Chapter 2

Mathematical Preliminaries

This chapter is a review of the mathematical details that underpin many of the topics of the previous chapter. These include elements of probability theory, stochastic calculus and other issues that play an important role in the development of financial mathematics in general, and the Black–Scholes and EMM methods in particular. For a more detailed account on stochastic calculus and sde's, the reader is referred to the texts of Lamberton and Lapeyre [50], Karatzas and Shreve [40] and Øksendal [59]. A recent very readable text that is not too technical, and hence in the same vein as this book, is that by Wiersema [75].

Another useful reference which covers almost everything in this chapter, and the previous one, is the book by Jeanblanc *et al* [38]. This text is much more technical and as such is perhaps more suitable for those with sufficiently advanced mathematical expertise. However, any serious student of Quantitative Finance ought to be familiar with this important contribution to the field.

We shall not present a fully integrated theory, which would take us far from our goal of pricing exotic options, but prefer to step lightly through various topics, identifying important concepts and earmarking important equations. Thus the reader should treat this chapter as a list of mathematical tools, rather than a coherent set of propositions and theorems. We offer only a few formal proofs in this chapter. Others, in most cases, can be found in the cited texts. It will take some skill to utilize the tools presented in this chapter, to price exotic options. Hopefully, therefore, readers of this book will not fall into the trap epitomized by the adage: *a bad craftsman blames his tools.*

2.1 Probability Spaces

We assume a market in which future prices up to a finite time horizon T, are random variables associated with a probability space $(\Omega, \mathbb{F}, P)$. Ω is the set of all possible price outcomes; $\mathbb{F}$ is a $\sigma-$algebra (a family of subsets of Ω) containing all sets pertaining to future prices, and P is a probability measure (called the *real-world measure*) which determines the probability of any event in $\mathbb{F}$. We also equip this probability space with a filtration $\mathcal{F}_t$. This is a non-decreasing family of sub-σ-algebras of $\mathbb{F}$, such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathbb{F} \quad \text{for all } 0 < s < t < T.$$

It is convenient to think of $\mathcal{F}_t$ as the price information available for all times up to and including t. We assume that the resulting filtered probability space $(\Omega, \mathbb{F}, P, \mathcal{F}_t)$ satisfies the so-called *usual conditions*. Technically, this means: $\mathbb{F}$ is $P-$complete, $\mathcal{F}_0$ contains all $P-$null sets of Ω and $\mathcal{F}_t$ is right-continuous.

A *stochastic price process* X_t is then a family of random variables defined on $(\Omega, \mathbb{F}, P, \mathcal{F}_t)$. X_t is said to be *adapted* to $\mathcal{F}_t$ if X_t is $\mathcal{F}_t-$ measurable. Basically, this simply means that X_t is known with certainty at time t.

An important operator on stochastic processes is the *conditional expectation* denoted by $\mathbb{E}\{X_T|\mathcal{F}_t\}$. By this we mean the expected value of X_T given all information up to and including t, for $t \leq T$. The expectation operator is linear, so that if (α_t, β_t) are processes adapted to $\mathcal{F}_t$, then for any processes (X_T, Y_T) with $t < T$,

$$\mathbb{E}\{\alpha_t X_T + \beta_t Y_T|\mathcal{F}_t\} = \alpha_t \mathbb{E}\{X_T|\mathcal{F}_t\} + \beta_t \mathbb{E}\{Y_T|\mathcal{F}_t\} \tag{2.1}$$

The Tower Law

Let X_t be a stochastic process; then for all $s \leq t \leq T$,

$$\mathbb{E}\{X_T|\mathcal{F}_s\} = \mathbb{E}\{\mathbb{E}\{X_T|\mathcal{F}_t\}|\mathcal{F}_s\}. \tag{2.2}$$

This is also known as the *Law of Iterated Expectations* and demonstrates how information can be nested through a non-decreasing filtration sequence. The tower law has important ramifications for option pricing, as we saw in Section 1.9 and as we shall also demonstrate in the applications.

2.2 Brownian Motion

A standard $\mathcal{P}$ Brownian motion (or standard Wiener process) B_t for $t > 0$ is the stochastic process satisfying

1.	B_t is continuous and $B_0 = 0$
2.	B_t has stationary and independent increments
3.	For fixed $t > 0$, $B_t \stackrel{d}{=} N(0, t)$ under probability measure $\mathcal{P}$

where $\stackrel{d}{=}$ means "equal in distribution." This characterization of Brownian motion is due to Kolmogorov, and is known to define it uniquely.

REMARK 2.1 We list here a number of properties of Brownian motion. We do not necessarily use all these properties, but it is nevertheless helpful to be familiar them.

1\. Although the sample paths of B_t are continuous, they are (with probability one) nowhere differentiable. The sample paths are therefore fractal and have similarity dimension $H = \frac{1}{2}$. That is, for all $c > 0$,

$$B_{ct} \stackrel{d}{=} c^H B_t = \sqrt{c}\, B_t.$$

2\. The sample paths of B_t are of unbounded variation, but have bounded quadratic variation. This means that if $\Delta_n = \{t_i^{(n)} : i = 1, 2, \ldots, n\}$ be a sequence of partitions of the interval $[0, t]$ such that

$$\lim_{n\to\infty} \max_i |t_{i+1}^{(n)} - t_i^{(n)}| = 0$$

with $t_0^{(n)} = 0$ and $t_{n+1}^{(n)} = t$, then it can be shown that

$$\lim_{n\to\infty} \sum_{i=1}^{n} |B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}}| \to \infty$$

while

$$\lim_{n\to\infty} \sum_{i=1}^{n} |B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}}|^2 = t.$$

The quadratic variation of B_t is therefore equal to t.

3\. It can also be shown that:

$$\begin{aligned} \lim_{t\to\infty} \frac{B_t}{t} &= 0 \quad \text{a.s.} \\ \limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} &= 1 \quad \text{a.s.} \\ \liminf_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} &= -1 \quad \text{a.s.} \end{aligned}$$

The last two equations are collectively called the *Law of the Iterated Logarithm.* The symbol a.s. stands for "almost surely," which means the equations are probabilistic statements, with probability one.

4. The property of independent increments means that for any non-overlapping intervals (s, t) and (u, v), the increment $(B_t - B_s)$ is independent of the increment $(B_v - B_u)$. In particular, since $B_0 = 0$, the increment $(B_t - B_s)$ is independent of B_s for all $s < t$.

The stationarity of these increments means that for any $h + s \geq 0$, we have $B_t - B_s \stackrel{d}{=} B_{t+h} - B_{s+h}$. In particular, taking $h + s = 0$ leads to the important conclusion

$$\boxed{B_t - B_s \stackrel{d}{=} B_{t-s}} \tag{2.3}$$

5. For any fixed $t > 0$, B_t is Gaussian with zero mean and variance equal to t. Hence it is possible to write

$$B_t \stackrel{d}{=} \sqrt{t}\,Z \qquad \text{where} \;\; Z \sim N(0,1). \tag{2.4}$$

6. B_t is a Markov process. That is, for all $s \leq t$,

$$\mathbb{P}\{B_t | \mathcal{F}_s\} = \mathbb{P}\{B_t | B_s\}$$

where $\mathcal{F}_s$ denotes the filtration of the probability space induced by B_s. Hence the Markov property implies that all the information at time $s \leq t$ is contained in the value of B_s alone, independent of the history prior to time s. In this sense, Brownian motion is said to be a *zero-memory* process.

7. The covariance of Brownian motion is given by

$$\text{cov}\{B_s, B_t\} = \min(s, t) \tag{2.5}$$

which leads to the correlation structure

$$\boxed{\text{corr}\{B_s, B_t\} = \sqrt{s/t} \quad \text{for all } s < t} \tag{2.6}$$

Thus while non-overlapping Brownian increments are independent, the above implies that overlapping Brownian increments are dependent, with covariance equal to the duration of the overlap. The correlation coefficient (2.6), for Brownian motion at two distinct instants of time, plays an important role in dual and multi-period exotic options considered in later chapters. ▯

2.3 Stochastic DE's

A very readable account of sde's, and their numerical solution can be found in Kloeden and Platen [43] or Wiersema [75]. A more technical approach is given in the book by Øksendal [59]. We present here a simplified account.

The stochastic process X_t is said to be an Itô process with instantaneous drift $\mu_t = \mu(X_t, t)$ and instantaneous variance $\sigma_t^2 = \sigma^2(X_t, t)$ if it satisfies the stochastic differential equation (sde)

$$dX_t = \mu_t dt + \sigma_t \, dB_t \tag{2.7}$$

where B_t is a $\mathcal{P}-$Brownian motion. Such Itô processes are also Markov processes.

Since $\mathbb{E}\{dB_t\} = 0$ and $\mathbb{V}\{dB_t\} = dt$, one way of interpreting this sde is to say that it is equivalent to the pair of statements:

$$\mu(x,t) = \lim_{dt \to 0} \frac{\mathbb{E}_P\{X_{t+dt} - X_t | X_t = x\}}{dt} \tag{2.8}$$

$$\sigma^2(x,t) = \lim_{dt \to 0} \frac{\mathbb{V}_P\{X_{t+dt} - X_t | X_t = x\}}{dt}. \tag{2.9}$$

Both limits are assumed to exist for well-defined Itô processes.

Associated with every sde of the form of Equation (2.7) there exists a transition probability density function (pdf) $f(x_0, t_0; x, t)$ that gives the probability that $X_t = x$ given $X_{t_0} = x_0$ for all $t_0 < t$. It can be shown that $f(x_0, t_0; x, t)$ satisfies a pair of pde's called the forward and backward Kolmogorov equations. The forward equation is

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}[\mu(x,t)f] + \frac{\partial^2}{\partial x^2}[\tfrac{1}{2}\sigma^2(x,t)f] \tag{2.10}$$

with initial condition $f(x_0, t_0; x, t) \to \delta(x - x_0)$ as $t \to t_0$; and the backward equation is

$$-\frac{\partial f}{\partial t_0} = \mu(x_0, t_0)\frac{\partial f}{\partial x_0} + \tfrac{1}{2}\sigma^2(x_0, t_0)\frac{\partial^2 f}{\partial x_0^2} \tag{2.11}$$

with initial condition $f(x_0, t_0 | x, t) \to \delta(x_0 - x)$ as $t_0 \to t$.

Since these pde's are of the diffusion type (i.e. parabolic), Itô processes are also called *diffusion processes.*

2.3.1 Arithmetic Brownian Motion

Arithmetic Brownian Motion or aBm is defined to be the process satisfying a sde for $t > t_0$ of the form

$$dX_t = \mu dt + \sigma dB_t; \qquad X_{t_0} = x_0 \tag{2.12}$$

where μ and σ are constants. In this case the solution of the sde, using Equation (2.3), can be written as

$$X_t \stackrel{d}{=} x_0 + \mu(t - t_0) + \sigma B_{t-t_0} \tag{2.13}$$

and the transition pdf of X_t is obviously the Gaussian density with mean (drift) $x_0 + \mu(t - t_0)$ and variance $\sigma^2(t - t_0)$:

$$f(x_0, t_0; x, t) = \frac{1}{\sigma\sqrt{t - t_0}}\phi\left[\frac{x - x_0 - \mu(t - t_0)}{\sigma\sqrt{t - t_0}}\right], \tag{2.14}$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ for $z \in \mathbb{R}$ is the density of a $N(0, 1)$ variate.

It is a straight forward matter to show that this $f(x_0, t_0; x, t)$ satisfies both the forward and backward Kolmogorov equations with constant μ and σ and the given initial conditions. It should also be clear that aBm is simply a re-scaled Brownian motion with non-zero deterministic drift.

2.4 Stochastic Integrals

The sde expressed by Equation (2.7) has an alternative meaning in terms of stochastic integrals. This is the representation

$$X_t = X_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dB_s. \tag{2.15}$$

The last integral is an example of a stochastic integral — an integral with respect to a Brownian motion. This integral is formally defined by the limit

$$\int_0^t \sigma_s\, dB_s = \lim_{n\to\infty} \sum_{i=0}^{n} \sigma(t_i^{(n)})\, [B(t_{i+1}^{(n)}) - B(t_i^{(n)})],$$

where $t_i^{(n)}$ determines the partition Δ_n of $[0, t]$ defined earlier. It is essential in this definition that $\sigma(t_i^{(n)})$ is evaluated at each left-hand point of the partition $[t_i^{(n)}, t_{i+1}^{(n)})$, because in this case, it is $\mathcal{F}_{t_i}$−measurable (i.e. adapted to the Brownian increment). It is this "non-anticipatory" feature that distinguishes the Itô integral from other stochastic integrals such as the Stratonovich integral. In any case, the Itô integral above is found to be the ideal one in financial applications. The next result, often referred to as Itô's Isometry, gives the mean and variance of a stochastic integral.

Itô's Isometry

If $f(t)$ is a deterministic function of time (in fact it can also be a stochastic process that is adapted to B_t), then $Z_t = \int_0^t f(s)\, dB_s$ for each fixed t, is a Gaussian random variable with zero mean and variance equal to

$$\mathbb{V}\{Z_t\} = \int_0^t |f(s)|^2\, ds. \tag{2.16}$$

2.5 Itô's Lemma

This is the stochastic extension of the chain rule for ordinary (deterministic) calculus. Let X_t satisfy the sde $dX_t = \alpha dt + \beta dB_t$ for arbitrary (predictable processes) $\alpha = \alpha(X_t, t)$ and $\beta = \beta(X_t, t)$. If $F(x, t)$ is any $\mathbb{C}_{2,1}$ function (i.e. twice differentiable in x and differentiable in t), then the random process $F(X_t, t)$ is an Itô process satisfying the sde

$$dF = (F_t + \alpha F_x + \tfrac{1}{2}\beta^2 F_{xx})dt + \beta F_x\, dB_t \tag{2.17}$$

where subscripts on the function $F(X_t, t)$ denote partial derivatives.

A useful alternate form of Itô's lemma is:

$$dF(X_t, t) = (F_t + \tfrac{1}{2}\beta^2 F_{xx})dt + F_x\, dX_t. \tag{2.18}$$

2.5.1 Geometrical Brownian Motion

A process X_t is said to follow geometrical Brownian motion (gBm) with constant drift rate μ and volatility σ if it satisfies the sde

$$dX_t = X_t(\mu dt + \sigma\, dB_t); \qquad X_{t_0} = x_0. \tag{2.19}$$

This sde can be solved explicitly by transforming it to simple aBm through Itô's Lemma. Let $F(X_t) = \log X_t$ independent of t. Then substituting

$$\alpha = \mu X_t;\ \beta = \sigma X_t;\ F_t = 0;\ F_x = 1/X_t;\ F_{xx} = -1/X_t^2$$

into Itô's formula Equation (2.17), there results the sde

$$dF = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma\, dB_t; \qquad F(X_{t_0}) = \log x_0.$$

This is aBm with solution

$$F(X_t) = \log X_t \overset{d}{=} \log x_0 + (\mu - \tfrac{1}{2}\sigma^2)(t - t_0) + \sigma B_{t-t_0}$$

The solution for X_t is now obtained by exponentiation to yield the result

$$X_t \stackrel{d}{=} x_0 \exp\{(\mu - \tfrac{1}{2}\sigma^2)(t - t_0) + \sigma B_{t-t_0}\}. \tag{2.20}$$

The result is important because gBm is the basic asset price model in the Black–Scholes framework. Since, under gBm X_t/x_0 for fixed t, is the exponential of a Gaussian rv with mean $m(t_0, t) = (\mu - \frac{1}{2}\sigma^2)(t - t_0)$ and variance $v^2(t_0, t) = \sigma^2(t - t_0)$, its pdf is log-normal (see Section 3.6) with parameters (m, v). The corresponding transition pdf of X_t is therefore

$$f(x_0, t_0; x, t) = \frac{1}{xv(t, t_0)}\phi\left[\frac{\log(x/x_0) - m(t_0, t)}{v(t_0, t)}\right]. \tag{2.21}$$

This transition pdf can also be shown to satisfy the forward and backward Kolmogorov equations with instantaneous drift μx and instantaneous variance $(\sigma x)^2$.

2.5.2 Itô's Product and Quotient Rules

Let X_t satisfy the sde $dX_t = \alpha dt + \beta dB_t$ and let $F(X_t, t)$ and $G(X_t, t)$ be two given $\mathbb{C}_{2,1}$ functions. Then Itô's product and quotient rules are given respectively by

$$d(FG) = (FdG + GdF) + \beta^2 F_x G_x \, dt \tag{2.22}$$

and

$$d(F/G) = \frac{GdF - FdG}{G^2} + \frac{\beta^2 G_x}{G^3}(FG_x - GF_x)\, dt. \tag{2.23}$$

These formulae (see Q6 in Exercise Problems) are interesting in that they show that stochastic calculus includes the extra dt terms. Observe that these terms vanish when $\beta = 0$, and we recover the standard product and quotient rules for ordinary Newtonian calculus.

2.6 Martingales

A $(P, \mathcal{F}_t)-$*martingale* M_t is a stochastic process satisfying

$$\mathbb{E}_P\{M_t|\mathcal{F}_s\} = M_s \quad \text{for all } s \leq t. \tag{2.24}$$

Martingales are important in financial mathematics because they are the stochastic entities that capture the notion of a "fair game." In other words, they are intimately associated with the no-arbitrage assumption of idealized markets. For example, if M_t denotes the time t random wealth of a gambler playing a fair game, then Equation (2.24) above tells us that expected future wealth is equal to current wealth. It is in this sense, that the game is said to be fair.

Example 2.1
All the following

$$B_t, \quad B_t^2 - t \quad \text{and} \quad e^{\sigma B_t - \frac{1}{2}\sigma^2 t}$$

are well-known martingales (see Q3(c) in Exercise Problems). ▯

2.6.1 Martingale Representation Theorem

This essentially states that if X_t satisfies the zero drift sde $dX_t = \sigma(X_t, t)dB_t$ then X_t is an $\mathcal{F}_t-$ martingale[1].

This is easily seen as follows. Assume $0 < s < t$, then in terms of stochastic integrals,

$$X_t = X_0 + \int_0^t \sigma(X_u, u)dB_u = X_s + \int_s^t \sigma(X_u, u)dB_u$$

Hence $\mathbb{E}\{X_t|\mathcal{F}_s\} = X_s$ since the last stochastic integral has zero mean.

The converse is also true. Thus if X_t and Y_t are $\mathcal{F}_t-$local martingales, there exists an $\mathcal{F}_t-$adapted process c_t such that $dY_t = c_t dX_t$.

Itô's Lemma now provides the following corollary. If $f(x,t)$ is a given $\mathbb{C}_{2,1}$ function, then $f(X_t, t)$, where $dX_t = \alpha(x,t)dt + \beta(x,t)dB_t$, is an $\mathcal{F}_t-$martingale if and only if $f(x,t)$ satisfies the pde

$$f_t + \alpha(x,t)f_x + \tfrac{1}{2}\beta^2(x,t)f_{xx} = 0. \tag{2.25}$$

This is just the condition that the drift term vanishes in the sde for $f(X_t, t)$.

The three processes in example 2.1 are now readily seen to be martingales because

$$f(x,t) = x, \quad f(x,t) = x^2 - t \quad \text{and} \quad f(x,t) = e^{\sigma x - \frac{1}{2}\sigma^2 t}$$

all satisfy the pde (2.25) with $(\alpha = 0, \beta = 1)$.

[1]Strictly speaking, only a *local* martingale, a distinction we shall not elucidate further.

2.7 Feynman–Kac Formula

Let $\alpha(x,t), \beta(x,t)$ and $r(x,t)$ be given functions of (x,t) for $t < T$ and $x \in D$, where D is some subset of $\mathbb{R}^+$, and let $\mathcal{A}$ be the differential operator defined by $\mathcal{A}u(x,t) = \alpha \dfrac{\partial u}{\partial x} + \frac{1}{2}\beta^2 \dfrac{\partial^2 u}{\partial x^2}$. Then, subject to technical conditions, the unique solution of the pde

$$\frac{\partial u}{\partial t} + \mathcal{A}u - ru = 0; \qquad x \in D, \quad 0 \le t \le T$$

with terminal value $u(x,T) = g(x)$, is given by

$$u(x,t) = \mathbb{E}\left\{ e^{-\int_t^T r(X_s,s)ds}\, g(X_T) | X_t = x \right\}, \tag{2.26}$$

where the expectation is taken with respect to the transition density induced by the sde $dX_t = \alpha(X_t,t)dt + \beta(X_t,t)dB_t$.

REMARK 2.2

1. The variables α, β^2 are the instantaneous mean and variance of an underlying Itô diffusion. The variable r plays the role of a time and state dependent interest rate.
2. The operator $\mathcal{A}$ is the infinitesimal generator of the diffusion X_t and is often called the *Dynkin operator*.
3. The technical conditions alluded to are required to make the proof, outlined below, rigorous. In particular, the conditions allow interchange of integration and differentiation and also allow us to bring the limit $t \to T$ inside the integration.
4. The FK-formula provides an important link between the two principal methods used to price options and derivatives. These are the PDE-method and the EMM-method alluded to in Chapter 1.

$\square$

Proof of FK-Formula
Let $f(x,t;X,T)$ denote the transition pdf of the process X_t. Then Equation (2.26) is equivalent to

$$u(x,t) = \int_D e^{-\int_t^T r_s ds}\, g(X) f(x,t;X,T)\, dX.$$

Formally differentiating inside the integral, we get

$$\begin{aligned}\frac{\partial u}{\partial t} + \mathcal{A}u - ru &= \int_D e^{-\int_t^T r_s ds}\, g(X)\left[rf + \frac{\partial f}{\partial t} + \mathcal{A}f - rf\right] dX \\ &= \int_D e^{-\int_t^T r_s ds}\, g(X)\left[\frac{\partial f}{\partial t} + \mathcal{A}f\right] dX \\ &= 0.\end{aligned}$$

The last line follows from the observation that $-\frac{\partial f}{\partial t} = \mathcal{A}f$ is the backward Kolmogorov equation (2.11). It remains to demonstrate that the terminal condition is satisfied.

$$\begin{aligned}u(x,T) &= \lim_{t\to T}\int_D e^{-\int_t^T r_s ds}\, g(X) f(x,t;X,T)\, dX \\ &= \int_D g(X) \lim_{t\to T} f(x,t;X,T)\, dX \\ &= \int_D g(X)\, \delta(X-x)\, dX \\ &= g(x) \qquad \text{for } x \in D.\end{aligned}$$

This completes the proof. □

It should now be clear that the FTAP described by equation (1.46) is just a special case of the FK-formula.

2.8 Girsanov's Theorem

If B_t is a standard $\mathcal{P}-$Brownian motion and $\lambda(t)$ is an adapted (i.e. $\mathcal{F}_t-$measurable) process satisfying the Novikov condition,

$$\mathbb{E}_P\{\exp(\tfrac{1}{2}\int_0^T \lambda^2(t)dt)\} < \infty,$$

then there exists a measure $\mathcal{P}^*$ such that

1.	$\mathcal{P}^*$ is equivalent to $\mathcal{P}$
2.	$\frac{d\mathcal{P}^*}{d\mathcal{P}} = \exp\left(-\int_0^T \lambda(t)dW_t - \frac{1}{2}\int_0^T \lambda^2(t)dt\right)$
3.	$B_t^* = B_t + \int_0^t \lambda(s)ds$ is a $\mathcal{P}^*-$Brownian motion

The converse states that if B_t is a $\mathcal{P}-$Brownian motion and $\mathcal{P}^*$ is a measure equivalent to $\mathcal{P}$, then there exists an adapted process $\lambda(t)$ such that

$$B_t^* = B_t + \int_0^t \lambda(s)ds$$

is a $\mathcal{P}^*-$Brownian motion.

REMARK 2.3
1. Two probability measures are said to be equivalent if they have the same null sets. That is $\mathcal{P}(A) = 0 \Leftrightarrow \mathcal{P}^*(A) = 0$.

2. The term $\mathcal{R}_t = \dfrac{d\mathcal{P}^*}{d\mathcal{P}}$ is called the Radon-Nikodym derivative and gives the factor needed in computing the change of measure formula: for any random process X_t

$$\mathbb{E}_{P^*}\{X_t|\mathcal{F}_s\} = \mathcal{R}_s^{-1}\mathbb{E}_P\{\mathcal{R}_t X_t|\mathcal{F}_s\} \quad \text{for all } s \leq t. \tag{2.27}$$

3. For the sde $dX_t = \mu_t dt + \sigma_t dB_t$, let $B_t^* = B_t + \int_0^t (\mu_s/\sigma_s)\, ds$. Then $dX_t = \sigma_t dB_t^*$ and X_t, by the martingale representation theorem, is therefore a P^*-martingale. ∎

Suppose we have a stochastic process satisfying an arbitrary sde with respect to a measure P. This process will not in general be a martingale under P. Girsanov's theorem shows, by changing the drift as above, how to find a new measure $\mathcal{P}^*$, equivalent to P, under which the process is a martingale.

2.9 Time Varying Parameters

Under the EMM, the sde for the asset price X_t when the risk-free rate r_t, dividend yield q_t and volatility σ_t are deterministic functions of t, is given by

$$dX_s = X_s\left[(r_s - q_s)\, ds + \sigma_s\, dB_s\right]; \quad (s > t,\ X_t = x). \tag{2.28}$$

We assume that r_s, q_s and σ_s are piecewise continuous. The solution of this sde is readily obtained as (e.g. see Problem 8(b) in the Exercise Problems)

$$X_T = x \exp\left[\int_t^T (r_s - q_s - \tfrac{1}{2}\sigma_s^2)ds + \int_t^T \sigma_s\, dB_s\right] \tag{2.29}$$

Now let us define

$$\bar{r} = \frac{1}{T-t}\int_t^T r_s\, ds; \qquad \bar{q} = \frac{1}{T-t}\int_t^T q_s\, ds \tag{2.30}$$

$$\hat{\sigma} = \left[\frac{1}{T-t}\int_t^T \sigma_s^2\, ds\right]^{\frac{1}{2}}. \tag{2.31}$$

Then, $(\bar{r}, \bar{q})$ are the mean risk-free rate and mean dividend yield over $[t, T]$, and $\hat{\sigma}$ is the root mean square (rms) volatility over $[t, T]$. Furthermore, by Itô's Isometry, (2.16), we have

$$\begin{aligned}\int_t^T \sigma_s\, dB_s &\stackrel{d}{=} N\left[0, \int_t^T \sigma_s^2\, ds\right] \\ &= N(0,\, \hat{\sigma}^2\tau); \quad \tau = (T-t) \\ &\stackrel{d}{=} \hat{\sigma}\sqrt{\tau}\, Z; \quad Z \sim N(0,1).\end{aligned}$$

Hence, under the EMM, the asset price at time T has the representation

$$X_T = x\, e^{(\bar{r}-\bar{q}-\frac{1}{2}\hat{\sigma}^2)\tau + \hat{\sigma}\sqrt{\tau}\, Z}. \tag{2.32}$$

Comparing this expression with the stock price formula (1.51) for conditions of constant parameters, we see that the case of time varying deterministic parameters has exactly the same mathematical structure. It follows, that if $V(x, t; r, q, \sigma)$ is the price of a European derivative with fixed expiry date T, under constant parameters (r, q, σ), then $V(x, t; \bar{r}, \bar{q}, \hat{\sigma})$ will be the corresponding price when the parameters are deterministic functions of time.

2.10 The Black–Scholes PDE

We mentioned in Section 1.7 that the BS-pde for $V(x, t)$

$$V_t = rV - rx\, V_x - \tfrac{1}{2}\sigma^2 x^2\, V_{xx}; \qquad V(x, T) = f(x)$$

can be transformed by two distinct variable changes into the the standard heat equation. We begin by converting the pde from a backward to a forward pde by through $\tau = (T-t)$, under which the time partial derivative becomes $V_t = -V_\tau$. Thus, $V(x, \tau)$ satisfies the *forward*, initial value problem (IVP)

$$V_\tau = -rV + rx\, V_x + \tfrac{1}{2}\sigma^2 x^2\, V_{xx}; \qquad V(x, 0) = f(x)$$

Scheme 1
Let

$$y = \log x; \qquad V(x, \tau) = U(y, \tau)\, e^{-\frac{1}{2}\alpha y - \beta\tau} \tag{2.33}$$

This is the transformation scheme of Wilmott et al. [76]. The factor $\frac{1}{2}$ in the exponent is introduced for later identification with the image prices defined in Section 7.2 on barrier options. To see how the transformation works, observe that when $y = \log x$, we have

$$V_x = V_y \frac{\partial y}{\partial x} = \frac{1}{x} V_y \qquad \Rightarrow \qquad x V_x = V_y \tag{2.34}$$

$$V_{xx} = -\frac{1}{x^2} V_y + \frac{1}{x^2} V_{yy} \quad \Rightarrow \quad x^2 V_{xx} = V_{yy} - V_y \tag{2.35}$$

In terms of the variable y, the new pde for $V(y, \tau)$ becomes

$$V_\tau = -rV + (r - \tfrac{1}{2}\sigma^2)\, V_y + \tfrac{1}{2}\sigma^2\, V_{yy}; \qquad V(y, 0) = f(e^y).$$

It is clear then, that the transformation $y = \log x$ converts the pde with non-constant coefficients into one with constant coefficients. Next, observe that in terms of $U(y, \tau)$,

$$\begin{aligned} V_\tau &= (U_\tau - \beta U)\, e^{-\frac{1}{2}\alpha y - \beta\tau} \\ V_y &= (U_y - \tfrac{1}{2}\alpha U)\, e^{-\frac{1}{2}\alpha y - \beta\tau} \\ V_{yy} &= (U_{yy} - \alpha U_y + \tfrac{1}{4}\alpha^2 U)\, e^{-\frac{1}{2}\alpha y - \beta\tau}. \end{aligned}$$

So substituting into the new pde for $U(y, \tau)$, we are free to choose the constant parameters (α, β) and do so by making the coefficients of U and U_y both equal to zero. This yields, after a little algebra,

$$\alpha = \frac{2r}{\sigma^2} - 1 \qquad \text{and} \qquad \beta = \frac{(r + \frac{1}{2}\sigma^2)^2}{2\sigma^2}. \tag{2.36}$$

The pde for $U(y, \tau)$ then reduces to

$$U_\tau = \tfrac{1}{2}\sigma^2\, U_{yy}; \qquad U(y, 0) = e^{\frac{1}{2}\alpha y} f(e^y). \tag{2.37}$$

This is the standard heat equation with "thermal conductivity" equal to $\frac{1}{2}\sigma^2$ and initial "temperature" equal to $e^{\frac{1}{2}\alpha y} f(e^y)$. Note that appropriate domain for this pde is $[\tau > 0;\ y \in \mathbb{R}]$, so even though $x > 0$ in the BS pde, the variable y may be any real number, positive or negative.

Scheme 2

Perhaps less well-known is this second scheme, which transforms the forward BS-pde via the new variables y and $U(y, \tau)$ through

$$y = \log x + (r - \tfrac{1}{2}\sigma^2)\tau; \qquad V(y, \tau) = e^{-r\tau}\, U(y, \tau). \tag{2.38}$$

We omit the details, which follow similar calculations as in Scheme 1. The new pde that results from this transformation is

$$U_\tau = \tfrac{1}{2}\sigma^2\, U_{yy}; \qquad U(y, 0) = f(e^y) \tag{2.39}$$

This is the same heat equation, as for Scheme 1, but with a different initial value.

REMARK 2.4 Some texts price exotic options by transforming the BS-pde to the heat equation, as above, and then solve the simpler looking pde that results. However, that is really an illusion. With the right tools, as we shall demonstrate throughout this book, it is just as easy to solve the BS-pde as it stands. This avoids the need to both forward and back transform the associated variables. Nevertheless, there are some situations where it is useful to consider the heat equation transformation.

Once we have transformed the BS-pde into the heat equation, we have at our disposal the vast theory that has been devoted to it. For example, the text by Cannon [11], gives many important results about the heat equation. While we don't often use these results explicitly in this book, we may do so implicitly. For example, we shall assume that the existence and uniqueness of solutions of the heat equation are inherited by the corresponding BS-pde. This means, that if we are able to construct a solution of the BS-pde by some clever tricks, then we can be assured that this is the only solution. We actually use this idea in several places in later chapters.

Another use of the heat equation is to get a better understanding of the image option, defined in Chapter 7 on barrier options. The image solution for the BS-pde looks rather complicated, but is easy to describe and understand for the heat equation. □

2.11 The BS Green's Function

Here is a specific example of where the heat equation transformation can be used to good effect. The *fundamental solution* or *Green's Function* for the heat equation $U_\tau = \frac{1}{2}\sigma^2 U_{yy}$ is well-known and is given by the formula

$$G(y,\tau;y_0) = \frac{1}{\sigma\sqrt{2\pi\tau}}\, e^{-\frac{(y-y_0)^2}{2\sigma^2\tau}} = \frac{1}{\sigma\sqrt{\tau}}\,\phi\left(\frac{y-y_0}{\sigma\sqrt{\tau}}\right). \tag{2.40}$$

This is the unique solution expressed in terms of the Gaussian density $\phi(z)$, with initial value $U(y,0) = \delta(y-y_0)$, where $\delta(y-y_0)$ is again the Dirac delta function.

If we now invert this expression for either of the above transformation schemes, we arrive at the Green's Function for the BS-pde. This leads to the important formula

$$\boxed{G(x,t;\,\xi,T) = \frac{e^{-r\tau}}{\sigma\sqrt{\tau}}\,\phi\left(\frac{\log(x/\xi) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)} \tag{2.41}$$

where we have written ξ for $\log y_0$. The terminal condition satisfied by this Green's Function is $G(x,T;\xi,T) = \delta(x/\xi - 1)$.

But as mentioned previously, given the right tools, we could also derive this result directly. We proceed to develop these tools in what follows.

2.12 Log-Volutions

We show in the present section how to obtain solutions of the BS-pde in terms of an operator we call the logarithmic convolution or *log-volution* for short.

Let $f(x), g(x)$ and $h(x)$ denote arbitrary functions defined on $x \in \mathbb{R}^+$. We allow these to be generalized functions such as Dirac delta functions and their derivatives.

DEFINITION 2.1 *The log-volution of $f(x)$ and $g(x)$, written as $f(x) \star g(x)$, is defined by*

$$\boxed{f(x) \star g(x) = \int_0^\infty f(y)\, g\left(\frac{x}{y}\right)\,\frac{dy}{y}} \tag{2.42}$$

The log-volution, as demonstrated next, is indeed a "logarithmic convolution." Let $x' = \log x$; $y' = \log y$ and write $F(x') = f(e^{x'})$; $G(x') = g(e^{x'})$. Then

$$f(x) \star g(x) = \int_{-\infty}^{\infty} F(y')G(x' - y')\,dy' = F(x') * G(x').$$

The expression $F(x') * G(x')$ is the usual way of writing a standard or linear convolution. Hence, the log-volution in the variable x is seen to be equal to a convolution in the variable $x' = \log x$, and hence the name.

The log-volution has many interesting properties, some of which are listed in the table on the next page. In this table, (α, β) are arbitrary scalars; $k > 0$ is a positive scalar and D denotes the differential operator

$$D = x\,\frac{d}{dx}. \tag{2.43}$$

Further, $L(x) = f(x) \star g(x)$ denotes the log-volution of $f(x)$ and $g(x)$.

	Log-Volution Properties
L1.	$f \star (\alpha g + \beta h) = \alpha(f \star g) + \beta(f \star h)$
L2.	$f \star g = g \star f$
L3.	$f \star (g \star h) = (f \star g) \star h$
L4.	$L(kx) = f(kx) \star g(x) = f(x) \star g(kx)$
L5.	$[x^\alpha f(x)] \star [x^\alpha g(x)] = x^\alpha\, [f(x) \star g(x)]$
L6.	$L(x^{-1}) = f(x^{-1}) \star g(x^{-1})$
L7.	$D[f \star g] = (Df) \star g = f \star (Dg)$
L8.	$f(x) \star \delta(x-1) = f(x)$

Thus log-volution is linear, commutative, and associative, and the identity element under log-volution is the Dirac delta function $\delta(x-1)$. The log-volution also has useful scaling, power, inversion, and derivative rules as given by properties L4 to L7 inclusively. The proofs of these properties are relatively straightforward using standard integration rules alone, but a simpler method utilizes the Mellin Transform, which we introduce later in this section.

Consider now the BS-pde given by Equation (1.50) and repeated here in terms of relative time $\tau = (T-t)$,

$$V_\tau = -rV + (r-q)x\, V_x + \tfrac{1}{2}\sigma^2 x^2\, V_{xx}.$$

Then if D is the operator defined by (2.43), it is a simple matter to show that

$$V_\tau = \tfrac{1}{2}\sigma^2 D^2 V + (r - q - \tfrac{1}{2}\sigma^2)DV - rV = Q(-D)V, \tag{2.44}$$

where $Q(s)$ is the quadratic function $Q(s) = \frac{1}{2}\sigma^2 s^2 - (r-q-\frac{1}{2}\sigma^2)s - r$. The minus sign in $Q(-D)$ is included for later convenience.

Thus, while the BS-pde for $V(x,\tau)$ in terms of the differential operator d/dx has non-constant coefficients, under the operator $D = xd/dx$, it now has constant coefficients.

The next theorem shows how to build solutions of the BS-pde from more basic ones using log-volutions.

THEOREM 2.1

Let $F(x)$ be any function independent of time τ defined on $x > 0$, such that the log-volution $F(x) \star U(x,\tau)$ exists. Then if $U(x,\tau)$ is a solution of the

BS-pde, so is

$$V(x,\tau) = F(x) \star U(x,\tau). \tag{2.45}$$

PROOF Consider formally the operator $\partial_\tau - Q(-D)$ applied to $V(x,\tau)$,

$$V_\tau - Q(-D)V = F(x) \star [U_\tau - Q(-D)U]$$

by the linearity and derivative properties of log-volution. However, this expression is equal to zero, since by assumption, $U(x,\tau)$ satisfies the BS-pde $U_\tau = Q(-D)U$. ∎

The next theorem shows how to price European derivatives with an arbitrary payoff $f(x)$, in terms of log-volutions.

COROLLARY 2.1
Let $G(x,\tau)$ denote the Green's Function of the BS-pde (2.44) with the initial value $G(x,0) = \delta(x-1)$. Then the solution of the BS-pde with initial value (i.e. payoff) $V(x,0) = f(x)$, is given by

$$V(x,\tau) = f(x) \star G(x,\tau). \tag{2.46}$$

PROOF The proof of (2.46) follows directly from theorem 2.1 and log-volution property L8. which yields, $V(x,0) = f(x) \star \delta(x-1) = f(x)$. ∎

2.12.1 The Mellin Transform

For any function $f(x)$ on $x > 0$, for which the integral below exists, the Mellin Transform of $f(x)$ denoted by $F(s)$, is defined by

$$F(s) = \mathcal{M}_s[f(x)] = \int_0^\infty f(x)x^{s-1}\,dx. \tag{2.47}$$

The parameter s is generally taken as a complex variable. The Mellin Transform is closely related to the Laplace Transforms, and just like the latter, has an inverse transform, which is given (with c real) by

$$f(x) = \mathcal{M}_x^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}\,ds. \tag{2.48}$$

Properties and tables of Mellin Transforms can be found in Erdélyi et al. [22]. We present some of the more important ones in the following. With the same notation we used in the log-volution table, we have the general properties shown in the next table. Of particular importance is property M6, that the transform of a log-volution of two functions is the product of their transforms. In property M7, the function ϕ denotes the Gaussian pdf of Equation (3.1).

Mellin Transform Properties

	x-domain function	*Mellin Transform*
M1.	$\alpha f(x) + \beta g(x)$	$\alpha F(s) + \beta G(s)$
M2.	$f(kx)$	$k^{-s}\,F(s)$
M3.	$f(x^\alpha)$	$\lvert\alpha\rvert^{-1}\,F(s/\alpha)$
M4.	$x^\alpha\,f(x)$	$F(s+\alpha)$
M5.	$(-D)^n\,f(x)$	$s^n\,F(s)$
M6.	$f(x)\star g(x)$	$F(s)\,G(s)$
M7.	$\phi(\log x)$	$e^{\frac{1}{2}s^2}$
M8.	$x^\gamma \mathbb{I}(x<k)$	$k^{s+\gamma}(s+\gamma)^{-1}$

The Mellin Transform provides a quick and efficient way of deriving an explicit expression for the Green's Function $G(x,\tau)$, defined in corollary 2.1. The function $G(x,\tau)$ satisfies the IVP

$$G_\tau = Q(-D)G; \qquad G(x,0) = \delta(x-1).$$

Taking the Mellin Transform of this pde leads to the ode, for $\hat{G}(s,\tau) = \mathcal{M}_s[G(x,\tau)]$,

$$\hat{G}_\tau = Q(s)\,\hat{G}(s,\tau); \qquad \hat{G}(s,0) = 1$$

which follows directly from the derivative property. This has unique solution $\hat{G}(s,\tau) = e^{Q(s)\tau}$, which can be written in the form,

$$\hat{G}(s,\tau) = e^{-r\tau}\cdot k^{-s}\cdot e^{\frac{1}{2}(\sigma\sqrt{\tau}s)^2}$$

with $k = e^{(r-q-\frac{1}{2}\sigma^2)\tau}$. Several of the Mellin Transform properties then lead to the result

$$G(x,\tau) = \frac{e^{-r\tau}}{\sigma\sqrt{\tau}}\,\phi\left(\frac{\log x + (r-q-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \tag{2.49}$$

This expression (which now includes the dividend yield q) agrees with the result (2.41) which we derived by first transforming the BS-pde to the heat equation.

PROPOSITION 2.1

Let $G(x,\tau)$ defined by (2.49) be the BS Green's Function. Then for any $k>0$,

$$x\mathbb{I}(x>k)\star G(x,t) = xe^{-q\tau}\,\mathcal{N}(d_1)$$
$$\mathbb{I}(x>k)\star G(x,t) = e^{-r\tau}\,\mathcal{N}(d_2)$$

where

$$d_{1,2}(x,\tau) = \frac{\log(x/k) + (r - q \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

We leave the proof as Q14 in the Exercise Problems to this chapter.

Sometimes we are interested in solving the inhomogeneous BS-pde (see Equation (2.50)), where the added term represents cash instalment rates per unit time to be paid if $h < 0$, or to be received if $h > 0$. The next theorem shows how to price derivatives with such continuous instalments.

THEOREM 2.2
Let $V(x,\tau)$ satisfy the inhomogeneous BS-pde

$$V_\tau = Q(-D)\,V + h(x,\tau); \qquad V(x,0) = f(x). \tag{2.50}$$

Then, subject to the existence of the log-volutions, the solution of this pde is given by

$$V(x,\tau) = f(x) \star G(x,\tau) + \int_0^\tau h(x,\tau') \star G(x,\tau-\tau')\,d\tau'. \tag{2.51}$$

PROOF Take the Mellin Transform of the pde to arrive at

$$\hat{V}_\tau = Q(s)\hat{V}(s,\tau) + \hat{h}(s,\tau); \qquad \hat{V}(s,0) = \hat{f}(s).$$

This ode has solution

$$\hat{V}(s,\tau) = \hat{f}(s)e^{Q(s)\tau} + \int_0^\tau e^{Q(s)(\tau-\tau')}\hat{h}(s,\tau')\,d\tau'.$$

Inverting the Mellin Transforms using, $\mathcal{M}_s[G(x,\tau)] = e^{Q(s)\tau}$, and several entries stated in table of Mellin Transform Properties, leads to the given result. ▯

Curiously, the second term in Equation (2.51) is simultaneously both a convolution (in time) and a log-volution (in price).

Example 2.2 (**American Call**)

Consider a standard American call option on a dividend paying asset, with constant dividend yield q. Then the price $V(x,\tau)$ with IV, $V(x,0) = f(x) = (x-k)^+$, satisfies

$$\begin{aligned} \mathcal{L}V = V_\tau - Q(-D)V = 0 &\quad \text{in } x < b(\tau) \\ V = x - k \qquad\qquad &\quad \text{in } x > b(\tau) \end{aligned}$$

where $\mathcal{L}$ denotes the BS-pde operator and $x = b(\tau) \geq k$ is the early exercise boundary. The domain $x < b(\tau)$ is called the *continuation region*, in which we continue to hold the option; the complementary domain $x > b(\tau)$ is the *stopping region* in which we exercise the option before expiry. Both equations are encompassed by the single inhomogeneous BS-pde

$$\mathcal{L}V(x,\tau) = -Q(-D)[x-k] \cdot \mathbb{I}(x > b(\tau)) = (qx - rk)\mathbb{I}(x > b(\tau)).$$

Using Theorem 2.2 and Proposition 2.1, we therefore have the solution

$$\begin{aligned} V(x,\tau) &= (x-k)^+ \star G(x,\tau) + \int_0^\tau [(qx - rk)\mathbb{I}(x > b(\tau - s)) \star G(x,s)]\, ds \\ &= C_k(x,\tau) + \int_0^\tau [qxe^{-qs}\mathcal{N}(z_1) - rke^{-rs}\mathcal{N}(z_2)]\, ds, \end{aligned}$$

where $C_k(x,\tau)$ denotes the European call option price, and

$$z_{1,2}(x,\tau,s) = \frac{\log(x/b(\tau - s)) + (r - q \pm \frac{1}{2}\sigma^2)s}{\sigma\sqrt{s}}.$$

The last expression for $V(x,\tau)$ determines the American call option price in terms of the (yet) unknown early exercise boundary $b(\tau)$. The continuity condition, $V(x,\tau)$ is continuous across the boundary $x = b(\tau)$, then yields the following integral equation for $b(\tau)$

$$b(\tau) - k = C_k(b(\tau),\tau) + \int_0^\tau [qb(\tau)e^{-qs}\mathcal{N}(z_1) - rke^{-rs}\mathcal{N}(z_2)]\, ds$$

where now, z_1 and z_2 are evaluated at $x = b(\tau)$.

This representation of the price of a vanilla American call option was first given by Kim [42] and Jamshidian [36]. A survey of similar results can be found in Chiarella *et al* [15]. It is obviously not a closed form solution because first the integral equation for $b(\tau)$ needs to be solved, and second the price also involves an unknown integral in terms of $b(\tau)$. ∎

2.13 Summary

This chapter dealt mainly with the mathematical and statistical tools needed to price options in the Black–Scholes framework. Most of these tools were introduced in a fairly non-technical way, consistent with the general approach taken in this book. Proofs of the well-known results were generally ignored, as these may be looked up in the quoted texts.

The most important topics we covered included:

Tower Law	Feynman–Kac Formula
Brownian motion	Girsanov's Theorem
SDE's	Time-Varying Parameters
Itô's Lemma	BS Green's Function
Martingales	Log-Volutions
	Mellin Transforms

The section on Time-Varying Parameters is important, as it has a direct consequence on the pricing of Asian options considered in Chapter 9.

The applications of the Mellin Transform and log-volutions to the BS-pde contain significant unpublished material, which also will have an impact on exotic option pricing in later chapters, particularly for reflecting barrier options.

We have seen that the FTAP applied to the BS model can be expressed entirely in terms of Gaussian rv's as the only stochastic variable required. It is therefore of important to have a good understanding of Gaussian rv's and the next chapter is devoted exclusively to their study. This chapter includes properties of univariate, bivariate and multi-variate Gaussian random variables, with corresponding extensions for the BS-pde and FTAP.

Exercise Problems

1. Let $V(x,t)$ denote the price of any European derivative on a single underlying asset X, for $t < T$, that pays $F(x)$ at expiry T. Now suppose you are offered a contract at time t which pays at time s, the amount $V(x,s)$ for $t < s < T$. Use the Tower Law to prove that the price of the derivative at time t should equal $V(x,t)$.

2. Let $\mathbb{E}_s\{X_t\}$ denote the conditional expectation $\mathbb{E}\{X_t|\mathcal{F}_s\}$ for $s < t$. If B_t is a standard Brownian motion, show that

$$\mathbb{E}_s\{B_t^3\} = B_s^3 + 3(t-s)B_t.$$

   Verify the Tower Law, for the example $\mathbb{E}_u\{\mathbb{E}_s\{B_t^3\}\} = \mathbb{E}_u\{B_t^3\}$ for $u < s < t$.

3. Prove by direct methods the following results for standard Brownian motions B_t:

   (a) $\mathrm{cov}\{B_s, B_t\} = \min(s,t)$ and for $s < t$, $\mathrm{corr}\{B_s, B_t\} = \sqrt{s/t}$

   (b) B_t is self-similar with similarity dimension $H = \frac{1}{2}$;
   i.e. $\quad B_{ct} \overset{d}{=} c^H B_t$ for all $c > 0$.

(c) $B_t,\ B_t^2 - t$ and $e^{\sigma B_t - \frac{1}{2}\sigma^2 t}$ are $\mathcal{F}_t-$martingales.

4. For the general linear sde

$$dX_t = (a_t X_t + \alpha_t)dt + (b_t X_t + \beta_t)dB_t; \qquad X_0 = x$$

show that

$$\frac{dE_t}{dt} = a_t E_t + \alpha_t; \qquad E_0 = x$$
$$\frac{dV_t}{dt} = (2a_t + b_t^2)V_t + (b_t E_t + \beta_t)^2; \qquad V_0 = 0$$

where $E_t = \mathbb{E}\{X_t\}$ and $V_t = \mathbb{V}\{X_t\}$.

5. Use Itô's Lemma to show that the sde

$$dX_t = \tfrac{1}{2}h(X_t)h'(X_t)dt + h(X_t)dB_t$$

is reducible to simple aBm by the transformation $Y_t = f(X_t)$ where $f(x) = \int^x [1/h(u)]du$. Hence solve the sde's

(a) $dX_t = \frac{1}{3}X_t^{1/3}dt + X_t^{2/3}dB_t$, and

(b) $dX_t = a^2\,dt + 2a\sqrt{X_t}\,dB_t$ with $a > 0$ and $X_0 = 1$. Hence find the probability that $X_t < 1$ at any time $t > 0$.

6. Let X_t satisfy the sde $dX_t = \alpha_t dt + \beta_t dB_t$ and suppose, $F(X_t, t)$ and $G(X_t, t)$ are two $\mathbb{C}_{2,1}$ functions. Derive Itô's Product and Quotient Rules

$$d(FG) = FdG + GdF + \beta^2(F_x G_x)dt$$
$$d(F/G) = \frac{GdF - FdG}{G^2} + \frac{\beta^2 G_x}{G^3}(FG_x - GF_x)dt.$$

7. Prove that $f(B_t, t)$, where B_t is a standard Brownian motion, is a (local) $\mathcal{F}_t$ martingale if $f(x,t)$ satisfies the pde $f_t + \frac{1}{2}f_{xx} = 0$.

(a) Hence show that $X_t = t^{n/2}\,H_n\left(\frac{B_t}{\sqrt{2t}}\right)$, where $H_n(x)$ are Hermite polynomials, is an $\mathcal{F}_t$ martingale.

(b) Obtain polynomials of orders one through four, in B_t that are $\mathcal{F}_t$ martingales.

(c) Use the Hermite polynomial generating function

$$e^{2xs - s^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) s^n$$

to show that $e^{\sigma B_t - \frac{1}{2}\sigma^2 t}$ is an $\mathcal{F}_t$ martingale.

8. (a) Use the 2D Itô's Lemma to prove that if $X = X_t$ and $Y = Y_t$ are two Itô processes relative to the same Brownian motion, with instantaneous variances σ_x^2 and σ_y^2, then

$$d(X/Y) = \frac{Y\,dX - X\,dY}{Y^2} + \frac{\sigma_y(\sigma_y X - \sigma_x Y)}{Y^3}\,dt.$$

(b) Let X_t satisfy the *general linear sde*

$$dX_t = (a_t X_t + \alpha_t)dt + (b_t X_t + \beta_t)\,dB_t$$

where $a_t, \alpha_t, b_t, \beta_t$ are deterministic functions of t. Show that the solution of this sde is given by

$$X_t = \Phi_0^t X_0 + \int_0^t (\alpha_s - b_s\beta_s)\Phi_s^t\,ds + \int_0^t \beta_s \Phi_s^t\,dB_s$$

where

$$\Phi_s^t = \exp\left(\int_s^t (a_u - \tfrac{1}{2}b_u^2)du + \int_s^t b_u\,dB_u\right).$$

Hint: Solve the sde $dY_t = Y_t(a_t dt + b_t dB_t)$; $Y_0 = 1$ and use Itô's Quotient Rule above, to solve for $Z_t = X_t/Y_t$.

9. The mean-reverting OU (Ornstein-Uhlenbeck) process is the solution of the sde

$$dX_t = a(\gamma - X_t)dt + \sigma dB_t; \qquad X_0 = x$$

where a, γ, σ are positive constants. Solve this sde and hence show that X_t is Gaussian with mean and variance

$$\mathbb{E}\{X_t\} = \gamma + (x - \gamma)e^{-at}$$
$$\mathbb{V}\{X_t\} = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

10. Suppose the underlying asset $X_t = x$ and an associated derivative $V_t = V(X_t, t)$ satisfy the sde's, in the real-world measure,

$$dX_t = \mu_X dt + \sigma_X dB_t; \qquad dV_t = \mu_V dt + \sigma_V dB_t$$

Show that, in order to avoid arbitrage,

$$\frac{\mu_X - rX_t}{\sigma_X} = \frac{\mu_V - rV_t}{\sigma_V}.$$

Hence derive the BS-pde when X_t follows gBm.

11. Show that the transformation (Scheme-2)

$$y = \log x + (r - \tfrac{1}{2}\sigma^2)\tau; \quad \tau = T - t; \quad V(x,t) = e^{-r\tau}U(y,\tau)$$

reduces the BS-pde for $V(x,t)$, with TV, $V(x,T) = f(x)$ to the IV heat equation

$$U_\tau = \tfrac{1}{2}\sigma^2 U_{yy}; \qquad U(y,0) = f(e^y).$$

12. Derive the log-volution properties L1 to L8 by

    (a) direct integration, and

    (b) using Mellin Transforms.

13. Derive the BS solutions as given in Section 2.9, for time-varying parameters, by applying the Mellin Transform directly to the the BS-pde.

14. Show that if $G(x,t)$ denotes the Green's Function for the BS-pde, then

$$\mathbb{I}(x > k) \star G(x,t) = e^{-r\tau}\,\mathcal{N}(d_2)$$
$$x\mathbb{I}(x > k) \star G(x,t) = xe^{-q\tau}\,\mathcal{N}(d_1)$$

where

$$d_{1,2} = \frac{\log(x/k) + (r - q \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

and q is the constant dividend yield.

—ooOoo—