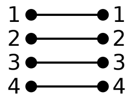
Chapter 2.1 Solutions to Exercises

Exercise 2. **(a)** **(i)** $\{(1,1),(2,2),(3,3),(4,4)\}$.

(ii)

	1	2	3	4
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

(iii)

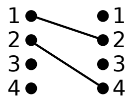


(b) **(i)** $\{(1,2),(2,4)\}$

(ii)

	1	2	3	4
1	0	1	0	0
2	0	0	0	1
3	0	0	0	0
4	0	0	0	0

(iii)

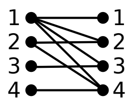


(c) **(i)** $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

(ii)

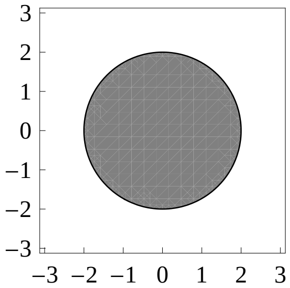
	1	2	3	4
1	1	1	1	1
2	0	1	0	1
3	0	0	1	0
4	0	0	0	1

(iii)

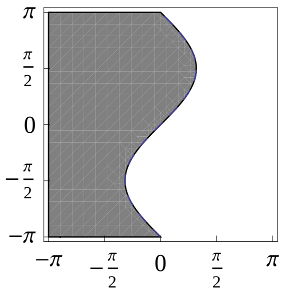


(d) See the answer to part (a).

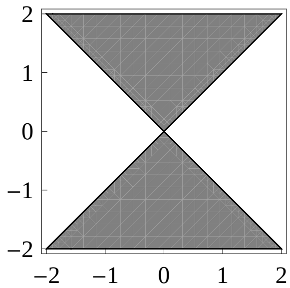
Exercise 4. **(a)**



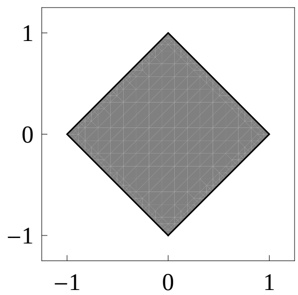
(b)



(c)



(d)



Exercise 6. **(a)** $\mathbb{R}^2$

(b) $\{(a,b) \in \mathbb{R}^2 : a \neq b\}$

(c) $\{(a,b) \in \mathbb{R}^2 : a < |b|$ or $a = b$ and $b > 0\}$

(d) $\mathbb{R}^2$

(e) $\{(a,b) \in \mathbb{R}^2 : b < a < -b\}$

(f) $\{(a,b) \in \mathbb{R}^2 : a = b$ and $b > 0\}$

Exercise 8. **(a)** $D = \mathbb{R} =$ range.

(b) $D = \mathbb{Z}$ and range $= \{$even integers$\}$.

(c) $D = \{x \in \mathbb{R} : x \neq \pm 2\}$. y is in the range if and only if

$$\frac{x^2 - 2}{x^2 - 4} = y$$

for some $x \in \mathbb{R}$. Solving for x we find that

$$x^2 = \frac{2 - 4y}{1 - y}$$

so y is in the range if and only if $(2 - 4y)/(1 - y) \geq 0$. Hence

$$\text{range} = \left\{ y \in \mathbb{R} : y \leq \frac{1}{2} \text{ or } 1 < y \right\}$$

(d) $D = \{(n,m) \in \mathbb{Z}^2 : n \geq 2m\}$. range $= \{\sqrt{k} : k \geq 0$ and $k \in \mathbb{Z}$.

Exercise 10. **(a)** If $0 \leq x < 1$ then $\lfloor x \rfloor = 0$ so f is not defined at x. Thus $D = (-\infty, 0) \cup [1, \infty)$.

(b) If $-1/3 < x \leq 0$ then $\lceil 3x \rceil = 0$ so f is not defined at x. Thus $D = (-\infty, -1/3] \cup (0, \infty)$.

(c) If $x \neq 0$ then $\lceil -x \rceil \neq \lceil x \rceil$ so $\lceil \pm x \rceil$ has two values. A function has only one value at each point x in its domain, so $D = \{0\}$.

(d) As in part (c), $D = \{0\}$.

Exercise 12. **(a)** $f(x) = (x-2)^2 - 10$. Its graph is a parabola so f is neither one-to-one nor onto. $f\left((-\infty, 0]\right) = [-6, \infty)$. $f^{-1}\left((-\infty, 0]\right) = [2 - \sqrt{10}, 2 + \sqrt{10}]$.

(b) The graph of g is the left half of the parabola from part (a). Thus g is strictly decreasing on its domain $(-\infty, 2]$ so g is one-to-one, and g maps $(-\infty, 2]$ to $[-10, \infty)$ so it is onto. $g\left((-\infty, 0]\right) = [-6, \infty)$. $g^{-1}\left((-1, 1]\right) = [2 - \sqrt{11}, -1)$.

(c) F is strictly decreasing on its domain so F is one-to-one. $F(x) \leq 0$ for all x so F is not onto. $F\left([-2, 2]\right) = [-2, 0]$. The range of F contains no positive values so $F^{-1}\left((-3, 3]\right) = F^{-1}\left((-3, 0]\right) = [-2, -\sqrt{5}]$.

(d) G is not one-to-one but it is onto since its range equals the codomain $[-1, 1]$. $G\left([0, \pi)\right) = (-1, 1]$. $G^{-1}\left(\{1\}\right) = \{2n\pi : n \in \mathbb{Z}\}$.

(e) L is one-to-one and onto. $L\left([e, e^3)\right) = [1, 3)$. $L^{-1}\left(\{1, 2, 3\}\right) = \{e, e^2, e^3\}$.

Exercise 14. **(a)** f is a bijection. $f^{-1}(y) = (y-1)/2$.

(b) g is not one-to-one so restricting the codomain cannot make it a bijection.

(c) h is not a bijection because its range contains only odd integers. If its codomain is replaced with $\{$odd integers$\}$ then h becomes a bijection and $h^{-1}(y) = (y-1)/2$.

(d) k is not a bijection because its range contains no negative numbers. If its codomain is replaced with $\{y \in \mathbb{R} : y \geq 0\}$ then k becomes a bijection and $k^{-1}(y) = y^2$.

Exercise 16. **(a)** $(g \circ f)(x) = \left(x^2 + x + 1\right)^2 + (x^2 + x + 1) + 1 = x^4 + 2x^3 + 4x^2 + 3x + 3$. $(g \circ f)(3) = g(f(3)) = g(13) = 183$. $(g \circ f)(-4) = g(f(-4)) = g(13) = 183$.

(b) $(g \circ f)(x) = \lfloor 2x + 1 \rfloor$. $(g \circ f)(3) = \lfloor 7 \rfloor = 7$. $-5.3 < f(-\pi) < -5.2$ so $(g \circ f)(-\pi) = \lfloor f(-pi) \rfloor = -6$.

Exercise 18. **(a)**

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 4 & 1 & 5 & 2 & 6 \end{pmatrix}$$

(b)

$$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

(c)

$$(\sigma \circ \tau)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 2 & 6 & 3 & 7 & 4 \end{pmatrix}$$

(d) Same answer as part (c).

Exercise 20. **(a)** True.

Proof: Let $c \in C$ be an arbitrary element. $g \colon B \to C$ is onto so there exists $b \in B$ such that $g(b) = c$. $f \colon A \to B$ is onto so there exists $a \in A$ such that $f(a) = b$. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$ so c is in the range of $(g \circ f)$. Since $c \in C$ was arbitrary this shows that $(g \circ f)$ is onto.

(b) False.

Counterexample: Let $A = B = \{0, 1\}$, $C = \{0\}$, and $f(0) = f(1) = g(0) = g(1) = 0$. $f \colon A \to B$ is not onto, but $(g \circ f) \colon A \to C$ is onto.

(c) True.

Proof: Let $c \in C$ be an arbitrary element. $(g \circ f)$ is onto so there exists $a \in A$ such that $(g \circ f)(a) = c$. Hence $g(f(a)) = c$ so c is in the range of g.

Exercise 22. **(a)** True.

Proof: Let $x \in \mathbb{R} - \mathbb{Z}$. x is not an integer so there exists an integer n such that $n < x < n + 1$. Thus $n + 1 < x + 1 < n + 2$, and $n + 1 = \lceil x \rceil = \lfloor x + 1 \rfloor$.

(b) False if $x < 0$ because $\sqrt{x}$ and $\sqrt{\lfloor x \rfloor}$ are not even real numbers. (Extending the floor function to complex numbers won't help because nonzero complex numbers have two square roots.)

However the statement is true if $x \geq 0$. For in that case $\sqrt{x}$ is a nonnegative real number so let

$$n = \lfloor \sqrt{x} \rfloor.$$

Then $n \leq \sqrt{x} < n + 1$; squaring we obtain $n^2 \leq x < (n + 1)^2$. n^2 is an integer so it follows that $n^2 \leq \lfloor x \rfloor$, hence $n^2 \leq \lfloor x \rfloor \leq x < (n + 1)^2$. Taking square roots we obtain $n \leq \sqrt{\lfloor x \rfloor} < (n + 1)$, so

$$n = \lfloor \sqrt{\lfloor x \rfloor} \rfloor.$$

Thus $\lfloor \sqrt{x} \rfloor = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$ if $n \geq 0$.

(c) True.

Proof: $x \le \lceil x \rceil$ and $y \le \lceil y \rceil$ hence

$$x + y \le \lceil x \rceil + \lceil y \rceil \tag{1}$$

$\lceil x \rceil$ and $\lceil y \rceil$ are integers so their sum is an integer, hence equation (1) says that $\lceil x + y \rceil \le \lceil x \rceil + \lceil y \rceil$.

(d) True.

Proof: If n is even then $n = 2k$ where $k \in \mathbb{Z}$. $k = \lceil n/2 \rceil$ and $k - 1 = \lfloor (n-1)/2 \rfloor$ so $\lceil n/2 \rceil = \lfloor (n-1)/2 \rfloor + 1$ when n is even.

If n is odd then $n = 2k + 1$ where $k \in \mathbb{Z}$. $k + 1 = \lceil n/2 \rceil$ and $k = \lfloor (n-1)/2 \rfloor$ so $\lceil n/2 \rceil = \lfloor (n-1)/2 \rfloor + 1$ when n is odd.

Exercise 24. **(a)** True for all real numbers. For if $n = \lceil x \rceil$ then $n - 1 < x \le n$ so $n - 2 < x - 1 \le n - 1$ hence $\lceil x - 1 \rceil = n - 1 = \lceil x \rceil - 1$.

(b) True for all real numbers. For if x is any real number there exists an integer n such that $n \le x < n + 1$. There are two cases to check.

If

$$n \le x < n + 1/2 \tag{2}$$

then $n + 1/2 \le x + 1/2 < n + 1$ so

$$n = \lfloor x \rfloor = \lfloor x + 1/2 \rfloor.$$

Multiply equation (2) by 2 to obtain $2n \le 2x < 2n + 1$, hence

$$2n = \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

in this case.

Otherwise

$$n + 1/2 \le x < n + 1 \tag{3}$$

so $n + 1 \le x + 1/2 < n + 3/2$ hence

$$n = \lfloor x \rfloor \quad \text{and} \quad n + 1 = \lfloor x + 1/2 \rfloor.$$

Multiply equation (3) through by 2 to obtain

$$2n + 1 \le 2x < 2n + 2.$$

Thus

$$2n + 1 = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor = \lfloor 2x \rfloor$$

so the formula works in this case too.

(c) The proposed formula is *never* true. The correct formula, which is true for all real numbers, is

$$\lceil x \rceil + \lceil x + 1/2 \rceil = \lceil 2x \rceil + 1.$$

The proof is similar to the proof of part (b): if x is any real number then there exists an integer n such that $n < x \le n + 1$. Again there are two cases.

If $n < x \le n + 1/2$ then $n + 1/2 < x \le n + 1$ and $2n < 2x \le 2n + 1$ so $n + 1 = \lceil x \rceil = \lceil x + 1/2 \rceil$ but $2n + 1 = \lceil 2x \rceil$ so $\lceil x \rceil + \lceil x + 1/2 \rceil = \lceil 2x \rceil + 1$.

Otherwise $n + 1/2 < x \le n + 1$, so $n + 1 < x + 1/2 \le n + 3/2$ and $2n + 1 < 2x \le 2n + 2$, hence $n + 1 = \lceil x \rceil$, $n + 2 = \lceil x + 1/2 \rceil$, and $2n + 2 = \lceil 2x \rceil$. Thus $\lceil x \rceil + \lceil x + 1/2 \rceil = \lceil 2x \rceil + 1$ in this case too.

Exercise 26. **(a)**

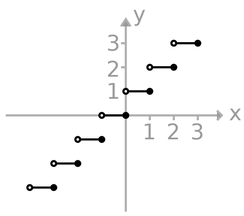
$$\begin{aligned}
&x \in f^{-1}(B_1 \sim B_2) \\
&\Leftrightarrow f(x) \in (B_1 \sim B_2) \\
&\Leftrightarrow f(x) \in B_1 \quad \text{and} \quad f(x) \notin B_2 \\
&\Leftrightarrow x \in f^{-1}(B_1) \quad \text{and} x \notin f^{-1}(B_2) \\
&\Leftrightarrow x \in \left(f^{-1}(B_1) \sim f^{-1}(B_2)\right).
\end{aligned}$$

(b) Any f that is not one-to-one spawns a counterexample. For example let $A = A_1\{0,1\}$, $A_2 = B = \{0\}$, and $f(0) = f(1) = 0$. $f(A_1) = f(A_2) = B$ so $f(A_1) \sim f(A_2) = \emptyset$. But $f(A_1 \sim A_2) = f(\{1\}) = B$ is nonempty.

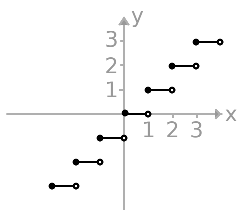
(c) Yes, in every case $(f(A_1) \sim f(A_2)) \subseteq f(A_1 \sim A_2)$. For if $y \in (f(A_1) \sim f(A_2))$ then $y \in f(A_1)$ but $y \notin f(A_2)$ so there exists $x \in A_1$ such that $f(x) = y$ but there is no $x' \in A_2$ such that $f(x') = y$. In particular $x \notin A_2$ so $x \in (A_1 \sim A_2)$. Hence $f(x) = y \in f(A_1 \sim A_2)$.

Exercise 28. For every $x \in A$, $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = ((h \circ g) \circ f)(x)$.

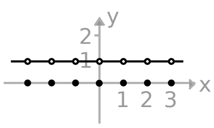
Exercise 30. **(a)**



(b)



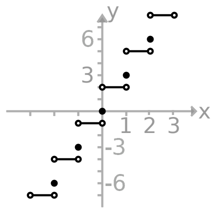
(c)



(d)

Exercise 32. **(a)** Assume that the k elements of the sequence $S = \big(i,\ \sigma(i),\ \sigma^2(i),\ \cdots,\ \sigma^{k-1}(i)\big)$ are all distinct. Since σ is a one-to-one function the sequence

$$\begin{aligned}
\sigma(S) &= \big(\sigma(i), \sigma(\sigma(i)), \sigma(\sigma^2(i)), \cdots, \sigma(\sigma^{k-1}(i))\big) \\
&= \big(\sigma(i), \sigma^2(i), \sigma^3(i), \cdots, \sigma^k(i)\big)
\end{aligned}$$



must also contain k distinct elements. Thus $\sigma^k(i)$ cannot equal any of the other elements $\sigma(i), \sigma^2(i), \cdots, \sigma^{k-1}(i)$ in $\sigma(S)$, so if $\sigma^k(i) \in S$ then $\sigma^k(i)$ must be i.

(b) Let $s_1, s_2, \cdots, s_k$ be k distinct elements in an n-element set S. The cycle $(s_1, s_2, \cdots, s_k)$ determines a permutation τ on S as follows. Set

$$\begin{aligned} \tau(s_j) &= s_{j+1} && \text{if } 1 \le j < k \\ \tau(s_k) &= s_1 \\ \tau(s) &= s && \text{if } s \notin \{s_1, s_2, \cdots, s_k\}. \end{aligned}$$

(c) Let $\tau = (a_1, a_2, \cdots, a_k)$ be a cycle of σ so $\sigma(a_i) = \tau(a_i)$ for each $i = 1, \cdots, k$. Let $j \in \{1, \cdots, n\}$. Choose $i \in \{1, \cdots, n\}$ such that $j \equiv i + 1 \pmod{k}$. Then

$$\sigma(a_i) = \tau(a_i) = a_j$$

so

$$(\sigma \circ \tau^{-1})(a_j) = \sigma\big(\tau^{-1}(a_j)\big) = \sigma\Big(\tau^{-1}\big(\tau(a_i)\big)\Big) = \sigma(a_i) = a_j$$

for every $j = 1, \cdots, n$.

(d) Proof by induction. Let n be a positive integer, and assume that every permutation on any set that has fewer than n elements is a composition of disjoint cycles. (This is trivially true if $n = 1$ since there are no permutations on the empty set.) Let S be a set with n elements, let $\sigma\colon S \to S$ be a permutation, and let $s \in S$. If the n elements $s, \sigma(s), \sigma^2(s), \cdots, \sigma^{n-1}(s)$ are distinct then $\sigma = \big(s, \sigma(s), \sigma^2(s), \cdots, \sigma^{n-1}(s)\big)$ is a cycle. Otherwise let k be the largest number such that $1 \le k < n$ and $s, \sigma(s), \cdots, \sigma^{k-1}(s)$ are distinct. σ maps the k-element set $S' = \{s, \sigma(s), \cdots, \sigma^{k-1}(s)\}$ onto itself.

Let $\sigma' = \sigma|_{S'}$ be the restriction of σ to the set S'. Part (a) of this exercise shows that $\sigma'\colon S' \to S'$ is a cycle: $\sigma' = \big(s, \sigma(s), \sigma^2(s), \cdots, \sigma^{n-1}(s)\big)$. Following part (b) of this exercise, extend σ' to a permutation on all of S by letting it act as the identity on the complement $S'' = S \sim S'$.

Since $\sigma\colon S \to S$ is a one-to-one function and $\sigma(S') = S'$ it follows that $\sigma(S'') = S''$. S'' has $n - k < n$ elements so, by the inductive hypothesis, the restriction $\sigma'' = \sigma|_{S''}$ of σ to S'' is a product of disjoint cycles. Again using part (b) of this exercise we may extend σ'' to all of S by letting it act on the identity on the complement S' of S''.

Consider the composition $\sigma' \circ \sigma''$. By construction σ' is a cycle that is disjoint from the cycles in σ'', so $\sigma' \circ \sigma''$ is a composition of disjoint cycles, so the proof by induction will be complete once we check the following

Claim. $\sigma = (\sigma' \circ \sigma'')$.

Proof. Let $s \in S$ be an arbitrary element.

Suppose $s \in S'$. σ'' acts as the identity on S' so $\sigma''(s) = s$ hence

$$(\sigma' \circ \sigma'')(s) = \sigma'(\sigma''(s)) = \sigma'(s) = \sigma(s)$$

since σ agrees with σ' on S'.

Suppose $s \in S''$. σ'' maps S'' onto S'' and σ' acts as the identity on S'' so $\sigma'(\sigma''(s)) = \sigma''(s)$ hence

$$(\sigma' \circ \sigma'')(s) = \sigma'(\sigma''(s)) = \sigma''(s) = \sigma(s)$$

since σ'' agrees with σ on S''.

Thus $(\sigma' \circ \sigma'')(s) = \sigma(s)$ for all $s \in S$. This proves the claim. $\square$

1. Chapter 2.2 Solutions to Exercises

Exercise 2. **(a)** Not an equivalence relation because reflexivity and transitivity fail.

(b) Equivalence relation.

(c) Not an equivalence relation because transitivity fails.

(d) Equivalence relation.

Exercise 4. **(a)** Equivalence relation.

(b) Equivalence relation.

(c) Not an equivalence relation because transitivity fails.

(d) Not an equivalence relation because transitivity fails.

Exercise 6. **(a)** Equivalence relation by prop. 2.3. $[\pi] = (3, 4]$.

(b) Equivalence relation. The equivalence classes are $\{1, 2\}$, $\{3\}$ and $\{4\}$. aRb if and only if $f(a) = f(b)$ where f is the function that assigns each element to its equivalence class so prop. 2.3 says this is an equivalence relation. $[2] = \{1, 2\}$.

(c) Equivalence relation by prop. 2.3 because $S \sim T$ if and only if $g(S) = g(T)$ where g is the map on $\mathscr{P}$ defined by $g(S) = S \cap \{1, 2, 3\}$. $[\{2, 3, 4\}] = \{\{2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 3, 4, 5\}, \{2, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 3, 4, 5, 6\}\}$.

(d) Equivalence relation by prop. 2.3 because aRb if and only if $h(a) = h(b)$ where h is the function that maps the name of each state to the first letter of the name.

Exercise 8. $(x, y) \sim (x', y')$ if and only if $f((x, y)) = f((x', y'))$ where f is the projection function $f((a, b)) = a$. Thus prop. 2.3 says $\sim$ is an equivalence relation. The equivalence classes are vertical lines.

Exercise 10. **(a)** Symmetric, but not reflexive or transitive.

(b) Prop. 2.3 says this is an equivalence relation. The equivalence classes are $\{a, b\}$, $\{c\}$, $\{d, f\}$, and $\{e\}$, and xRy if and only if $f(x) = f(y)$ where f is the function that assigns each element to its equivalence class.

(c) nRm if and only if $r(n^2) = r(m^2)$ where, for each integer k, $0 \le r(k^2) < 12$ is the remainder obtained when one divides k^2 by 12 using the method taught in elementary school. Clearly the map $k \mapsto r(k^2)$ is a function so prop. 2.3 says R is an equivalence relation.

Exercise 12. **(a)** No. The only reflexive, symmetric relations are $\{(1, 1), (2, 2)\}$ and $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$. Both of these are transitive.

(b) Yes. $\{(1, 1), (2, 2), (1, 2)\}$ is reflexive and transitive but not symmetric.

(c) Yes. The empty relation $\{\}$ is (trivially) symmetric and transitive but not reflexive.

Exercise 14. Obviously it is reflexive and symmetric. But it is not transitive because $\{1, 2\}R\{2, 3\}$ and $\{2, 3\}R\{3, 4\}$ but $\{1, 2\}\not R\{3, 4\}$.

Exercise 16. **(a)** Two, because the two partitions $\{\{1, 2\}\}$ and $\{\{1\}, \{2\}\}$ are the only ways to partition $\{1, 2\}$.

(b) Fifteen, because there are fifteen ways to partition a 4-element set: one way to partition it into four classes of size 1, $\binom{4}{2} = 6$ ways to partition it into one class of size 2 and two classes of size 1, $\binom{4}{2}/2 = 3$ ways to partition it into two classes of size 2, four ways to partition it into one class of size 3 and one class of size 1, and one way to partition it into one class of size 4.

Exercise 18. Let $\sim$ be an equivalence relation. The argument in Theorem 2.1, case 1, shows that if $a \sim b$ then $[b] \subseteq [a]$. But $b \sim a$ also follows from $a \sim b$ by the symmetry property, so the same argument with the symbols a and b swapped shows that $[a] \subseteq [b]$. Therefore, if $a \sim b$ then $[a] = [b]$.

Exercise 20. **(a)** R is not reflexive.

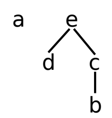
(b) R is not antisymmetric because $(1, 2)R(2, 1)$ and $(2, 1)R(1, 2)$ but $(1, 2) \neq (2, 1)$.

(c) R is not reflexive (e.g. $2 \not R 2$), nor antisymmetric (e.g. $2R3$ and $3R2$ but $2 \neq 3$), nor transitive (e.g. $2R3$ and $3R4$ but $2 \not R 4$).

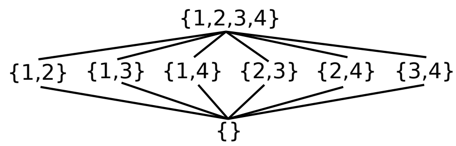
(d) R is reflexive and transitive but not antisymmetric (e.g. $1R10$ and $10R1$ but $1 \neq 10$).

Exercise 22. The Hasse diagrams show that these relations are partial orders.

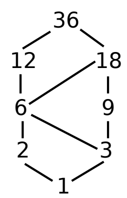
(a)



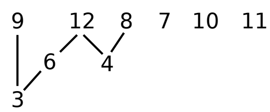
(b)



(c)



(d)



Exercise 24. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A \times B$.

Reflexivity: $a_1 \preceq_A a_1$ and $b_1 \preceq_B b_1$ because $\preceq_A$ and $\preceq_B$ are partial orders. Thus $(a_1, b_1) \preceq (a_1, b_1)$, so $\preceq$ is reflexive.

Antisymmetry: Suppose $(a_1, b_1) \preceq (a_2, b_2)$ and $(a_1, b_1) \preceq (a_2, b_2)$. Then $a_1 \preceq_A a_2$ and $a_2 \preceq_A a_1$, also $b_1 \preceq_B b_2$ and $b_2 \preceq_B b_1$. Therefore $a_1 = a_2$ and $b_1 = b_2$ because $\preceq_A$ and $\preceq_B$ are partial orders. Thus $(a_1, b_1) = (a_2, b_2)$ so $\preceq$ is antisymmetric.

Transitivity: Suppose $(a_1, b_1) \preceq (a_2, b_2)$ and $(a_2, b_2) \preceq (a_3, b_3)$. Then $a_1 \preceq_A a_2$ and $a_2 \preceq_A a_3$, also $b_1 \preceq_B b_2$ and $b_2 \preceq_B b_3$. Therefore $a_1 \preceq_A a_3$ and $b_1 \preceq_B b_3$ because $\preceq_A$ and $\preceq_B$ are partial orders. Therefore $(a_1, b_1) \preceq (a_3, b_3)$, so $\preceq$ is transitive.

Thus $\preceq$ is reflexive, antisymmetric, and transitive, so it is a partial order.

However there are examples where $\preceq$ is not a linear order even though $\preceq_A$ and $\preceq_B$ are linear orders. For example, let $A = B = \mathbb{Z}$, the set of all integers, and let $\preceq_A = \preceq_B = \leq$ be the usual "less than or equal" relation. $\leq$ is a linear order on $\mathbb{Z}$. However the relation $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \preceq (a', b')$ iff $a \leq a'$ and $b \leq b'$ is not a *linear* order because, for example, $(1, 0), (0, 1) \in \mathbb{Z} \times \mathbb{Z}$ and $(1, 0) \not\preceq (0, 1)$ and $(0, 1) \not\preceq (1, 0)$ so $(1, 0)$ and $(0, 1)$ are not comparable with the relation $\preceq$.

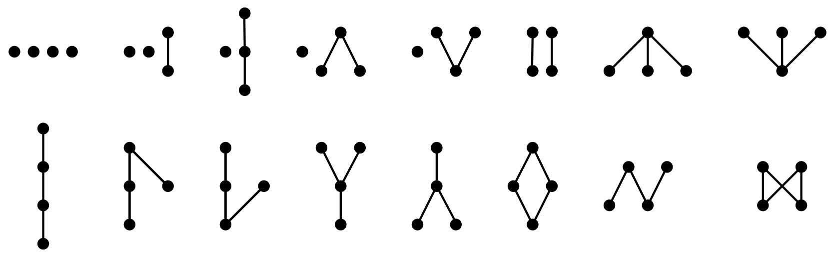
Exercise 26. Suppose R is both an equivalence relation and a partial order on a set A. If aRb then bRa because equivalence relations are symmetric hence so $a = b$ because partial orders have the antisymmetry property. Conversely, if $a = b$ then aRb because equivalence relations are reflexive.

Exercise 28. **(a)** There are three: the constant function $f(1) = f(2) = 1$, the constant function $f(1) = f(2) = 2$, and the identity function $f(1) = 1, f(2) = 2$.

(b) There are six: the constant function $f(1) = f(2) = 1$, the constant function $f(1) = f(2) = 2$, the constant function $f(1) = f(2) = 3$, the function $f(1) = 1, f(2) = 2$, the function $f(1) = 2, f(2) = 3$, and the function $f(1) = 1, f(2) = 3$.

(c) There are eight: the constant function $f(1) = f(2) = 1$, the constant function $f(1) = f(2) = 2$, the constant function $f(1) = f(2) = 3$, the constant function $f(1) = f(2) = 4$, the function $f(1) = 1, f(2) = 2$, the function $f(1) = 1, f(2) = 3$, the function $f(1) = 1, f(2) = 4$, and the function $f(1) = 2, f(2) = 4$.

Exercise 30.



Exercise 32. **(a)** The antisymmetry property says if $x \preceq a$ and $x \neq a$ then $x \not\preceq a$.

(b) Let x be a least element y a minimal element of a poset A. Then part (a) says that x is a minimal element, and

$$
\begin{aligned}
& x \preceq y && \text{because } x \text{ is a least element} \\
& x \neq y \Rightarrow x \not\preceq y && \text{because } y \text{ is a minimal element} \\
& \therefore x = y &&
\end{aligned}
$$

(c) $\mathbb{Z}$ with the usual order is an example of a poset with no minimal element.

Suppose $(A, \preceq)$ is a finite poset. For each $a \in A$ let

$$S(a) = \{x \in A : x \preceq a\}.$$

The transitive property says

$$a \preceq b \Rightarrow S(a) \subseteq S(b).$$

A is finite so there are only finitely many of these subsets $S(a) \subseteq A$, so (at least) one of them, say $S(a')$, has the smallest number of elements.

I claim $S(a') = \{a'\}$, so a' is a minimal element.

For let $x \in S(a')$. Then

$$x \preceq a' \tag{4}$$

so $S(x) \subseteq S(a')$, hence $S(x) = S(a')$ because $S(a')$ has the least number of elements. Therefore $a' \in S(x)$, so

$$a' \preceq x.$$

Hence (4) and the antisymmetry property say that $a' = x$.

Exercise 34. **(a)** a, d, b are minimal elements and a, e are maximal. There are no least or greatest elements.

(b) $\emptyset$ is the minimal and least element, $\{1, 2, 3, 4\}$ is the maximal and greatest element.

(c) 1 is the minimal and least element, 36 is the maximal and greatest element.

(d) $3, 7, 10, 11$ are minimal elements and $7, 8, 9, 10, 1112$ are maximal. There are no least or greatest elements.

Exercise 36. **(a)** $\emptyset$ is the minimal and least element, $\{1, 2, 3\}$ is the maximal and greatest element.

(b) $(1, 1)$ is the minimal and least element, $(2, 3)$ is the maximal and greatest element.

Exercise 38. **(a)** True.

Proof. Let $a \in A$ and $b \in B$ be minimal elements, and let $(a', b') \preceq (a, b)$ be an element of C. Then $a \preceq a'$, so $a = a'$ because a is minimal. Thus $b \preceq b'$ so $b = b'$ because b is minimal.

(b) True.

Proof. Let $a \in A$ and $b \in B$ be least elements and let $(a', b') \in C$ be an arbitrary element. Then $a \preceq a'$ and $b \preceq b'$. Therefore $(a, b) \preceq (a'b')$.

(c) False.

Counterexample: let $A = \{1\}$ with the ordering $1 \preceq 1$ and $B = Z$ with the usual $\leq$ ordering. 1 is a minimal element of A but C is isomorphic to B by the map $(1, n) \mapsto n$ so C has no minimal element.

(d) False.

Counterexample: let $A = Z$ with the usual $\leq$ ordering and $B = \{1\}$ with the ordering $1 \preceq 1$. 1 is the least element of B but C is isomorphic to A by the map $(n, 1) \mapsto n$ so C has no least element.

Chapter 2.3 Solutions to Exercises

Exercise 2.

(a) 0 **(b)** 1 **(c)** 0 **(d)** 0

Exercise 4. **(a)**

x	y	$x+\bar{y}$	$\bar{x}+y$	$f(x,y)=(x+\bar{y})(\bar{x}+y)$
0	0	1	1	1
0	1	0	1	0
1	0	1	0	0
1	1	1	1	1

(b)

x	y	$x+\bar{y}$	$\bar{x}+y$	$\overline{x+\bar{y}}$	$\overline{\bar{x}+y}$	$g(x,y)=\overline{(x+\bar{y})(\bar{x}+y)}$
0	0	1	1	0	0	0
0	1	0	1	1	0	0
1	0	1	0	0	1	0
1	1	1	1	0	0	0

(c)

x	y	$x+\bar{y}$	$\bar{x}+y$	$(x+\bar{y})(\bar{x}+y)$	$h(x,y)=\overline{(x+\bar{y})(\bar{x}+y)}$
0	0	1	1	1	0
0	1	0	1	0	1
1	0	1	0	0	1
1	1	1	1	1	0

(d)

x	y	$x+\bar{y}$	$\overline{x+\bar{y}}$	$\bar{x}+y$	$k(x,y)=\overline{(x+\bar{y})}(\bar{x}+y)$
0	0	1	0	1	0
0	1	0	1	1	1
1	0	1	0	0	0
1	1	1	0	1	0

Exercise 6. **(a)** The value of z does not affect the result so it is the same as the table in problem (4a).

x	y	$x+\bar{y}$	$\bar{x}+y$	$f(x,y,z)=(x+\bar{y})(\bar{x}+y)$
0	0	1	1	1
0	1	0	1	0
1	0	1	0	0
1	1	1	1	1

(b)

w	x	y	z	$xyzw$	$wy\bar{z}$	$\overline{wy\bar{z}}$	$g(w,x,y,z) = xyzw + \overline{wy\bar{z}}$
0	0	0	0	0	0	1	1
0	0	0	1	0	0	1	1
0	0	1	0	0	0	1	1
0	0	1	1	0	0	1	1
0	1	0	0	0	0	1	1
0	1	0	1	0	0	1	1
0	1	1	0	0	0	1	1
0	1	1	1	0	0	1	1
1	0	0	0	0	0	1	1
1	0	0	1	0	0	1	1
1	0	1	0	0	1	0	0
1	0	1	1	0	0	1	1
1	1	0	0	0	0	1	1
1	1	0	1	0	0	1	1
1	1	1	0	0	1	0	0
1	1	1	1	1	0	1	1

(c)

w	x	y	z	$x+y+z$	$\overline{x+y+z}$	wz	$h(w,x,y,z) = (x+y+z)(wz)$
0	0	0	0	0	1	0	0
0	0	0	1	1	0	0	0
0	0	1	0	1	0	0	0
0	0	1	1	1	0	0	0
0	1	0	0	1	0	0	0
0	1	0	1	1	0	0	0
0	1	1	0	1	0	0	0
0	1	1	1	1	0	0	0
1	0	0	0	0	1	0	0
1	0	0	1	1	0	1	0
1	0	1	0	1	0	0	0
1	0	1	1	1	0	1	0
1	1	0	0	1	0	0	0
1	1	0	1	1	0	1	0
1	1	1	0	1	0	0	0
1	1	1	1	1	0	1	0

(d)

w	x	y	z	$k(w,x,y,z) = w + x + y + z$
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

Exercise 8. **(a)**

x	y	z	$x+y$	$y+z$	$x+z$	$(x+y)(y+z)(x+z)$	xy	yz	xz	$xy+yz+xz$
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	1	0	0	0	0	0
0	1	0	1	1	0	0	0	0	0	0
0	1	1	1	1	1	1	0	1	0	1
1	0	0	1	0	1	0	0	0	0	0
1	0	1	1	1	1	1	0	0	1	1
1	1	0	1	1	1	1	1	0	0	1
1	1	1	1	1	1	1	1	1	1	1

(b)

$$\begin{aligned}
&(x+y)(y+z)(x+z) = 0 \\
\Leftrightarrow &x+y = 0 \text{ or } y+z = 0 \text{ or } x+z = 0 \\
\Leftrightarrow &x = y = 0 \text{ or } y = z = 0 \text{ or } x = z = 0.
\end{aligned}$$

So $(x+y)(y+z)(x+z) = 0$ iff at least two of the variables x, y, z equals 0. Hence $(x+y)(y+z)(x+z) = 1$ iff at most one of the variables x, y, z equals 0.

On the other hand

$$\begin{aligned}
&xy + yz + xz = 1 \\
\Leftrightarrow &xy = 1 \text{ or } yz = 1 \text{ or } xz = 1 \\
\Leftrightarrow &x = y = 1 \text{ or } y = z = 1 \text{ or } x = z = 1.
\end{aligned}$$

So $xy + yz + xz = 1$ iff at most one of the variables x, y, z equals 0. Hence $xy + yz + xz = 0$ if and only if at least two of the variables x, y, z equals 0.

Thus the Boolean functions $(x+y)(y+z)(x+z)$ and $xy + yz + xz$ have the same value at each point $(x, y, z) \in \{0,1\} \times \{0,1\} \times \{0,1\}$.

(c) Expanding out, using the associative, distributive, and commutative law several times,

$$
\begin{aligned}
&(x+y)(y+z)(x+z) = (x+y)(y+z)x + (x+y)(y+z)z \\
&= (x+y)(yx+zx) + (x+y)(yz+zz) \\
&= (x+y)yx + (x+y)zx + (x+y)yz + (x+y)zz \\
&= xyx + yyx + xzx + yzx + xyz + yyz + xzz + yzz \\
&= xxy + xyy + xxz + xyz + xyz + yyz + xzz + yzz \\
&= xy + xy + xz + xyz + xyz + yz + xz + yz && \text{idempotence law} \\
&= x(y+y) + x(z+z) + y(z+z) + xy(z+z) \\
&= xy + xz + yz + xy && \text{idempotence} \\
&= x(y+y) + yz + xz \\
&= xy + xz + yz. && \text{idempotence}
\end{aligned}
$$

Exercise 10. **(a)** The right hand side of the equation is in disjunctive normal form so it's enough to make a table for the left hand side.

$\bar{x}$	$\bar{y}$	$\bar{z}$	x	$x+\bar{z}$	$\bar{y}(x+\bar{z})$	$\bar{x}+\bar{y}(x+\bar{z})$
0	0	0	1	1	0	0
0	0	1	1	1	0	0
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	1	0	1
1	1	0	0	0	0	1
1	1	1	0	1	1	1

Thus $\bar{x} + \bar{y}(x + \bar{z} = 1$ iff $x\bar{y} = 1$ or $\bar{x} = 1$, which agrees with the disjunctive normal form on the right side of the equation.

(b)

$$
\begin{aligned}
&\bar{x} + \bar{y}(x+\bar{z}) = 0 \\
&\Leftrightarrow \bar{x} = 0 \text{ and } [\bar{y} = 0 \text{ or } \{x = 0 \text{ and } \bar{z} = 0\}]
\end{aligned}
$$

But if $\bar{x} = 0$ then $x = 1$ so

$$
\begin{aligned}
&\bar{x} + \bar{y}(x+\bar{z}) = 0 \\
&\Leftrightarrow \bar{x} = 0 \text{ and } \bar{y} = 0 \\
&\Leftrightarrow x = 1 \text{ and } y = 1.
\end{aligned}
$$

Since $\bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} + x\bar{y}z + x\bar{y}\bar{z}$ is in disjunctive normal form it is easy to check its value is 0 exactly when $x = y = 1$, so the right and left sides agree.

Exercise 12. **(a)** **(i)**

$$(f \cdot g)(w,x,y,z) = (x+\bar{y})(\bar{x}+y)(xyzw + \overline{wy\bar{z}})$$

Claim. *$(f \cdot g)(w,x,y,z)$ simplifies to $(xy + \bar{x}\bar{y})(\bar{w} + \bar{x} + z)$.*

PROOF.

$$\begin{aligned} f(x,y,z) &= (x+\bar{y})(\bar{x}+y) \\ &= x\bar{x} + xy + \bar{y}\bar{x} + \bar{y}y \\ &= 0 + xy + \bar{y}\bar{x} + 0 && \text{zero identity law} \\ &= xy + \bar{x}\bar{y} && \text{identity law} \end{aligned}$$

Also

$$\begin{aligned} g(w,x,y,z) &= xyzw + \overline{wx\bar{z}} \\ &= xyzw + \bar{w} + \bar{x} + z && \text{De Morgan's law} \\ &= \bar{w} + \bar{x} + (xywz + z) \\ &= \bar{w} + \bar{x} + z && \text{absorption law} \end{aligned}$$

□

(ii) $(f+\bar{h})(w,x,y,z) = (x+\bar{y})(\bar{x}+y) + \overline{\overline{(x+y+z)}(wz)}$.

CLAIM. $(f+\bar{h})(w,x,y,z)$ *simplifies to* 1.

PROOF.

$$\begin{aligned} \overline{h(w,x,y,z)} &= (x+y+z) + \overline{(wz)} && \text{De Morgan's law} \\ &= x+y+z+\bar{w}+\bar{z} && \text{De Morgan's law} \\ &= x+y+\bar{w}+(z+\bar{z}) \\ &= x+y+\bar{w}+1 && \text{unit identity law} \\ &= 1 && \text{dominance law} \end{aligned}$$

Thus

$$(f+\bar{h})(w,x,y,z) = f(x,y,z) + 1 = 1.$$

□

(b) From the tables in exercise (6) it follows that

$$\begin{aligned} f(x,y,z) &= xy + \bar{x}\bar{y} \\ g(w,x,y,z) &= \bar{w} + w\bar{x}\bar{y} + w\bar{x}yz + wx\bar{y} + wxyz \\ h(w,x,y,z) &= 0 \\ k(w,x,y,z) &= w+x+y+z \end{aligned}$$

(d)

$$\begin{aligned} f^d(x,y,z) &= (x+y)(\bar{x}+\bar{y}) \\ g^d(w,x,y,z) &= \bar{w}(w+\bar{x}+\bar{y})(w+\bar{x}+y+z)(w+x+\bar{y})(w+x+y+z) \\ h^d(w,x,y,z) &= 1 \\ k^d(w,x,y,z) &= wxyz \end{aligned}$$

Exercise 14. **(a)** The identity dual to the identity in exercise (8) is

$$xy + yz + xz = (x+y)(y+z)(x+z).$$

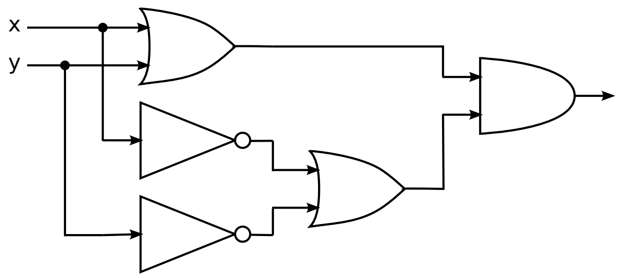
"Dualizing" does not change this particular identity.

(b,c,d) see the solutions to parts (a-c) of exercise (8).

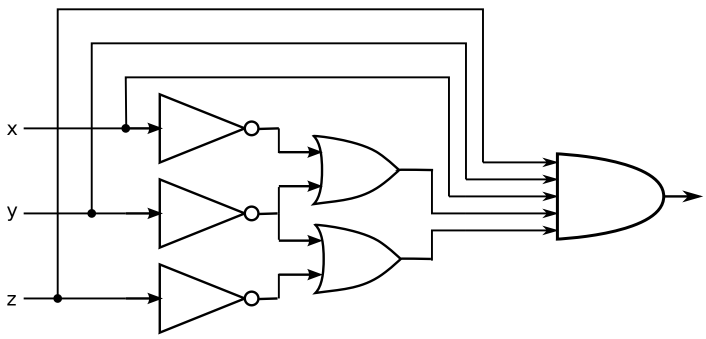
Exercise 16. $(x+\bar{y})(\bar{\bar{x}}+y)$

Exercise 18. $\overline{(x+y)}(x+z)\overline{(y+z)}$

Exercise 20. **(a)**

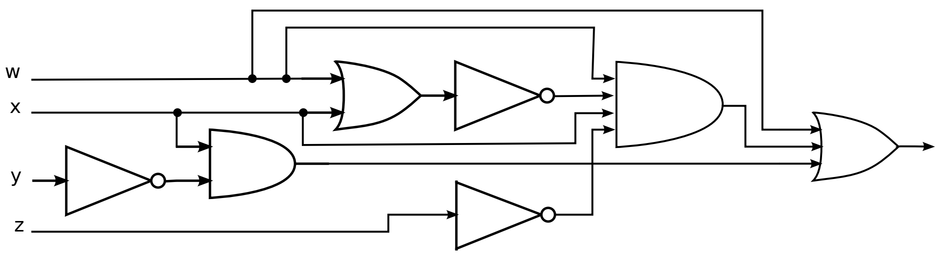


(b)



In fact the function produces 0 for every input so a "circuit" with no components at all would produce the same result.

(c)



Exercise 22. Call the five inputs u, w, x, y, z as in problem (24). The Boolean function

$$f(u,w,x,y,z) = uwx + uwy + uwz + uxy + uxz + uyz + wxy + wxz + wyz + xyz$$

computes the outcome. One might build a circuit for this using ten "and" elements with three inputs each, then collect their outputs with a single "or" element with ten inputs. Alternatively, one could use the distributive law to rewrite the formula this way

$$f(u,w,x,y,z) = (u+w)xy + (x+y)uw + [(u+w)(x+y) + uw + xy]z$$

and regard f of it as the composition of two functions

$$f = g \circ h$$

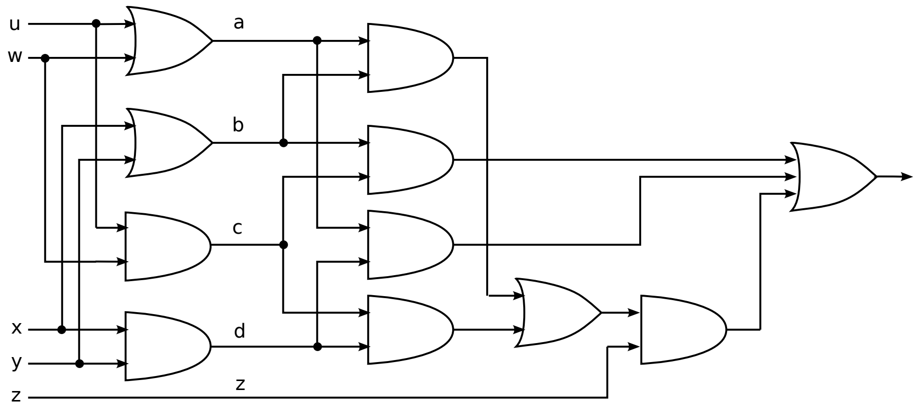
where

$$h(u, w, x, y, z) = (u + w, x + y, uw, xy, z) \qquad = (a, b, c, d, z)$$

and

$$g(a, b, c, d, z) = ad + bc + (ab + cd)z$$

This enables one to break the circuit into two pieces, one for g and one for h. Plug the output of h into the input of g to obtain f.



Exercise 24. See solution to exercise (22).

Exercise 26. **(a)**

	y	$\bar{y}$
x	1	
$\bar{x}$		1

(b)

	y	$\bar{y}$
x		
$\bar{x}$		

(c)

	y	$\bar{y}$
x		1
$\bar{x}$	1	

(d)

	y	$\bar{y}$
x		
$\bar{x}$	1	

Exercise 28. **(a)** See exercise (26a).

(b)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	1		1	1
$w\bar{x}$	1		1	1
$\bar{w}\bar{x}$	1	1	1	1
$\bar{w}x$	1	1	1	1

So $g(w, x, y, z) = yz + \bar{w} + \bar{y}$.

(c)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx				
$w\bar{x}$				
$\bar{w}\bar{x}$				
$\bar{w}x$				

and $h(w, x, y, z) = 0$.

(d)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	1	1	1	1
$w\bar{x}$	1	1	1	1
$\bar{w}\bar{x}$	1	1		1
$\bar{w}x$	1	1	1	1

$k(w, x, y, z) = w + x + y + z.$

Exercise 30. **(a)**

	y	$\bar{y}$
x	1	1
$\bar{x}$		

The simplified formula is x so the "circuit" is a single wire:

$x \longrightarrow$

(b) The circuit in fig. 2.25b produces 0 for every input so there are no 1s in the Karnaugh map and the simplified circuit contains no components.

Exercise 32. **(a)** Adding the "don't care" condition does not simplify the formula $g(w, x, y, z) = yz + \bar{w} + \bar{y}$ that we obtained in exercise 28b.

(b) The Boolean function in exercise 20b produces 0 for every input so it can't be simplified.

(c) The expression in exercise 20c simplifies to $w + x\bar{y}$. Adding the "don't care" condition does not simplify that expression.

Exercise 34. **(a)** $(\bar{x} + y)(x + \bar{y})$.

(b) $(\bar{w} + x + \bar{y} + z)(\bar{w} + \bar{x} + \bar{y} + z)$

(c) 0

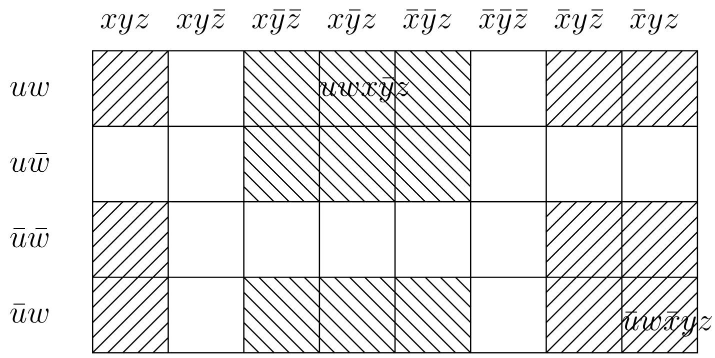
(d) $w + x + y + z$

Exercise 36. **(a)** Each cell corresponds to a minterm $y_1y_2y_3y_4y_5$ where $y_i = x_i$ or $y_i = \bar{x}_i$ and x_i are Boolean variables, $i = 1, \cdots, 5$. Adjacent cells correspond to minterms that differ in exactly one literal. There are five adjacent cells because each minterm has five literals.

(b) See part (c).

(c)

The figure below shows $uwx\bar{y}z$ and its four adjacent cells (downward-sloping crosshatch), also $\bar{u}w\bar{x}yz$ and its four adjacent cells (upward-sloping crosshatch).



Exercise 38. **(a)**

$$
\begin{aligned}
& x\bar{y} + y\bar{z} + \bar{x}z \\
&= x\bar{y}(z+\bar{z}) + (x+\bar{z})y\bar{z} + \bar{x}(y+\bar{y})z \\
&= x\bar{y}z + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z} + \bar{z}yz + \bar{x}\bar{y}z \\
&= (x+\bar{x})\bar{y}z + x(\bar{y}+y)\bar{z} + \bar{x}y(\bar{z}+z) \\
&= \bar{y}z + x\bar{z} + \bar{x}y
\end{aligned}
$$

(b)

$$
\begin{aligned}
& \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} + x\bar{y}z + x\bar{y}\bar{z} \\
&= \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}z + (\bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}) + x\bar{y}z + (x\bar{y}\bar{z} + x\bar{y}\bar{z}) \\
&= (\bar{x}yz + \bar{x}y\bar{z}) + (\bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}) + (x\bar{y}z + x\bar{y}\bar{z}) + (x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}) \\
&= \bar{x}y(z+\bar{z}) + \bar{x}\bar{y}(z+\bar{z}) + x\bar{y}(z+\bar{z}) + (x+\bar{x})\bar{y}\bar{z} \\
&= \bar{x}y + \bar{x}\bar{y} + x\bar{y} + \bar{y}\bar{z} \\
&= \bar{x}(y+\bar{y}) + \bar{y}(x+\bar{z}) \\
&= \bar{x} + \bar{y}(x+\bar{z})
\end{aligned}
$$