^2} = 3.97\times10^{-15}\ \mathrm{W\ m^{-2}\ Hz^{-1}sr^{-1}}$$

The Rayleigh-Jeans approximation is off by 5.0%, which is better than 10%. By trial and error, you can find that at 77K, the Rayleigh-Jeans approximation is off by 10%.

c. Following the same procedure as in question b, but this time fixing the temperature at 2.73 K and solving for the frequency we find:
$h\nu/kT = 0.1$ when $\nu = 5.77\times10^9$ Hz. When we substitute these into the Planck function and Rayleigh-Jeans approximation, we get:
Planck function: $B_\nu(2.73\ \mathrm{K}) = 2.65\times10^{-20}\ \mathrm{W\ m^{-2}\ Hz^{-1}\ sr^{-1}}$ and
Rayleigh-Jeans: $B_\nu(2.73\ \mathrm{K}) = 2.79\times10^{-20}\ \mathrm{W\ m^{-2}\ Hz^{-1}\ sr^{-1}}$, which is off by only 5.2%. Here the Rayleigh-Jeans approximation is off by 10% when $\nu = 1.07\times10^{10}$ Hz, or $\lambda = 2.8$ cm.

10. The brightness temperature is given by setting the intensity of the radiation equal to the Planck function and solving for the temperature. At longer wavelengths and higher temperatures, so that the Rayleigh-Jeans approximation applies, $T_B = \frac{\lambda^2}{2k}I_\nu$ and so is directly proportional to the intensity. Its physical significance is that it directly gives an estimate of the temperature of an opaque, thermally radiating source, since the brightness temperature of the emitted radiation will equal the temperature of the radiating body. If the body is transparent, emitting thermally, and has no background radiation source, then the brightness temperature of the radiation is a lower limit to the body's temperature. It is primarily used at radio frequencies because the Rayleigh-Jeans approximation makes the conversion from intensity to brightness very simple (by multiplying by $\frac{\lambda^2}{2k}$) while at higher frequencies, the conversion requires solving for T in the Planck function.

11. a. Since the CMB is a perfect blackbody, we use the Planck function (Equation 2.5). The frequency of the observation was

$$\nu = \frac{3.00\times10^{8}\ \mathrm{m\ s^{-1}}}{0.0735\ \mathrm{m}} = 4.08\times10^{9}\ \mathrm{Hz}$$

and so the Planck function gives an intensity of

$$B_\nu(2.73\ \mathrm{K}) = \frac{2(6.626\times10^{-34}\ \mathrm{J\ s})(4.08\times10^{9}\ \mathrm{Hz})^3}{(3.00\times10^{8}\ \mathrm{m\ s^{-1}})^2}\times$$

$$\frac{1}{\exp[(6.626\times10^{-34}\ \mathrm{J\ s})(4.08\times10^{9}\ \mathrm{Hz})/(1.38\times10^{-23}\ \mathrm{J\ K^{-1}})(2.73\ \mathrm{K})]-1}$$

$$= 1.35 \times 10^{-20} \text{ W m}^{-2} \text{ Hz}^{-1} \text{ sr}^{-1}$$

b. Using Equation 2.9,

$$\nu_{\text{peak}} = (5.88 \times 10^{10} \text{ Hz K}^{-1})\ 2.73 \text{ K} = 1.61 \times 10^{11}\text{Hz}$$

which corresponds to a wavelength of

$$\lambda_{\text{peak}} = \frac{3.00 \times 10^{8} \text{ m s}^{-1}}{1.61 \times 10^{11}\text{Hz}} = 0.00186 \text{ m} = 1.86 \text{ mm}.$$

c. $T + \Delta T = 2.73$ K $+2.73 \times 10^{-5}$ K $= 2.7300273$K. The intensities at these two temperatures are (using Equation 2.5)

$$B_\nu(2.7300273 \text{ K}) = 1.3453028 \times 10^{-20} \text{ W m}^{-2} \text{ Hz}^{-1} \text{ sr}^{-1}$$

and

$$B_\nu(2.7300000 \text{ K}) = 1.3452888 \times 10^{-20} \text{ W m}^{-2} \text{ Hz}^{-1} \text{ sr}^{-1}$$

To measure this temperature difference, then, one needs to be able to accurately measure an intensity smaller than 1.48×10^{-25} W m^{-2} Hz^{-1} sr^{-1}. With this radiation filling a typical radio telescope beam of order 10^{-5} sr in diameter, this corresponds to measuring fractions of a mJy in the change of the detected flux density.

12. a. By Equation 2.23, the net linear polarization is $L = \sqrt{Q^2 + U^2}$, where Q and U are given by Equations 2.20 and 2.21. Since the radiation's net polarization is horizontal, we set $U = 0$. The net horizontal polarization of 5% then means that

$$0.05 = L/I = \sqrt{Q^2 + U^2}/I = (I_x - I_y)/(I_x + I_y),$$

and hence $1.05 I_y = 0.95 I_x$ and so $I_y/I_x = 0.9048$. The horizontal polarization intensity relative to the total intensity is

$$I_x/I_{\text{total}} = I_x/(I_x + I_y) = I_x/(1.9048 I_x) = 0.525$$

and that of the vertical polarization is

$$I_y/I_{\text{total}} = (I_{\text{total}} - I_x)/I_{\text{total}} = 1 - 0.525 = 0.475$$

b. The unpolarized intensity it $I - L$, and so the unpolarized intensity relative to the total intensity is 95%.

c. We need that 95% of the detected radiation travels straight to our telescope while 5% of the detected power has been scattered off the screen. We can imagine, for example, a situation where the screen is the same distance as the emission source, but off to one side, and catches about a quarter of the emitted radiation, but reflects about 15% of the incident radiation in our direction. This will yield a net linear polarization of 5% because 15% of the 25% equals 5% of the 75% that does not meet the screen.

13. a. Equation 2.23: $L = \sqrt{Q^2 + U^2}$ where $Q = I_x - I_y$ (Equation 2.20) and $U = I_a - I_b$ (Equation 2.21), where the a-axis is tilted 45° relative to the x-axis. We consider radiation that is linearly polarized at angle θ relative to the x-axis. The electric field

amplitude along the x-axis, then is $E\cos\theta$, and that along the y-axis is $E\sin\theta$. The electric field amplitude along the a-axis is

$$E\cos(\theta-\pi/4) = E[\cos\theta\cos(\pi/4)+\sin\theta\sin(\pi/4)] = (E/\sqrt{2})(\cos\theta+\sin\theta)$$

and that along the b-axis is

$$E\sin(\theta-\pi/4) = E[\sin\theta\cos(\pi/4)-\cos\theta\sin(\pi/4)] = (E/\sqrt{2})(\sin\theta-\cos\theta).$$

The total intensity of this radiation is, simply, E^2.

The intensity in Q and U, then, are

$$Q = (E\cos\theta)^2-(E\sin\theta)^2 = E^2(\cos^2\theta-\sin^2\theta)$$

and

$$U = [(E/\sqrt{2})(\cos\theta+\sin\theta)]^2-[(E/\sqrt{2})(\sin\theta-\cos\theta)]^2 = 2E^2\sin\theta\cos\theta$$

Relative to I, these are $Q/I = \cos^2\theta-\sin^2\theta$ and $U/I = 2\sin\theta\cos\theta$.
For L/I, then, we have

$$\begin{aligned} L/I = \sqrt{Q^2+U^2} &= \sqrt{\cos^4\theta-2\sin^2\theta\cos^2\theta+\sin^4\theta+4\sin^2\theta\cos^2\theta} \\ &= \sqrt{\cos^4\theta+2\sin^2\theta\cos^2\theta+\sin^4\theta} = \cos^2\theta+\sin^2\theta = 1 \end{aligned}$$

and hence, we get that the radiation is 100% polarized, just as we started with.

b. If the radiation is linearly polarized at an angle of 45° relative to the x-axis, then $I_x = E^2\cos^2(\pi/4) = E^2/2$ and $I_y = E^2\sin^2(\pi/4) = E^2/2$ and so $Q = I_x - I_y = 0$, and hence using Q without U would, incorrectly, suggest that the radiation is unpolarized, when in fact it is 100% polarized.

14. a. The total intensity is I, which is given by:

$$I = 5.00\times10^{-18}\ \text{W Hz}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}.$$

b. Since both Q and U are equal to zero, there is no linear polarization ($L = \sqrt{Q^2+U^2}$, Equation 2.23), so the answer to this question is 0.

c. By Equation 2.22,

$$V = I_R - I_L = 2.50\times10^{-20}\ \text{W Hz}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1},$$

and using the total intensity, we know that

$$I_R + I_L = 5.00\times10^{-18}\ \text{W Hz}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}.$$

By summing these two equations, and dividing by two, we get

$$I_R = 2.513\times10^{-20}\ \text{W Hz}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}.$$

Using either equation we can then solve for I_L. We get

$$I_L = 2.487\times10^{-20}\ \text{W Hz}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}.$$