Chapter 2

2.1. *A plane wave has an electric field given by* $\underline{\vec{E}} = E_o \sin\left(kz - \omega t + \phi\right)\hat{x}$ *is incident on a material with a susceptibility given by* $\chi = \left(1 + \sqrt{3}i\right)\chi_o / 2$.

a) *What is the complex amplitude of the electric field?*

$$A = E_o e^{i\left(\phi - \frac{\pi}{2}\right)}\hat{x} \tag{S2.1}$$

b) *What is the phase shift between a polarization induced by the field and the incident field?*

The relationship between the polarization's complex amplitude and the field's complex amplitude is,

$$P = \varepsilon_o \chi A \tag{S2.2}$$

Therefore the phase difference is the phase of the complex susceptibility, which is $\frac{\pi}{3}$.

c) *What is the real polarization in this medium induced by the field?*

$$\underline{\vec{P}} = \varepsilon_o \chi_o E_o \sin\left(kz - \omega t + \phi + \frac{\pi}{3}\right)\hat{x} \tag{S2.3}$$

or equivalently,

$$\underline{\vec{P}} = \varepsilon_o \chi_o E_o \cos\left(kz - \omega t + \phi - \frac{\pi}{6}\right)\hat{x} \tag{S2.4}$$

2.2. *A plane wave in a vacuum has an electric field given by,* $\underline{\vec{E}} = E_o \cos(kz - \omega t + \varphi)\hat{x}$.

a) *What is* $\underline{\vec{B}}$*?*

We use Maxwell's equation, 2.2, to relate the curl of the electric field to the time derivative of the magnetic field. For a field of the form given here, $\nabla \rightarrow ik\hat{z}$ and $\frac{\partial}{\partial t} \rightarrow -i\omega$. Therefore,

$$\underline{\vec{B}} = \frac{E_o}{c}\cos\left(kz - \omega t + \varphi\right)\hat{y} \tag{S2.5}$$

b) *What is the complex amplitude of* $\underline{\vec{B}}$*?*

$$\vec{B} = \frac{E_o}{c} e^{i\phi}\hat{y} \tag{S2.6}$$

c) *What are* $\underline{\vec{S}}$ and $\left\langle \underline{\vec{S}} \right\rangle$*?*

$$\underline{\vec{S}} = \underline{\vec{E}} \times \underline{\vec{H}} = \frac{\underline{\vec{E}} \times \underline{\vec{B}}}{\mu_o} = \frac{E_o^2}{\mu_o c}\cos^2\left(kz - \omega t + \varphi\right)\hat{z} \tag{S2.7}$$

$$\left\langle \underline{\vec{S}} \right\rangle = \frac{E_o^2}{2\mu_o c}\hat{z} \tag{S2.8}$$

2.3. *Consider two monochromatic quantities,* $\underline{\vec{F}} = \frac{1}{2}\vec{F}e^{i\omega t} + \text{c.c.}$ *and* $\underline{\vec{G}} = \frac{1}{2}\vec{G}e^{i\omega t} + \text{c.c.}$. *Prove that* $\left\langle \underline{\vec{F}} \bullet \underline{\vec{G}} \right\rangle = \frac{1}{2}\operatorname{Re}\left(\vec{F} \bullet \vec{G}^*\right)$, *where* $\langle\ \rangle$ *denotes a time average.*

$$\underline{\vec{F}}\bullet\underline{\vec{G}} = \frac{1}{4}\left(\vec{F}\bullet\vec{G}e^{i2\omega t} + \vec{F}\bullet\vec{G}^* + c.c.\right) \qquad \text{(S2.9)}$$

Note that any complex number, z, plus its complex conjugate, z^*, is $z+z^*=2\mathrm{Re}(z)$. Therefore,

$$\underline{\vec{F}}\bullet\underline{\vec{G}} = \frac{1}{2}\left(\mathrm{Re}\left(\vec{F}\bullet\vec{G}e^{i2\omega t}\right) + \mathrm{Re}\left(\vec{F}\bullet\vec{G}^*\right)\right). \qquad \text{(S2.10)}$$

The first term on the right hand side is proportional to cos(2ωt) and it therefore time-averages to zero. Hence,

$$\left\langle \underline{\vec{F}}\bullet\underline{\vec{G}} \right\rangle = \frac{1}{2}\mathrm{Re}\left(\vec{F}\bullet\vec{G}^*\right). \qquad \text{(S2.11)}$$

2.4. *Use Poynting's theorem and Equation 2.39 to determine the phase difference between an incident monochromatic plane-wave field and the induced polarization that leads to (a) maximum attenuation, and (b) maximum gain of the incident beam. Use this result to comment on whether the polarization should lead or lag with respect to the incident field for 1) attenuation and 2) gain.*

Taking the time-average of Poynting's theorem gives

$$\left\langle \int_{\substack{\text{closed}\\\text{surface}}} \underline{\vec{S}}\bullet d\vec{a} \right\rangle = -\left\langle \int \underline{\vec{E}}\cdot\frac{\partial\underline{\vec{P}}}{\partial t}dV \right\rangle \qquad \text{(S2.12)}$$

The left-hand side is the time-averaged power flow out of a closed surface. Hence, when

$$\left\langle \underline{\vec{E}}\cdot\frac{d\underline{\vec{P}}}{dt} \right\rangle < 0 \qquad \text{(S2.13)}$$

more power flows out of the surface than flows in (gain). When this expression is greater than zero net power flows into the surface (attenuation). For monochromatic plane waves,

$$\left\langle \underline{\vec{E}}\cdot\frac{d\underline{\vec{P}}}{dt} \right\rangle = \frac{1}{2}\mathrm{Re}\left(\vec{A}\cdot\left(\frac{d\vec{P}}{dt}\right)^*\right) = \frac{1}{2}\mathrm{Re}\left(i\omega\vec{A}\cdot\vec{P}^*\right) \qquad \text{(S2.14)}$$

(see Problem 2.3) where $\vec{A}$ and $\vec{P}$ are complex amplitudes. Let's simplify the expression by substituting

$\vec{A} = \vec{E}_o e^{i\phi_E}$ and $\vec{P} = \vec{P}_o e^{i\phi_P}$ so that

$$\frac{1}{2}\mathrm{Re}\left(i\omega\vec{A}\cdot\vec{P}^*\right) = \frac{\omega}{2}\vec{A}_o\cdot\vec{P}_o\,\mathrm{Re}\left(ie^{i(\phi_E-\phi_P)}\right) = \frac{\omega}{2}\vec{A}_o\cdot\vec{P}_o\sin\left(\phi_P-\phi_E\right) \qquad \text{(S2.15)}$$

Maximum gain and attenuation occur when P and A are 90^o out of phase. a) Maximum gain occurs when $\left(\phi_P-\phi_E\right) = -\frac{\pi}{2}$ and b) maximum attenuation occurs when $\left(\phi_P-\phi_E\right) = \frac{\pi}{2}$.

Equation (S2.15) also shows that when the polarization phase is greater than the field (ie leads in phase) then attenuation occurs, and gain occurs when the polarization lags with respect to the field.

2.5. *Show that absorption is linear in the sense that when two inputs at different frequencies are present in a medium that*

$$\left\langle \underline{\vec{E}}_{total}\cdot\frac{d\underline{\vec{P}}_{total}}{dt} \right\rangle = \left\langle \underline{\vec{E}}_1\cdot\frac{d\underline{\vec{P}}_1}{dt} \right\rangle + \left\langle \underline{\vec{E}}_2\cdot\frac{d\underline{\vec{P}}_2}{dt} \right\rangle. \qquad \textit{2.140}$$

Use monochromatic plane waves for $\underline{\vec{E}}_1(\omega_1)$ *and* $\underline{\vec{E}}_2(\omega_2)$. *The polarization's complex amplitude is related to the field via* $\vec{P} = \varepsilon_o\chi\vec{A}$. *Assume that the susceptibility is the same for both frequencies.*

The total field is given by,

$$\underline{\vec{E}}_{Tot} = \frac{1}{2}\left[\vec{A}_1 e^{-i\omega_1 t} + \vec{A}_2 e^{-i\omega_2 t} + \text{c.c.}\right]. \tag{S2.16}$$

The time derivative of the total polarization is, given our assumptions,

$$\frac{d\underline{\vec{P}}_{Tot}}{dt} = \frac{1}{2}\left[-i\varepsilon_o\chi\omega_1\vec{A}_1 e^{-i\omega_1 t} + -i\varepsilon_o\chi\omega_2\vec{A}_2 e^{-i\omega_2 t} + \text{c.c.}\right]. \tag{S2.17}$$

We take the dot-product of Equations (S2.16) and (S2.17),

$$\underline{\vec{E}}_{Tot} \bullet \frac{d\underline{\vec{P}}_{Tot}}{dt} = \frac{1}{4}\left[\vec{A}_1 e^{-i\omega_1 t} + \vec{A}_2 e^{-i\omega_2 t} + \text{c.c.}\right] \bullet \left[-i\varepsilon_o\chi\omega_1\vec{A}_1 e^{-i\omega_1 t} - i\varepsilon_o\chi\omega_2\vec{A}_2 e^{-i\omega_2 t} + \text{c.c.}\right]. \tag{S2.18}$$

Taking the time average of Equation (S2.18) leaves us with

$$\left\langle \underline{\vec{E}}_{Tot} \bullet \frac{d\underline{\vec{P}}_{Tot}}{dt} \right\rangle = \frac{1}{4} i\varepsilon_o\left(\chi^* - \chi\right)\left[\omega_1\left|\vec{A}_1\right|^2 + \omega_2\left|\vec{A}_2\right|^2\right] = \frac{1}{2}\varepsilon_o \operatorname{Im}\left(\chi\right)\left[\omega_1\left|\vec{A}_1\right|^2 + \omega_2\left|\vec{A}_2\right|^2\right]. \tag{S2.19}$$

To evaluate the right-hand side of 2.140, we use Equation (S2.11) for each term, for example,

$$\left\langle \underline{\vec{E}}_1 \cdot \frac{d\underline{\vec{P}}_1}{dt} \right\rangle = \frac{1}{2}\operatorname{Re}\left(i\varepsilon_o\omega_1\chi^*\left|\vec{A}_1\right|^2\right) = \frac{1}{2}\varepsilon_o\omega_1 \operatorname{Im}\left(\chi\right)\left|\vec{A}_1\right|^2. \tag{S2.20}$$

Hence

$$\left\langle \underline{\vec{E}}_1 \cdot \frac{d\underline{\vec{P}}_1}{dt} \right\rangle + \left\langle \underline{\vec{E}}_2 \cdot \frac{d\underline{\vec{P}}_2}{dt} \right\rangle = \frac{1}{2}\varepsilon_o \operatorname{Im}\left(\chi\right)\left[\omega_1\left|\vec{A}_1\right|^2 + \omega_2\left|\vec{A}_2\right|^2\right], \tag{S2.21}$$

and this equation is equal to the left hand side of Equation 2.140 as shown in (S2.19).

2.6. Show that, when the electric field is of the form $\operatorname{Re}\left\{E_o e^{i(\vec{k}\bullet\vec{r}-\omega t)}\hat{E}\right\}$

$$\nabla \bullet \underline{\vec{E}} = \operatorname{Re}\left(i\vec{k}\bullet\vec{E}_o e^{i\left(\vec{k}\bullet\vec{r}-\omega t\right)}\right)$$

$$\nabla \times \underline{\vec{E}} = \operatorname{Re}\left(i\vec{k}\times\vec{E}_o e^{i\left(\vec{k}\bullet\vec{r}-\omega t\right)}\right)$$

$$\nabla^2 \underline{\vec{E}} = \operatorname{Re}\left(-\left|\vec{k}\right|^2 \vec{E}_o e^{i\left(\vec{k}\bullet\vec{r}-\omega t\right)}\right)$$

That is, for monochromatic plane waves $\nabla \to i\vec{k}$ *and* $\frac{\partial}{\partial t} \to -i\omega$.

These relations may be directly written out. Another approach is to consider that any combination of $\vec{\nabla}$ operations results in a function of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \text{and} \frac{\partial}{\partial z}$, which we will call $f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. To see the result of this function operating on a monochromatic field of the form, $E_o e^{i(\vec{k}\bullet\vec{r}-\omega t)}\hat{E}$, we note that the only spatial dependence comes in the $\vec{k}\bullet\vec{r}$ term. Hence, $f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ operating on $E_o e^{i(\vec{k}\bullet\vec{r}-\omega t)}\hat{E}$ is written as

$$\left(E_o\hat{E}\right) f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \exp\left\{i\left(k_x x + k_y y + k_z z - \omega t\right)\right\}. \tag{S2.22}$$

The action of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \text{and} \frac{\partial}{\partial z}$ on the exponential term is illustrated by considering

$$\frac{\partial}{\partial x}\left[\exp i\left(k_x x + k_y y + k_z z - \omega t\right)\right] = ik_x\left[\exp i\left(k_x x + k_y y + k_z z - \omega t\right)\right].$$

Hence $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \text{and} \frac{\partial}{\partial z}$ operating on $e^{i(\vec{k}\bullet\vec{r}-\omega t)}$ give back $e^{i(\vec{k}\bullet\vec{r}-\omega t)}$ multiplied by ik_x, ik_y, and ik_z respectively. Since the operations return the $e^{i(\vec{k}\bullet\vec{r}-\omega t)}$ multiplied by a constant, successive operations of partial derivatives result in $e^{i(\vec{k}\bullet\vec{r}-\omega t)}$ multiplied by the appropriate constants. Therefore $f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)e^{i(\vec{k}\bullet\vec{r}-\omega t)} = f\left(ik_x, ik_y, ik_z\right)e^{i(\vec{k}\bullet\vec{r}-\omega t)}$. Similarly,

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\mathrm{Re}\left(\vec{E}_o e^{i(\vec{k}\bullet\vec{r}-\omega t)}\right) = \mathrm{Re}\left(f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\vec{E}_o e^{i(\vec{k}\bullet\vec{r}-\omega t)}\right) = \mathrm{Re}\left(f\left(ik_x, ik_y, ik_z\right)\vec{E}_o e^{i(\vec{k}\bullet\vec{r}-\omega t)}\right). \quad \text{(S2.23)}$$

Hence, we may identify,

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \rightarrow ik_x\hat{x} + ik_y\hat{y} + ik_z\hat{z} = i\vec{k} \quad \text{(S2.24)}$$

A similar argument yields, $\frac{\partial}{\partial t} \rightarrow -i\omega$.

2.7. Use the Kronecker-delta properties to prove that $\chi_{\alpha\beta} = R_{i\alpha}R_{j\beta}\chi'_{ij}$ *can be "inverted" to give* $\chi'_{ij} = R_{i\alpha}R_{j\beta}\chi_{\alpha\beta}$.

Operate on both sides,

$$R_{i\alpha'}\chi_{\alpha\beta} = R_{i\alpha'}R_{i\alpha}R_{j\beta}\chi'_{ij} = \delta_{\alpha\alpha'}R_{j\beta}\chi'_{ij} \quad \text{(S2.25)}$$

The Kronecker-delta gives α=α',

$$R_{i\alpha}\chi_{\alpha\beta} = R_{j\beta}\chi'_{ij} \quad \text{(S2.26)}$$

Similarly, we operate on both sides again,

$$R_{j\beta'}R_{i\alpha}\chi_{\alpha\beta} = R_{j\beta'}R_{j\beta}\chi'_{ij} = \delta_{\beta\beta'}\chi'_{ij} \quad \text{(S2.27)}$$

The Kronecker-delta gives β=β', and we have the final result,

$$R_{j\beta}R_{i\alpha}\chi_{\alpha\beta} = \chi'_{ij}. \quad \text{(S2.28)}$$

2.8. Use the constitutive relationship, Equation 2.5, along with Equation 2.45 to prove that $\varepsilon_{ij} = \varepsilon_o\left(\delta_{ij} + \chi_{ij}\right)$.

Substituting Equation 2.45 into Equation 2.5 (written in component form) gives

$$D_i = \varepsilon_o A_i + \varepsilon_o \chi_{ij} A_j \quad \text{(S2.29)}$$

This is equivalent to

$$D_i = \varepsilon_o \delta_{ij} A_j + \varepsilon_o \chi_{ij} A_j = \varepsilon_o\left(\delta_{ij} + \chi_{ij}\right)A_j. \quad \text{(S2.30)}$$

Furthermore, $D_i = \varepsilon_{ij}A_j$ so that we may equate

$$\varepsilon_{ij} = \varepsilon_o\left(\delta_{ij} + \chi_{ij}\right) \quad \text{(S2.31)}$$

2.9. In Chapter 3, we show that the second order nonlinear polarization is given in component form as

$$P_i^{(2)} = \varepsilon_o \chi_{ijk}^{(2)} A_j A_k, \quad \textit{2.145}$$

where $\chi_{ijk}^{(2)}$ *is the second order susceptibility and* A_j *is a component of the incident field's complex amplitude. Use Equation2.145 to prove that* $\chi_{ijk}^{(2)}$ *is a rank 3 tensor.*

We prove that $\chi_{ijk}^{(2)}$ is a rank 3 tensor by showing that it transforms like a tensor. We start by noting that the polarization and field are vectors and hence we can rewrite them according to tensor transformations. Specifically, we may write them with respect to a different coordinate system. For the purposes of this example, ijk refer to one coordinate system and αβγ to the other. A given polarization transforms like,

$$P_i^{(2)} = R_{i\alpha} P_\alpha^{(2)} . \tag{S2.32}$$

We transform the polarization and fields in Equation 2.145,

$$R_{i\alpha} P_\alpha^{(2)} = \varepsilon_o \chi_{ijk}^{(2)} \left(R_{j\beta} A_\beta \right) \left(R_{k\gamma} A_\gamma \right) = \varepsilon_o R_{j\beta} R_{k\gamma} \chi_{ijk}^{(2)} A_\beta A_\gamma \tag{S2.33}$$

Next operate on both sides,

$$R_{i'\alpha} R_{i\alpha} P_\alpha^{(2)} = \delta_{i'i} P_\alpha^{(2)} = \varepsilon_o R_{i'\alpha} R_{j\beta} R_{k\gamma} \chi_{ijk}^{(2)} A_\beta A_\gamma \tag{S2.34}$$

Hence,

$$P_\alpha^{(2)} = \varepsilon_o \left(R_{i\alpha} R_{j\beta} R_{k\gamma} \chi_{ijk}^{(2)} \right) A_\beta A_\gamma \tag{S2.35}$$

In the transformed coordinate system we identify,

$$R_{i\alpha} R_{j\beta} R_{k\gamma} \chi_{ijk}^{(2)} = \chi_{\alpha\beta\gamma}^{(2)} \tag{S2.36}$$

Next we "invert" the relationship by operating on both sides (see also Problem 2.7),

$$R_{i\alpha'} R_{i\alpha} R_{j\beta} R_{k\gamma} \chi_{ijk}^{(2)} = R_{i\alpha'} \chi_{\alpha\beta\gamma}^{(2)} \tag{S2.37}$$

Note that $R_{i\alpha'} R_{i\alpha} = \delta_{\alpha\alpha'}$ so that $R_{j\beta} R_{k\gamma} \chi_{ijk}^{(2)} = R_{i\alpha} \chi_{\alpha\beta\gamma}^{(2)}$. Similarly, we operate on both sides of this equation with $R_{j\beta'}$ and $R_{k\gamma'}$ yielding

$$\chi_{ijk}^{(2)} = R_{i\alpha} R_{j\beta} R_{k\gamma} \chi_{\alpha\beta\gamma}^{(2)} . \tag{S2.38}$$

Therefore the second order susceptibility transforms as a rank 3 tensor.

2.10. *Find the principal indices of refraction (eigenvalues) and the direction of the principal axes (eigenvectors) for the following relative dielectric tensor,* $\vec{\vec{\varepsilon}}_r$ *,*

$$\begin{bmatrix} 2.5 & .5 & 0 \\ .5 & 2.5 & 0 \\ 0 & 0 & 4 \end{bmatrix} . \tag*{2.146}$$

The eigenvalues are found from

$$\begin{vmatrix} 2.5-\lambda & .5 & 0 \\ .5 & 2.5-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \tag{S2.39}$$

Solving for λ gives λ_1=2, λ_2=3, and λ_3=4.

The corresponding indices are $n_1 = \sqrt{2}$, $n_2 = \sqrt{3}$, and $n_3 = 2$. The eigenvector corresponding to

$$\lambda_1 \text{ is } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} , \tag{S2.40}$$

$$\text{for } \lambda_2 \text{ is } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} , \tag{S2.41}$$

$$\text{and for } \lambda_3 \text{ is } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{S2.42}$$

2.11. *An electric field in a material that has a dielectric tensor as given in Problem 2.10 has a complex amplitude,* $\vec{E} = \left(\hat{x} + \hat{y} + \hat{z} \right) \frac{E_o}{\sqrt{3}}$. *What is the complex amplitude of the displacement vector,* $\vec{D}$*? Show that* $\vec{D}$ *and* $\vec{E}$ *are not parallel, and find the angle between them.*

We use $D_i = \varepsilon_o \varepsilon_{ij} E_j$, which in matrix form is,

$$\vec{D} = \varepsilon_o \begin{bmatrix} 2.5 & .5 & 0 \\ .5 & 2.5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{E_o}{\sqrt{3}} = \varepsilon_o \frac{\left(3\hat{x} + 3\hat{y} + 4\hat{z}\right)}{\sqrt{3}} E_o \tag{S2.43}$$

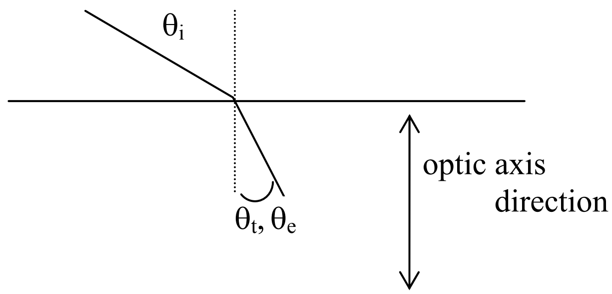
Clearly D and E are not parallel. We find the angle between the two vectors by taking the dot-product of their unit-vectors:

$$\cos\rho = \hat{D}\bullet\hat{E} = \frac{\left(3\hat{x}+3\hat{y}+4\hat{z}\right)}{\sqrt{34}} \bullet \frac{\left(\hat{x}+\hat{y}+\hat{z}\right)}{\sqrt{3}} = \frac{10}{\sqrt{102}} = 0.99$$

The angle is 0.14 rads or 8.0^o.

2.12. *A uniaxial crystal of indices n_o and n_z is cut so that the optic axis is perpendicular to the surface. Show that for a beam incident on the interface from air at an angle of incidence, θ_i, that the angle of refraction of the e-wave is*

$$\tan\theta_e = \frac{n_z}{n_o} \frac{\sin\theta_i}{\sqrt{n_z^{\,2} - \sin^2\theta_i}} \,. \tag{2.147}$$



Since the optic axis and the surface normal are parallel, $\theta_e = \theta_t$. Snell's law is

$$n_i \sin\theta_i = n_t \sin\theta_t \left(= n_e \sin\theta_e\right)$$

where n_i=1 and we substitute in for the extraordinary index,

$$\sin\theta_i = \left(\frac{\cos^2\theta_e}{n_o^2} + \frac{\sin^2\theta_e}{n_z^2}\right)^{-1/2} \sin\theta_e \tag{S2.44}$$

Rearranging gives,

$$\left(\frac{1}{\tan^2\theta_e n_o^2} + \frac{1}{n_z^2}\right) = \frac{1}{\sin^2\theta_i} \tag{S2.45}$$

Solving for $\tan\theta_e$,

$$\tan\theta_e = \frac{n_z}{n_o} \frac{\sin\theta_i}{\sqrt{n_z^{\,2} - \sin^2\theta_i}} \tag{S2.46}$$

2.13. *The law of reflection in crystals is not always intuitively obvious. In a derivation similar to that for Snell's law, reflections satisfy the equation* $n_i \sin\theta_i = n_r \sin\theta_r$*, where θ_i and θ_r are the incident angle and reflected angle with respect to the surface normal respectively; n_i and n_r are the indices for the incident and reflected beams. As the following example shows, in crystals it is possible that* $n_i \neq n_r$ *and therefore* $\theta_r \neq \theta_i$. *A right prism, shown in Figure 2.13, is made out of a uniaxial birefringent crystal with the optic axis as shown in the figure. The principal indices are n_o=2.2 and n_z=2.1 and the prism angles are 45 ^{o} and 90^o. The incident beam of light is linearly polarized in the vertical direction (parallel to the optic axis). Calculate the reflected angle with respect to the reflected surface normal (dotted line). Note that it is not 45^o!*

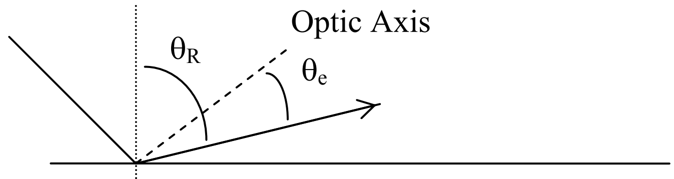


Figure S2.1. Geometry for Problem 2.13.

Figure S2.1 shows the geometry with respect to the reflecting interface. The incident angle is 45°. Let's assume that the reflected angle is slightly larger than 45° as shown in Figure S2.1. We have

$$n_i \sin\theta_i = n_R \sin\theta_R = n_R(\theta_e)\sin\left(\theta_e + \frac{\pi}{4}\right). \tag{S2.47}$$

The incident beam is polarized along the crystal's z-axis so $n_i=n_z$. The incident angle is $\theta_i=\pi/4$, and we use a standard trig identity to expand,

$$\sin\left(\theta_e + \frac{\pi}{4}\right) = \frac{\sin\theta_e + \cos\theta_e}{\sqrt{2}} \tag{S2.48}$$

Making the above substitutions along with the extraordinary index of the reflected wave in Equation (S2.47) gives

$$n_z = \left(\frac{\cos^2\theta_e}{n_o^2} + \frac{\sin^2\theta_e}{n_z^2}\right)^{-1/2}(\sin\theta_e + \cos\theta_e) \tag{S2.49}$$

After some algebraic manipulation we obtain,

$$\tan\theta_e = \frac{n_z^2 - n_o^2}{2n_o^2} \tag{S2.50}$$

Substituting in n_o=2.2 and n_z=2.1 gives θ_e=-44 mrad (=-2.5°). So the reflected angle is θ_R=42.5°.

2.14. *A linear-optical effect that plays a role in ultrashort-pulse nonlinear optics is the group velocity mismatch between different carrier frequencies. For example, in second harmonic generation the fundamental and second harmonic will temporally separate due to group velocity mismatch. The group velocity is given by* $v_g = \frac{\partial\omega}{\partial k}$*, however, it is easier to calculate inverse group velocity* $v_g^{-1} = \frac{\partial k}{\partial\omega}$*. Plot the group velocity as a function of wavelength for an o-wave traveling through BBO (see Appendix B for Sellmeier Equations. Based on the group velocity evaluated at 0.532 and 1.064μm, over what propagation distance in the BBO will a 100 fs pulse at 0.532 temporally separate from a 100 fs 1.064μm pulse? Define temporal separation when the pulses are separated by 100 fs.*

The inverse group velocity is given by

$$v_g^{-1} = \frac{\partial k}{\partial\omega} = \frac{n}{c}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right) \tag{S2.51}$$

Using the Sellmeier Equations in Appendix B allows us to obtain n(λ) and dn/dλ. A plot of the group velocity is given in Figure S2.2. A direct calculation at 1.064 and 0.532 μm gives for the group velocity,

$$v_g(1.064\mu m) = 1.972\times10^8\, m/s$$

$$v_g(0.532\mu m) = 1.742\times10^8\, m/s$$

A 100 fsec pulse corresponds to approximately 30 μm spatial extent (in vacuum). Therefore the pulses are separated when their peaks are spaced roughly 30 μm apart. The separation between the two pulses is

$$\Delta x = |v_1 - v_2|t \tag{S2.52}$$

where v_1 and v_2 are the group velocities and t=0 corresponds to when the two pulses are coincident in time. Using Δx=30μm and using the group velocities given above, we obtain $t=6.05\times10^{-12}$s as the period of time it takes the two pulses to separate. In this time the 1.064 μm pulse travels 1.08 mm and the 0.532 μm pulse travels 1.05 mm. Therefore after 1 mm the two pulses separate. Note that a longer interaction length is possible by timing the pulses so that the 0.532 μm pulse arrives first (it travels slower). In this way, a 2 mm interaction length is possible.

c) From the plot, λ_F=2.03 μm and λ_{SHG}=1.015 have the same group velocity.

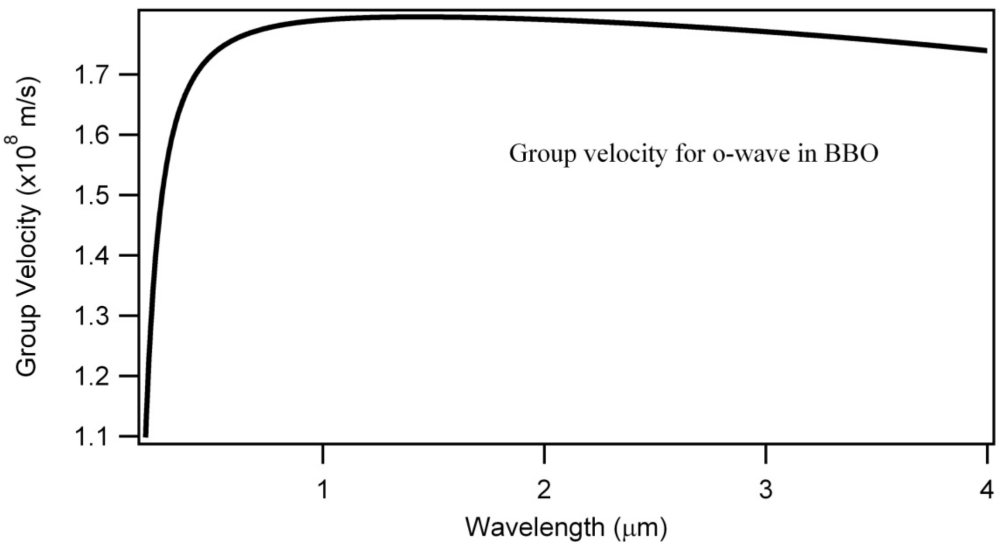


Figure S2.2. Group velocity in BBO.

2.15. Derive the wave equation in an isotropic medium for the case where j_f is not equal to zero. Assume monochromatic plane waves and use Ohm's law to relate the complex amplitudes of the free current density and the field, ie. $\vec{j}_F = \sigma\vec{A}$.

a) Show that the resultant index of refraction is complex.

b) Show that the imaginary part of the index leads to attenuation as the field propagates through the medium.

a) The wave equation is

$$\nabla\times\left(\nabla\times\vec{E}\right) = -\mu_o \frac{\partial}{\partial t}\nabla\times\vec{H} \,. \tag{S2.53}$$

Similar to the derivation for the lossless case, this equation becomes

$$-\nabla^2\vec{E} = -\mu_o \frac{\partial}{\partial t}\left(j_f + \frac{\partial\vec{D}}{\partial t}\right). \tag{S2.54}$$

The complex amplitude of $\vec{D}$ is related to that of the field, $\vec{E}$ via $\vec{D} = \varepsilon_o\varepsilon_r\vec{A}$. For monochromatic plane waves of the form $\vec{A}e^{i\left(\vec{k}\bullet\vec{r}-\omega t\right)}$ we replace $\vec{\nabla}\to i\vec{k}$ and $\frac{\partial}{\partial t}\to -i\omega$. After making these substitutions we arrive at,

$$\left|\vec{k}\right|^2 = \left(i\omega\mu_o\sigma_o + \frac{\varepsilon_r\omega^2}{c^2}\right) = \frac{\varepsilon_r\omega^2}{c^2}\left(1 + i\frac{\sigma_o}{\varepsilon_o\varepsilon_r\omega}\right). \tag{S2.55}$$

We see that the plane wave solution requires a complex k-vector. From the relationship, $\tilde{k}=\frac{\tilde{n}\omega}{c}$, we obtain the complex index,

$$\tilde{n}=\frac{\sqrt{\varepsilon_r}\,\omega}{c}\left(1+i\frac{\sigma_o}{\varepsilon_o\varepsilon_r\omega}\right)^{1/2}. \tag{S2.56}$$

b) Rewriting $\tilde{n}=n+i\kappa$ gives $\tilde{k}=(n+i\kappa)\frac{\omega}{c}$. Therefore the plane wave solution is of the form

$$\vec{A}\exp i\left((n+i\kappa)\frac{\omega}{c}z-\omega t\right)=\vec{A}\exp(-\kappa z)\exp i\left(\frac{n\omega}{c}z-\omega t\right) \tag{S2.57}$$

where we have chosen the direction of propagation to be the z-direction. This solution shows that the field exponentially decays with a decay constant given by the imaginary part of the index.

2.16. *A new transparent material has been discovered and before nonlinear optics experiments can be performed, we need to know it's dispersion curve. Measurements of the index of refraction are made at five wavelengths (using five different lasers). Table 2.1below gives the results of the measurements.*

Table2.1 Index as a function of wavelength

Wavelength (μm)	Index of refraction
0.330	2.47379
0.532	2.23421
0.6328	2.20271
1.064	2.15601
1.57	2.1375
2.804	2.1029

Fit the above data using a two-pole Sellmeier equation,

$$n^2(\lambda)=A_o+\frac{\lambda^2 A_1}{\lambda^2\text{-}A_2}+\frac{\lambda^2 A_3}{\lambda^2\text{-}A_4}$$

where λ is kept in units of microns. The stability of the fitting routine requires reasonable guesses for the fitting parameters. A_2 and A_4 correspond to the IR and UV poles, so set the starting guesses accordingly. If we assume that the dispersion is relatively "slow" then the index should not be a strong function of wavelength. From the data we see that it has a value centered on approximately 2.2. Hence a good guess for A_o is ~2.2^2. After performing the fitting routine, give the values for the fit parameters A_0 through A_4. Plot the measured and fitted index of refraction as a function of wavelength on the same graph. Check to be sure that A_2 and A_4 give poles in the IR and UV respectively.

The fit and fitting parameters are shown in Figure S2.3. From the fit values we see a UV pole at 0.211μm and an IR pole at 114 μm.

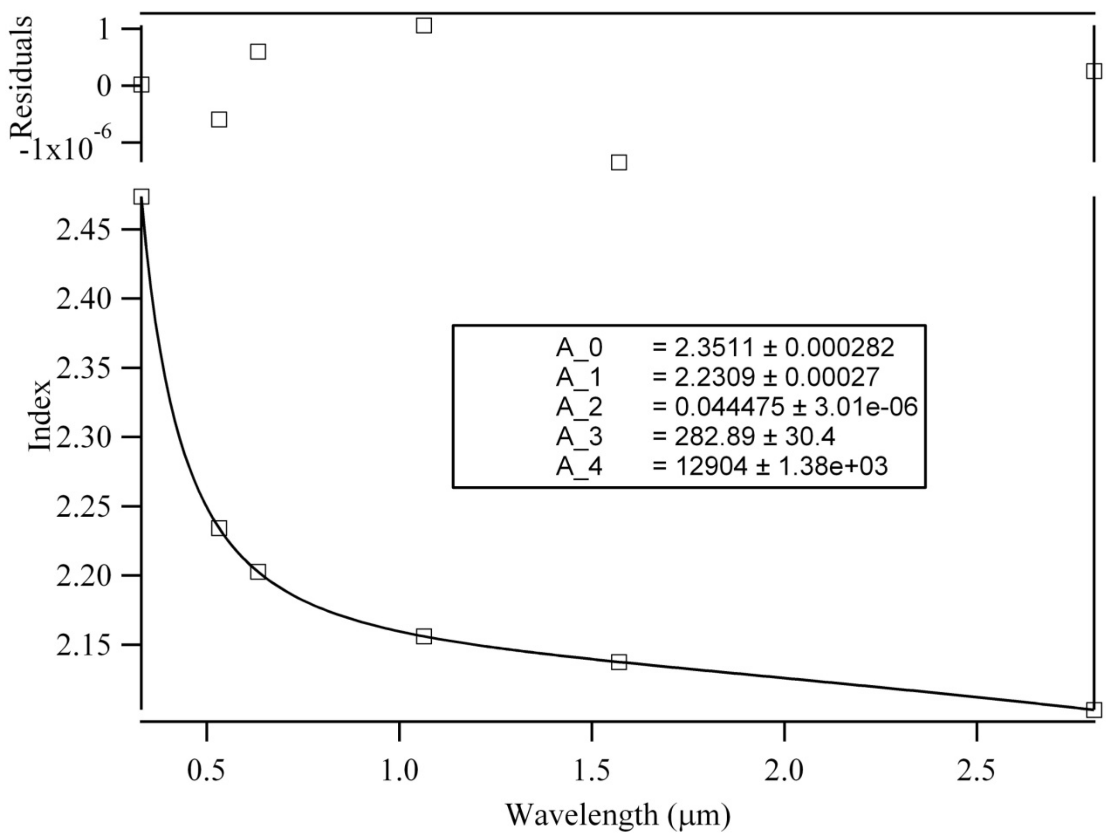


Figure S2.3. Data and fit solution for Problem 2.16.

2.17. Use Equation2.100 to show that the two eigen-polarizations are orthogonal. Hint, show that the D-vectors corresponding to the two eigen-indices are orthogonal.

Fresnel's equation states:

$$\frac{s_x^2}{\frac{1}{n_x^2}-\frac{1}{n^2}}+\frac{s_y^2}{\frac{1}{n_y^2}-\frac{1}{n^2}}+\frac{s_z^2}{\frac{1}{n_z^2}-\frac{1}{n^2}}=0\text{, or }\sum_i \frac{s_i^2}{\frac{1}{n_i}-\frac{1}{n^2}}=0 \qquad \text{(S2.58)}$$

This equation has two solutions, which we call n_1, and n_2. The D-vectors associated with these values of n are perpendicular by showing that their dot-product is zero.
Equation 2.100 is,

$$D_i=\frac{\left(\hat{k}\bullet\vec{E}\right)s_i}{\mu c^2\left[\frac{1}{n_i^2}-\frac{1}{n^2}\right]} \qquad \text{(S2.59)}$$

And hence

$$\vec{D}_{n1}\bullet\vec{D}_{n2}=\sum_i\frac{1}{\mu c^2}\frac{\left(\hat{k}\bullet\vec{E}\right)s_i^2}{\left(\frac{1}{n_o^2}-\frac{1}{n_1^2}\right)\left(\frac{1}{n_i^2}-\frac{1}{n_2^2}\right)} \qquad \text{(S2.60)}$$

Using the method of partial fractions,

$$\vec{D}_{n1}\cdot\vec{D}_{n2}=\frac{(\hat{k}\bullet\vec{E})}{\mu c^2}\frac{n_1^2 n_2^2}{n_2^2\text{-}n_1^2}\sum_i s_i^2\left[\frac{1}{\left(\frac{1}{n_i^2}\text{-}\frac{1}{n_1^2}\right)}-\frac{1}{\left(\frac{1}{n_i^2}\text{-}\frac{1}{n_2^2}\right)}\right]=\frac{(\hat{k}\bullet\vec{E})}{\mu c^2}\frac{n_1^2 n_2^2}{n_2^2\text{-}n_1^2}\left[\sum_i\frac{s_i^2}{\left(\frac{1}{n_i^2}\text{-}\frac{1}{n_1^2}\right)}-\sum_i\frac{s_i^2}{\left(\frac{1}{n_i^2}\text{-}\frac{1}{n_2^2}\right)}\right]$$

Notice that both sums in the square brackets are solutions to Fresnel's equation (n_1 and n_2) and hence are both equal to zero. Therefore $\vec{D}_{n1}\cdot\vec{D}_{n2}=0$ proving that $\vec{D}_{n1}$ is perpedicular to $\vec{D}_{n2}$.

2.18. Calculate the angle where n_e=n_o in the xz plane of a biaxial crystal. This angle is the angle of the optic axis with respect to the z-axis. Use the Sellmeier equations for KTP to plot the angle of the optic axis with respect to the z-axis as a function of wavelength from 0.5 to 4.0 μm.

The intersection is given by

$$\frac{\cos^2\theta}{n_x^2}+\frac{\sin^2\theta}{n_z^2}=\frac{1}{n_y^2} \tag{S2.61}$$

Solving for angle gives,

$$\sin^2\theta=\frac{n_z^2}{n_y^2}\left(\frac{n_x^2-n_y^2}{n_x^2-n_z^2}\right) \tag{S2.62}$$

The dispersion of the optic-axis angle is shown in Figure S2.4.

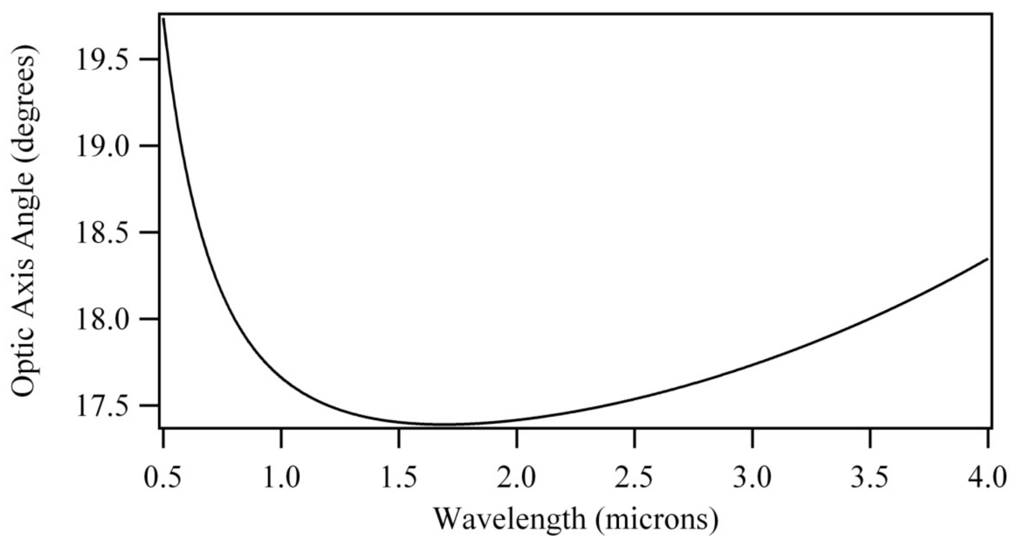


Figure S2.4. Dispersion of the optic axis angle.

2.19. Give expressions for n_e in each of the principal planes of a biaxial crystal. In the xz- and yz- planes, assume that the k-vector propagates at an angle, θ, with respect to the z-axis. In the xy-plane, assume that the k-vector propagates at an angle, ϕ, with respect to the x-axis.

xy-plane:

$$n_e(\phi)=\left[\frac{\cos^2\phi}{n_y^2}+\frac{\sin^2\phi}{n_x^2}\right]^{-1/2} \tag{S2.63}$$

xz-plane:

$$n_e(\theta)=\left[\frac{\cos^2\theta}{n_x^2}+\frac{\sin^2\theta}{n_z^2}\right]^{-1/2} \tag{S2.64}$$

yz-plane:

$$n_e(\theta) = \left[\frac{\cos^2\theta}{n_y^2} + \frac{\sin^2\theta}{n_z^2}\right]^{-1/2} \qquad \text{(S2.65)}$$

2.20. Nonlinear optical interactions typically involve multiple beams at multiple wavelengths. If the beams have different polarizations, then they may walk off each other as the following example demonstrates. Two beams of light, one at 600 nm (o-polarized) and the other at 300 nm (e-polarized), are sent through a BBO crystal with their k-vectors at an angle of 40.5 degrees to the optic axis. The BBO crystal is 2 cm long. The beam diameters are the same and are given by "D". Determine the distance in the crystal where the two beams walk off each other. Define beam separation when the centers of the beams are displaced by one beam diameter. For what beam diameters is the full crystal length used? Sellmeier equations for BBO are found in Appendix B.

The beams walk off each other (using the definitions of walked-off as given in the problem statement) at a distance,

$$L = \frac{D}{\tan\rho} \qquad \text{(S2.66)}$$

where ρ is the walk-off angle of the e-wave. The walk-off angle is given by Equation 2.106,

$$\rho = \tan^{-1}\left(\frac{n_o^2}{n_z^2}\tan\theta\right) - \theta\,. \qquad \text{(S2.67)}$$

The walk-off angle is evaluated for the e-wave at 300 nm (the o-wave at 600 nm has no walk-off). The walk-off angle is, 82mrad (4.73^o). So the effective crystal length is 12D.

2.21. In the following, use the Sellemeier Equations for BBO found in Appendix B.

a) Make a plot $n_e(\theta)$ from θ=0 to $\pi/2$ for both λ=1.064 μm and λ=0.532μm. Include on the same graph $n_o(\lambda)$ for both wavelengths.

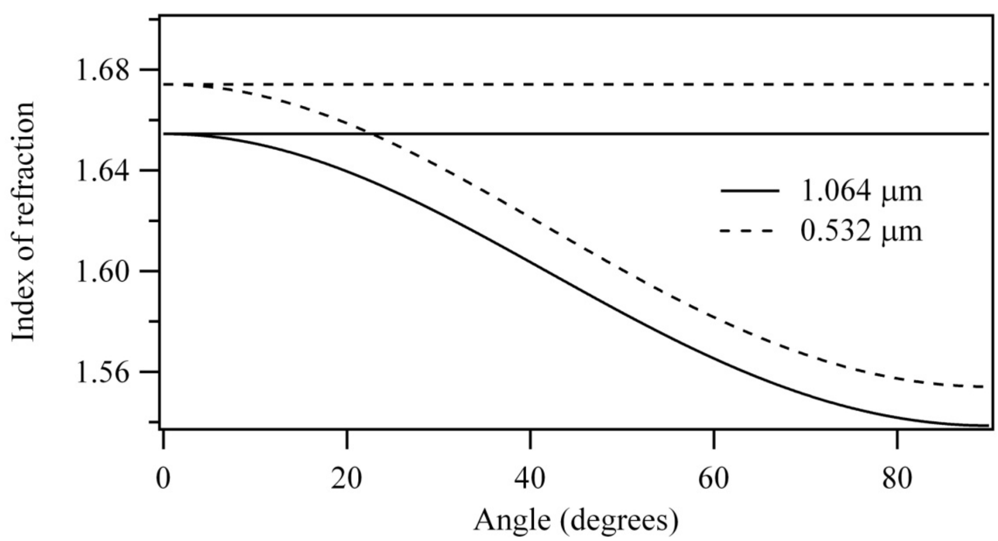


Figure S2.5. Plot of n_e and n_o for 1.064 and 0.532 μm. The horizontal lines correspond to the ordinary index.

b) Is BBO positive or negative uniaxial?

Negative uniaxial

c) For what polarization states and for what angle does n(1.064) = n(0.532)? Hint, this corresponds to the intersection of two of the curves in part (a). This angle is the phase matching angle for second harmonic generation.

When the 0.532 μm is an e-wave and the 1.064 μm beam is an o-wave the indices are the same at 22.7°.

2.22. For nonlinear interactions in birefringent crystals, walkoff is an important factor. Plot the walk-off angle (in degrees) as a function of crystal orientation (from 0 to 90°) for the crystal, BBO (see Appendix B for Sellemeier equations). Use a wavelength of 0.6μm. The orientation of the crystal is defined as the angle between the k-vector and the optic axis of the crystal. Where is walkoff zero?

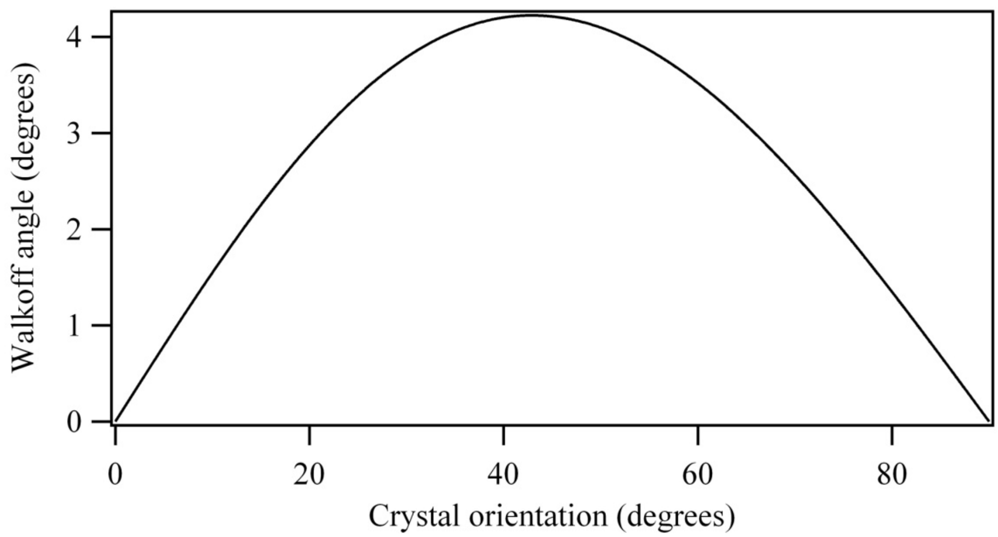


Figure S2.6. Plot of walkoff in BBO.
Walk-off is zero at 0 and 90°.

2.23. Assume a beam waist is incident on a lens of focal length, f. The beam propagates a distance, d, where it is incident on a crystal that has an index n. Provided that d is less than the focal length, prove that the focused spot size in the crystal is the same as the focused spot size without a crystal. Also assume that the initial beam waist incident on the lens is large enough that the focused spot in air is a distance of approximately f away from the lens.

The ABCD matrix for the beam transformation going from the initial beam waist to the focused beam location is,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f} - \frac{d_2}{nf} & d + \frac{d_2}{n} \\ -\frac{1}{nf} & \frac{1}{n} \end{bmatrix} \tag{S2.68}$$

The starting q is a beam waist, so it is purely imaginary and we write it as iz_{R0}, where $z_{R0} = \pi w_o^2 / \lambda$ The q parameter after transformation is (see Equation 2.125),

$$\begin{aligned} q &= \frac{i\left(1 - \frac{d}{f} - \frac{d_2}{nf}\right) z_{R0} + d + \frac{d_2}{n}}{-\frac{iz_{R0}}{nf} + \frac{1}{n}} = \frac{i\left(nf - nd - d_2\right) z_{R0} + dnf + d_2 f}{f - iz_{R0}} \\ &= \frac{\left(dnf + d_2 f\right) f - \left(nf - nd - d_2\right) z_{R0}^2}{f^2 + z_{R0}^2} + i \frac{\left(dnf + d_2 f\right) z_{R0} + \left(nf - nd - d_2\right) z_{R0} f}{f^2 + z_{R0}^2} \end{aligned} \tag{S2.69}$$

$$\frac{1}{q}=\frac{f-iz_{R0}}{(dnf+d_2f)+i(nf-nd-d_2)z_{R0}}$$
$$=\frac{f(dnf+d_2f)-(nf-nd-d_2)z_{R0}^2}{(dn+d_2)^2f^2+(nf-nd-d_2)^2z_{R0}^2}-i\frac{nf^2z_{R0}}{(dn+d_2)^2f^2+(nf-nd-d_2)^2z_{R0}^2} \tag{S2.70}$$

At the focus, the radius of curvature is infinite and hence the real part of 1/q is zero. Therefore,

$$d_2=\frac{(f-d)nz_{R0}^2-ndf^2}{(f^2+z_{R0}^2)} \tag{S2.71}$$

Substituting this expression back into Equation (S2.70) gives (we can ignore the real part since it is zero from the previous step)

$$\frac{1}{q}=-i\frac{f^2+z_{R0}^2}{nf^2z_{R0}}. \tag{S2.72}$$

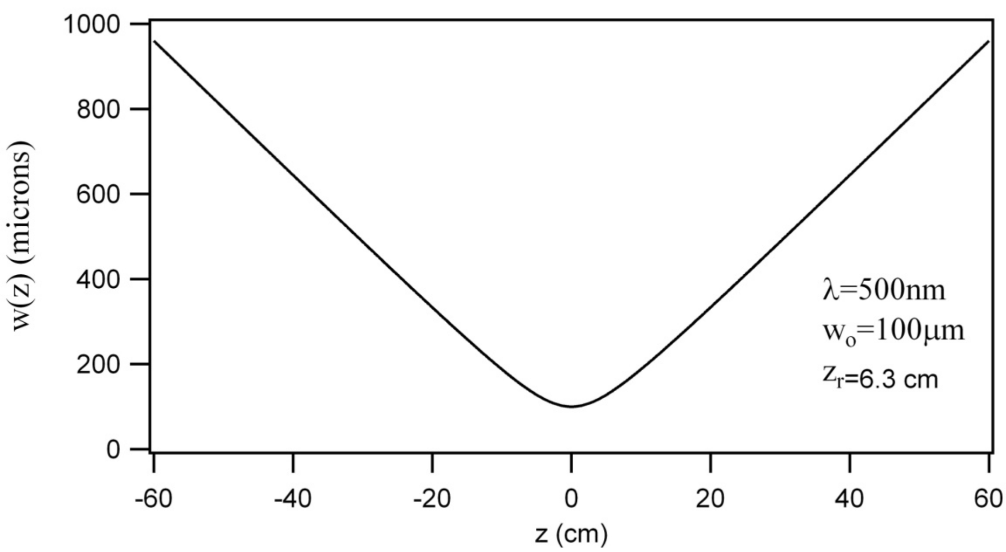
The beam size is found from Equation 2.122,

$$\frac{1}{q}=-i\frac{f^2+z_{R0}^2}{nf^2z_{R0}}=-i\frac{\lambda}{\pi n w_{\text{final}}^2} \tag{S2.73}$$

$$w_{\text{final}}^2=\frac{\lambda}{\pi}\frac{f^2z_{R0}}{f^2+z_{R0}^2} \tag{S2.74}$$

Thus we see that the focused spot size has no dependence on the index of refraction.

2.24. Plot w(z) and 1/R(z) for a Gaussian beam. Plotting the inverse of R(z) avoids problems near z=0 where the radius of curvature is infinite. Assume that the wavelength is 500 nm, and that w_o=100μm. Plot from z=-10z_R to z=+10z_R where z_R is the Rayleigh range.



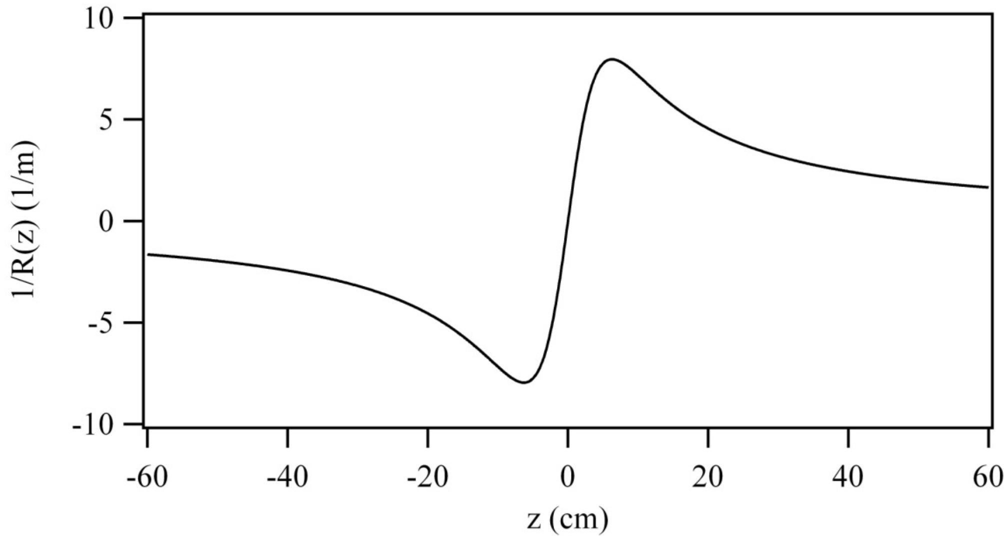


Figure S2.7. Top: w(z) and Bottom 1/R(z) for Problem 2.24.

2.25. *Three Gaussian laser beams are focused to have the same beam waist, w_o=200 μm, at z=0 (radius of curvature=∞). The wavelength of the three lasers are λ_1=250 nm, λ_2=500 nm, λ_3=1000 nm. Find the ratio of the Rayleigh ranges for laser1/laser2, and laser1/laser3, ie $\frac{z_{R1}}{z_{R2}}$, $\frac{z_{R1}}{z_{R3}}$. Use the result to comment on beam overlap issues that arise for nonlinear interactions that require the overlap of beams with different wavelength.*

The ratio of Rayleigh ranges for beams with the same w_o but different wavelengths is

$$\frac{z_{R1}}{z_{R2}} = \frac{\lambda_2}{\lambda_1}, \qquad \frac{z_{R1}}{z_{R3}} = \frac{\lambda_3}{\lambda_1} \tag{S2.75}$$

In the specific cases considered here,

$$\frac{z_{R1}}{z_{R2}} = \frac{\lambda_2}{\lambda_1} = 2, \qquad \frac{z_{R1}}{z_{R3}} = \frac{\lambda_3}{\lambda_1} = 4 \tag{S2.76}$$

Exact beam overlap is possible for only small distances since the different beams at different wavelengths diffract at different rates. Therefore this issue is important in nonlinear interactions where multiple beams at different wavelengths are interacting. The optimum spot sizes for the different beams is usually determined numerically.

2.26. *Consider the optical system shown in Figure 2.14. A Gaussian beam waist (w_o) is located a distance d_1 from a lens of focal length, f. A distance d_2 on the other side of the lens is a second beam waist.*

a) Use the Gaussian beam parameter, q, and the ABCD matrix approach to show that the distance, d_2, is given by the expression

$$d_2 = \frac{\left[\left(d_1 - f\right)d_1 + z_R^2\right]f}{\left(f - d_1\right)^2 + z_R^2}$$

where z_R is determined by the initial beam waist, w_o and is given by $z_R = \frac{\pi w_o^2}{\lambda}$. Hint, at a beam waist what is the radius of curvature and therefore what should the real part of 1/q be?

b) Show that the beam waist located at d_2 is given by the relationship,

$$w_{\text{final}}^2 = \left[\frac{f^2 z_R}{\left(f - d_1\right)^2 + z_R^2}\right]\frac{\lambda}{\pi}$$

c) Consider a laser operating at 1.064 μm that has a beam waist of 200 μm at a location of 1 cm after the laser exit port. A certain nonlinear application requires us to focus this beam to w=130μm. At what location(s) can we put a 10 cm focal length lens to obtain the desired spot size?

a) The ABCD matrix for the focusing system is,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{d_2}{f} & d_1 + d_2 - \frac{d_1 d_2}{f} \\ -\frac{1}{f} & 1-\frac{d_1}{f} \end{bmatrix} \tag{S2.77}$$

The Gaussian beam parameter is transformed according to

$$q_{\text{final}} = \frac{Aq_o + B}{Cq_o + D} = \frac{(f-d_2)q_o + (d_1+d_2)f - d_1 d_2}{f - d_1 - q_o} \tag{S2.78}$$

We note that because the beam starts at a beam-waist, $q_o = iz_R$. Taking $1/q_{\text{final}}$ and setting the real part to zero gives (since the radius of curvature is infinite at the beam waist),

$$(f-d_1)\left[(d_1+d_2)f - d_1 d_2\right] + (f-d_2)z_R^2 = 0 \tag{S2.79}$$

We solve this for d_2 as a function of d_1,

$$d_2 = \frac{\left[(d_1 - f)d_1 + z_R^2\right]f}{(f-d_1)^2 + z_R^2} \tag{S2.80}$$

Note that the geometric limit is obtained by setting z_R=0.

b) The imaginary part of $1/q_{\text{final}}$ is

$$\text{Im}\left[\frac{1}{q_{\text{final}}}\right] = \frac{-(f-d_1)(f-d_2)z_R - \left((d_1+d_2)f - d_1 d_2\right)z_R}{\left((d_1+d_2)f - d_1 d_2\right)^2 + (f-d_2)^2 z_R^2} = -\frac{\lambda}{\pi w_{\text{final}}^2} \tag{S2.81}$$

The following relationships help with the algebra required for Equation (S2.81) and using d_2 from Equation (S2.80):

$$f - d_2 = \frac{f^2(f-d_1)}{(f-d_1)^2 + z_R^2} \tag{S2.82}$$

and

$$(d_1+d_2)f - d_1 d_2 = \frac{f^2 z_R^2}{(f-d_1)^2 + z_R^2} \tag{S2.83}$$

Substituting these equations into (S2.81)gives,

$$\text{Im}\left[\frac{1}{q_{\text{final}}}\right] = -\frac{(f-d_1)^2 + z_R^2}{f^2 z_R} = -\frac{\lambda}{\pi w_{\text{final}}^2} \tag{S2.84}$$

Solving for w_{final},

$$w_{\text{final}}^2 = \left[\frac{f^2 z_R}{(f-d_1)^2 + z_R^2}\right]\frac{\lambda}{\pi} \tag{S2.85}$$

c) Equation (S2.85) is rearranged to solve for d_1 as a function of the given parameters,

$$d_1 = f \pm \sqrt{\frac{w_o^2}{w_{\text{final}}^2} f^2 - z_R^2}$$

For the parameters given, d_1= 19.9 cm or d_1=1.4 mm. The 19.9 cm solution is usually more practical to implement since it gives more working distance between the lens and the 130 μm beam waist.

2.27. *Table 2.1gives data for the beam radius (1/e field radius) of a laser beam as a function of position. The wavelength of the laser is 1.064 μm.*

Table2.1 Beam radius as a function of position

W(μm)	z(cm)
54.4	0.125
44.5	0.135
39.5	0.145
32.5	0.155
26.2	0.165
21.9	0.175
16.3	0.185
12.0	0.195
10.6	0.205
11.3	0.215
14.1	0.225
19.1	0.235
24.0	0.245
31.1	0.255
37.5	0.265
41.7	0.275
48.1	0.285
53.0	0.295
58.0	0.305

a) Plot W vs. z and on the same plot include the curve

$$W(z)=W_o\left[1+M^4\left(\frac{\lambda(z\text{-}z_o)}{\pi W_o^2}\right)^2\right]^{1/2}$$

For this part of the problem set M=1. For the values of W_o, and z_o estimate them based on the data. z_o is the location of the minimum beam waist and W_o is the minimum radius. Tweak the values of W_o and z_o to get a best fit for all the data. Include the best values of z_o and W_o on the plot.

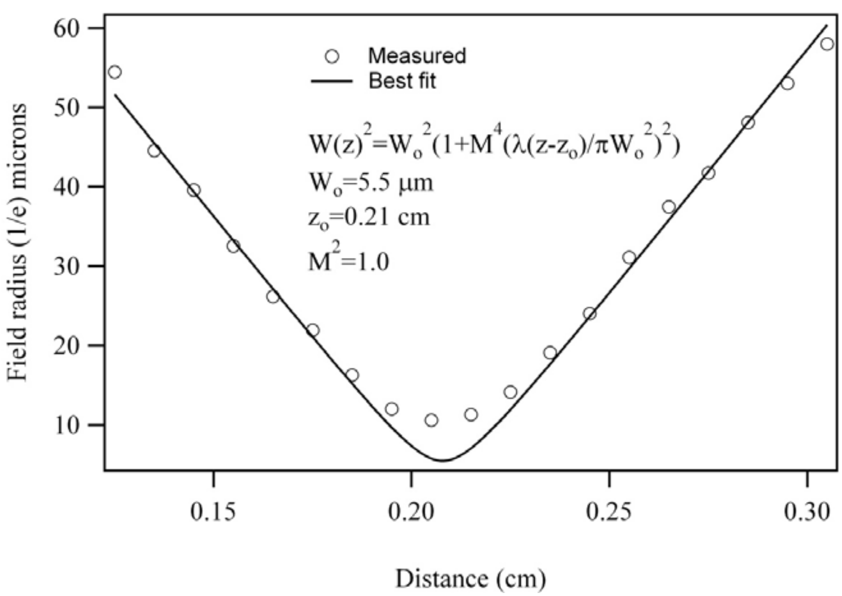


Figure S2.8. Plot for Problem 2.27 (a).

b) Using the results of part (a) as a starting point, perform a fit of the function W(z) above to the data to determine best values for W_o, z_o, and M^2.

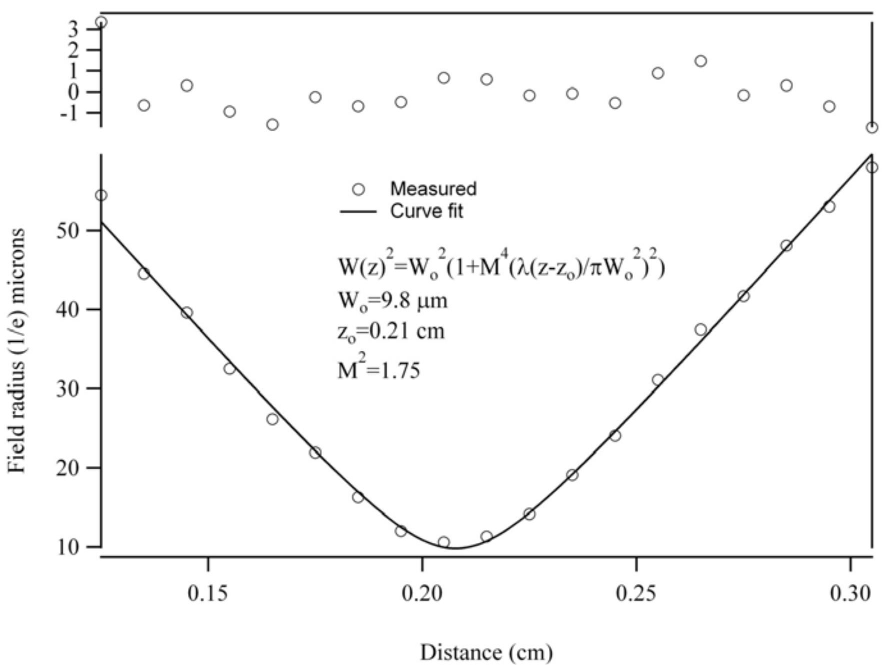


Figure S2.9. Plot for Problem 2.28 (b). The top part of the plot shows the residuals between the fit and the data.