Chapter 2

Particle Motion in One Dimension

2.2.1

$$\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$$
$$\frac{1}{\lambda'} = \frac{1}{\lambda + \frac{h}{mc}(1-\cos\theta)}$$
$$\frac{\nu'}{c} = \frac{1}{\frac{c}{\nu} + \frac{h}{mc}(1-\cos\theta)}$$
$$\nu' = \frac{\nu}{1 + \frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}}$$
$$\frac{\nu'-\nu}{\nu} = \left[\frac{1}{1+\frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}} - 1\right]$$
$$\frac{\Delta\nu}{\nu} = \frac{-\frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}}{1+\frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}}$$

This is exact. But for visible light $\frac{h\nu}{mc^2} << 1$, so

$$\frac{|\Delta\nu|}{\nu} \approx \frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}$$

2.3.1 (a) We found in (1.53) that the condition for the experiment to work was

$$(\Delta p_z)_+ - (\Delta p_z)_- > \delta p_z$$
$$\Rightarrow \left|r\hbar\frac{\partial H}{\partial z}\right|\frac{L}{v} > \delta p_z$$

If we say that this δp_z corresponds to the first minimum of a single slit Fraunhaffer pattern, then

$$\Rightarrow \delta p_z \approx \frac{\hbar}{\delta z}$$

Combining these yields

$$\delta z > \frac{v}{L\left|r\dfrac{\partial H}{\partial z}\right|}$$

Plugging in all values gives

$$v = \sqrt{\frac{3(1.38\times 10^{-16})10^3}{1.79\times 10^{-22}}} \approx 4.8\times 10^4 \text{ cm s}^{-1}.$$

So

$$\delta z > \frac{4.8\times 10^4}{10\times 10^7}\frac{1}{10^4}$$

$$\delta z \geq 5\times 10^{-8} \text{ cm} = 5\mathring{A}$$

(b) Since $v = \left(\dfrac{3kT}{m}\right)^{1/2}$, the only thing which changes is $m_{Ag} \to m_{ele}$, so

$$v_{ele} = v_{Ag}\left(\frac{m_{Ag}}{m_{ele}}\right)^{1/2} = v_{Ag}\times(4.4\times 10^2)$$

$$m_{electron} = 9.1\times 10^{-28}\,\text{gm}$$

$$\Rightarrow \delta z_{ele} \geq \delta z_{Ag}\times 440 = 2.2\times 10^{-5}\,\text{cm}$$

It is much wider!

2.4.1

$$E = T + V = \frac{\vec{p}^2}{2m} - \frac{k}{r^{2+\epsilon}}, \quad \epsilon > 0.$$

For zero angular momentum $\vec{p} \to p_r$. The text explains it is reasonable to expect

$$\delta p_r \delta r \geq \hbar.$$

The state with the smallest possible uncertainty will have

$$p_r \sim \frac{\hbar}{r}.$$

The energy is

$$E \approx \frac{1}{2m}\left(\frac{\hbar}{r}\right)^2 - \frac{k}{r^{2+\epsilon}}$$

When is

$$\frac{\partial E}{\partial r} = 0\ ?$$

$$\frac{\partial E}{\partial r} = -\frac{2}{2m}\frac{\hbar^2}{r^3} + \frac{(2+\epsilon)k}{r^{3+\epsilon}} = 0$$

$$\Rightarrow r^\epsilon = \frac{m}{\hbar^2}(2+\epsilon)k.$$

However, notice

$$
\begin{aligned}
\frac{\partial^2 E}{\partial r^2}\Big|_{r^\epsilon=\frac{m}{\hbar^2}(2+\epsilon)k} &= \frac{3\hbar^2}{mr^4} - \frac{(3+\epsilon)(2+\epsilon)k}{r^{4+\epsilon}} \\
&= \left(\frac{3\hbar^2}{m} - \frac{(3+\epsilon)(2+\epsilon)k}{r^\epsilon}\right)\frac{1}{r^4} \\
&= \left(\frac{3\hbar^2}{m} - \frac{(3+\epsilon)(2+\epsilon)k\hbar^2}{m(2+\epsilon)k}\right)\frac{1}{r^4} \\
&= -\frac{\epsilon\hbar^2}{m}\frac{1}{r^4} \quad < 0.
\end{aligned}
$$

Since $\dfrac{\partial^2 E}{\partial r^2} < 0$ when $\dfrac{\partial E}{\partial r} = 0$, we have an **unstable** maximum.

2.4.2

$$
\begin{aligned}
p_x = \frac{\hbar}{2} \quad &\Rightarrow \quad p = \frac{\hbar}{2x} \\
E = \frac{1}{2m}&\left(\frac{\hbar}{2x}\right)^2 + \frac{1}{2}m\omega^2x^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial x} &= 0 = -\frac{\hbar^2}{4mx^3} + m\omega^2 x \\
\Rightarrow \quad x^2 &= \frac{\hbar}{2m\omega}.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow E &= \frac{1}{2m}\left(\frac{\hbar}{2x}\right)^2 + \frac{1}{2}m\omega^2x^2 \\
&= \frac{\hbar^2 m\omega}{4m\hbar} + \frac{\hbar m\omega^2}{4m\omega} \\
&= \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega.
\end{aligned}
$$

2.5.1 (a)

$$
\begin{aligned}
rmv_n &= n\hbar \qquad n = 1, 2, 3, ... \\
\Rightarrow v_n &= \frac{n\hbar}{rm} \\
r^2 &= \frac{v_n^2}{\omega^2} = \frac{n^2\hbar^2}{\omega^2 r^2 m^2} \\
\Rightarrow r^4 &= \frac{n^2\hbar^2}{\omega^2 m^2} \\
\Rightarrow r &= \sqrt{\frac{n\hbar}{\omega m}}.
\end{aligned}
$$

(Alternate approach: Given $v_n = \dfrac{n\hbar}{rm}$, then

$$E = \underbrace{\frac{1}{2}m\omega^2 r^2}_{\text{potential}} + \underbrace{\frac{1}{2}mv_x^2}_{\text{kinetic}}$$

Take $\dfrac{\partial E}{\partial r} = 0$

$$\Rightarrow m\omega^2 r^4 = \frac{n^2\hbar^2}{m}$$
$$\Rightarrow r = \sqrt{\frac{n\hbar}{\omega m}},$$

as before.)
(b)

$$\begin{aligned} E &= \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 r^2 \\ &= \frac{1}{2}m\frac{n^2\hbar^2}{r^2m^2} + \frac{1}{2}m\omega^2\frac{n\hbar}{\omega m} \\ &= \frac{1}{2}\omega n\hbar + \frac{1}{2}\omega n\hbar \\ &= n\hbar\omega. \end{aligned}$$

2.6.1 (a)

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, g(k)\, e^{ikx} \\ &= \frac{N}{\sqrt{2\pi}}\int_{-k}^{k} dk\, e^{ikx} \\ &= \frac{-iN}{\sqrt{2\pi}}\frac{e^{ikx} - e^{-ikx}}{x} \\ &= \frac{2N}{\sqrt{2\pi}}\frac{\sin kx}{x}. \end{aligned}$$

(b) Obviously $\delta k \sim k$. A reasonable definition of the width of $f(x)$ gives $\delta x = \dfrac{\pi}{k}$ (first node in $f(x)$, so $\delta k\, \delta x = k\dfrac{\pi}{k} = \pi \sim 1$

2.7.1

$$\frac{\partial^2}{\partial x^2} f(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, g(k)\, e^{ix(kx-\omega t)}(-k^2)$$
$$\frac{\partial^2}{\partial t^2} f(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, g(k)\, e^{ix(kx-\omega t)}(-\omega^2)$$

Assume $\omega^2 = \hbar^2 c^2 + \mu^2 c^2 \qquad \left(\mu \equiv \frac{mc}{\hbar}\right)$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t) - \mu^2 f(x,t).$$

This is the so called Klein-Gordon equation.

2.8.1

$$C(t) \equiv \int_{-\infty}^{\infty} dx' \, \frac{1}{(2\pi)^{1/4}} \frac{1}{\sqrt{\delta x}} e^{-\frac{(x')^2}{4\delta x^2}} \times \frac{1}{(2\pi)^{1/4}} \frac{1}{\sqrt{\delta x(t)}} e^{-\frac{(x')^2}{4\delta x \, \delta x(t)}}$$

$$\left(\delta x(t) = \delta x + \frac{i\hbar t}{2m\delta x}\right)$$

Note $C(0) = 1$. Simplify:

$$C(t) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\sqrt{\delta x \, \delta x(t)}} \int_{-\infty}^{\infty} dx' \, e^{-\frac{(x')^2}{4\delta x}\left(\frac{1}{\delta x} + \frac{1}{\delta x(t)}\right)}$$

Use the Gaussian integrals in Section 2.6.

$$\Rightarrow \quad C(t) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\sqrt{\delta x \, \delta x(t)}} \sqrt{\frac{\pi}{\frac{1}{4\delta x}\left(\frac{1}{\delta x} + \frac{1}{\delta x(t)}\right)}}$$

$$C(t) = \sqrt{\frac{2\delta x}{\delta x + \delta x(t)}}$$

$$\Rightarrow \quad |C(t)| = \left(\frac{\delta x^2}{\delta x^2 + \left(\frac{\hbar t}{4m\delta x}\right)^2}\right)^{1/4}.$$

Set $|C(t_c)| = 1/2$

$$\Rightarrow \quad \frac{1}{16} = \frac{\delta x^2}{\delta x^2 + \left(\frac{\hbar t_c}{4m\delta x}\right)^2}$$

$$\Rightarrow \quad t_c = \frac{4\sqrt{15} m \delta x^2}{\hbar}.$$

$2\sqrt{15}$ larger than (2.90). Can make the above smaller by choosing $|C(t_c)|$ closer to one.

2.9.1

$$\int_{-\infty}^{\infty} dx\ |f(x)|^2 = \frac{N^2}{2\pi}\int_{-\infty}^{\infty} dx \int_{-k}^{k} dk\, e^{ikx}\int_{-k}^{k} dk'\, e^{-ik'x}$$
$$= \frac{N^2}{2\pi}\int_{-k}^{k} dk \int_{-k}^{k} dk' \underbrace{\int_{-\infty}^{\infty} dx\, e^{ix(k-k')}}_{2\pi\,\delta(k-k')} = 2kN^2 = 1$$
$$\Rightarrow N = \frac{1}{\sqrt{2k}}.$$

2.9.2 We will only get contributions from $\delta(f(x))$ each time $f(x) = 0$. Assuming there are n (non-repeating or simple) roots to this equation, we expect that

$$\delta(f(x)) = \delta\left(\sum_{i=1}^{n} \left.\frac{df}{dx}\right|_i (x - x_i)\right)$$

from a Taylor series expansion of $f(x)$ near $x = x_i$. This gives

$$\delta(f(x)) = \sum_{i=1}^{n} \delta\left(\left.\frac{df}{dx}\right|_i (x - x_i)\right),$$

since the sum of separate zeros of $f(x)$ gives separate delta function contributions. Then using (2.106), we have

$$\delta\left(\left.\frac{df}{dx}\right|_i (x - x_i)\right) = \frac{1}{\left|\left.\frac{df}{dx}\right|_i\right|}\delta(x - x_i),$$
$$\Rightarrow \delta\left(f(x)\right) = \sum_{i=1} \frac{1}{\left|\left.\frac{df}{dx}\right|_i\right|}\delta(x - x_i).$$

2.10.1 (a)

$$\psi_(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{i\frac{xp_x}{\hbar}}\,\psi_(p_x,t),$$
$$\Rightarrow \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-i\frac{xp'_x}{\hbar}}\,\psi_(x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp_x\,\psi_(p_x,t)\int_{-\infty}^{\infty} dx\, e^{i\frac{x}{\hbar}(p_x - p'_x)}.$$

But from (2.120)

$$\int_{-\infty}^{\infty} dx\, e^{i\frac{x}{\hbar}(p_x - p'_x)} = 2\pi\hbar\delta(p_x - p'_x),$$

$(p'_x \to p_x)$

$$\Rightarrow \psi_(p_x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-i\frac{xp_x}{\hbar}}\,\psi_(x,t).$$

(b) We have that

$$\psi_g(x,0) = \frac{1}{\sqrt{\sqrt{2\pi}\delta x}} e^{\left(\frac{ix}{\hbar}\overline{p}x - \frac{1}{4\delta x}\frac{x^2}{\delta x}\right)}$$

$$\psi_g(p_x,0) = \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\sqrt{2\pi}\delta x}} \int_{-\infty}^{\infty} dx\, e^{\left(\frac{ix}{\hbar}(\overline{p}-p_x) - \frac{x^2}{4\delta x^2}\right)}$$

can write (complete the square)

$$-\frac{x^2}{4\delta x^2} + \frac{ix}{\hbar}(\overline{p}-p_x) = \frac{1}{4\delta x^2}\left[x + \frac{2i\,\delta x^2}{\hbar}(p_x-\overline{p})\right]^2 - \frac{\delta x^2}{\hbar^2}(p_x-\overline{p})^2$$

$$\Rightarrow \psi_g(p_x,0) = \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\sqrt{2\pi}\delta x}} e^{\left(-\frac{\delta x^2}{\hbar^2}(p_x-\overline{p})^2\right)} \int_{-\infty}^{\infty} dx\, e^{\left(-\frac{1}{4\delta x^2}\left[x+\frac{2i\,\delta x^2}{\hbar}(p_x-\overline{p})^2\right]\right)}.$$

Use $\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$, so

$$\psi_g(p_x,0) = \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{\sqrt{2\pi}\delta x}} e^{\left(-\frac{\delta x^2}{\hbar^2}(\overline{p}-p_x)^2\right)} \sqrt{4\pi\,\delta x^2}$$

$$\psi_g(p_x,0) = \frac{\sqrt{2\delta x}}{(2\pi\hbar^2)^{\frac{1}{4}}} e^{-\frac{\delta x^2}{\hbar^2}(\overline{p}-p_x)^2}.$$

By inspection $p_x = \overline{p}$ maximizes $\psi_g(p_x,0)$. Width of $e^{-\alpha k^2}$ is $\frac{1}{2\sqrt{\alpha}}$, so width of $e^{-\frac{\delta x^2}{\hbar^2}(\overline{p}-p_x)^2}$ is $\frac{\hbar}{2\delta x}$.

2.10.2

$$\begin{aligned}\frac{\langle p_x \rangle}{m} &= \frac{1}{m}\int_{-\infty}^{\infty} dp_x\, p_x\, |\psi(p_x,0)|^2 \\ &= \frac{2\delta x}{m(2\pi\hbar^2)^{1/2}} \int_{-\infty}^{\infty} dp_x\, p_x\, e^{-2\frac{\delta x^2}{\hbar^2}(\overline{p}-p_x)^2}.\end{aligned}$$

Define $p'_x = p_x - \overline{p}$

$$\frac{\langle p_x \rangle}{m} = \frac{2\delta x}{m(2\pi\hbar^2)^{1/2}} \int_{-\infty}^{\infty} dp'_x\, (p'_x + \overline{p})\, e^{-2\left(\frac{\delta x}{\hbar}\right)^2 p_x'^{\,2}}$$

(integral involving p'_x vanishes)

$$= \frac{2\delta x}{m(2\pi\hbar^2)^{1/2}} \overline{p} \int_{-\infty}^{\infty} dp'_x\, e^{-2\left(\frac{\delta x}{\hbar}\right)^2 p_x'^{\,2}}.$$

Let $z = \sqrt{2}\frac{\delta x}{\hbar}p'_x \;\Rightarrow\; dp'_x = \frac{\hbar}{\sqrt{2}\delta x}dz.$

$$\Rightarrow \frac{\langle p_x \rangle}{m} = \frac{2\delta x}{m(2\pi\hbar^2)^{1/2}}\overline{p}\frac{\hbar}{\sqrt{2}\delta x}\underbrace{\int_{-\infty}^{\infty} dz\, e^{-z^2}}_{=\sqrt{\pi}(\text{Sect. 2.6})}$$

$$\Rightarrow \frac{\langle p_x \rangle}{m} = \frac{\overline{p}}{m}.$$

2.10.3

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}$$

Use $\psi(p_x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-ixp_x/\hbar}\psi(x,t)$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-ixp_x/\hbar}\psi(x,t)\right) = -\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{-ixp_x/\hbar}\frac{\partial^2\psi(x,t)}{\partial x^2}\right).$$

Integrating by parts twice:

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, e^{-ixp_x/\hbar}\frac{\partial^2\psi(x,t)}{\partial x^2} &= \int_{-\infty}^{\infty} dx - \frac{\partial}{\partial x}\left(e^{-ixp_x/\hbar}\right)\frac{\partial\psi(x,t)}{\partial x} \\ &= \int_{-\infty}^{\infty} dx\frac{\partial^2}{\partial x^2}\left(e^{-ixp_x/\hbar}\right)\psi(x,t) \\ &= -\frac{p_x^2}{\hbar^2}\int_{-\infty}^{\infty} dx e^{-ixp_x/\hbar}\psi(x,t).\end{aligned}$$

Therefore, we have

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\psi(p_x,t) &= \frac{\hbar^2 p_x^2}{2m\hbar^2}\psi(p_x,t) \\ &= \frac{p_x^2}{2m}\psi(p_x,t).\end{aligned}$$

2.15.1

$$\langle\, x'|(\,\overbrace{p_x x}^{\text{Term 1}} - \overbrace{x p_x}^{\text{Term 2}}\,)$$

$$\begin{aligned}
\text{Term 1}: \quad \langle\, x'|p_x x &= \frac{\hbar}{i}\left(\frac{\partial}{\partial x'}\langle\, x'|\right) x \qquad \text{using: (2.170)}\\
&= \frac{\hbar}{i}\left(\frac{\partial}{\partial x'}\langle\, x'|x\right) \qquad \text{using: Rule 2}\\
&= \frac{\hbar}{i}\frac{\partial}{\partial x'}\left(x'\langle\, x'|\right) \qquad \text{using: (2.155)}\\
&= \frac{\hbar}{i}\left(\langle\, x'| + x'\frac{\partial\langle\, x'|}{\partial x'}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Term 2}: \quad -\langle\, x'|x p_x &= -x'\langle\, x'|p_x \qquad \text{using: (2.155)}\\
&= -x'\frac{\hbar}{i}\frac{\partial\langle\, x'|}{\partial x'} \qquad \text{using: (2.170)}
\end{aligned}$$

$$\Rightarrow \quad \langle\, x'|(p_x x - x p_x) = \frac{\hbar}{i}\langle\, x'|.$$

2.16.1

$$\begin{aligned}
\langle A(p_x)\rangle_{\psi,t} &= \langle\psi,t|A(p_x)|\psi,t\rangle = \langle\psi,t|1\cdot A(p_x)|\psi,t\rangle\\
&= \int dx'\ \langle\psi,t|x'\rangle\,\langle x'|A(p_x)|\psi,t\rangle
\end{aligned}$$

$$\left(\langle x'|A(p_x) = A\left(\frac{\hbar}{i}\frac{\partial}{\partial x'}\right)\langle x'|\right)$$

$$\begin{aligned}
&= \int dx'\ \langle\psi,t|x'\rangle\, A\left(\frac{\hbar}{i}\frac{\partial}{\partial x'}\right)\langle|\psi,t\rangle\\
&= \int dx'\ \psi^*(x',t)A\left(\frac{\hbar}{i}\frac{\partial}{\partial x'}\right)\psi(x',t).
\end{aligned}$$

2.16.2

$$\begin{aligned}
\langle p_x\rangle_{\psi,t} &= \langle\psi,t|p_x|\psi,t\rangle\\
&= \int dx'\ \langle\psi,t|x'\rangle\left\langle x'|\underbrace{p_x}_{\hookleftarrow}|\psi,t\right\rangle\\
&= \int dx'\ \psi^*(x',t)\frac{\hbar}{i}\frac{\partial}{\partial x'}\psi(x',t).
\end{aligned}$$

Write this as:

$$\langle p_x \rangle_{\psi,t} = \frac{1}{2}\int_{-\infty}^{\infty} dx'\, \psi^*(x',t)\frac{\hbar}{i}\frac{\partial}{\partial x'}\psi(x',t) + \frac{1}{2}\int_{-\infty}^{\infty} dx'\, \psi^*(x',t)\ \underbrace{\frac{\hbar}{i}\frac{\partial}{\partial x'}\psi(x',t)}_{\leftarrow \text{ integ. by parts}}$$

$$\langle p_x \rangle_{\psi,t} = \frac{1}{2}\int_{-\infty}^{\infty} dx'\, \psi^*(x',t)\frac{\hbar}{i}\frac{\partial}{\partial x'}\psi(x',t) + \frac{1}{2}\frac{\hbar}{i}|\psi(x',t)|^2\Big|_{-\infty}^{\infty}$$
$$- \frac{1}{2}\int_{-\infty}^{\infty} dx'\, \frac{\hbar}{i}\frac{\partial}{\partial x'}\psi^*(x',t)\psi(x',t)$$

$$\text{but}\quad j(x',t) = \frac{i\hbar}{2m}\left(\frac{\partial \psi^*}{\partial x'}\psi - \psi^*\frac{\partial \psi}{\partial x'}\right),$$

So,

$$\begin{aligned}\frac{\langle p_x \rangle_{\psi,t}}{m} &= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} dx'\, \left(\frac{\partial \psi^*}{\partial x'}\psi - \psi^*\frac{\partial \psi}{\partial x'}\right)\\ &= \int_{-\infty}^{\infty} dx'\, j(x',t).\end{aligned}$$

2.16.3

(a)

$$1 = |A|^2\int_{-\infty}^{\infty} dp'_x\, \frac{1}{p_x'^2 + a^2} = \frac{|A|^2}{|a|}\underbrace{\tan^{-1}\left(\frac{2p'_x}{2|a|}\right)\Big|_{-\infty}^{\infty}}_{\frac{\pi}{2}+\frac{\pi}{2}=\pi}$$

$$|A|^2 = \frac{|a|}{\pi}$$

(b)

$$\begin{aligned}\langle p_x \rangle &= \frac{a}{\pi}\int_{-\infty}^{\infty} dp'_x\, \frac{e^{-p'_x\bar{x}/\hbar}}{\sqrt{p_x'^2 + a^2}}p'_x\frac{e^{-p'_x\bar{x}/\hbar}}{\sqrt{p_x'^2 + a^2}}\\ &= \frac{a}{\pi}\int_{-\infty}^{\infty} dp'_x\, \frac{p'_x}{\sqrt{p_x'^2 + a^2}} = 0.\end{aligned}$$

(c)

$$\psi(p_x,t) = \psi(p_x)e^{-i\omega t}, \quad \omega = \frac{E}{\hbar} = \frac{p_x^2}{2m\hbar^2}$$

2.16.4 (a) From (1.134)

$$S_y = \frac{\hbar}{2}\sigma_y \qquad (\sigma_y = \sigma_2)$$

$$\text{where} \quad \sigma_y = i\left(|-+| \ - \ |+-|\right).$$

$$\text{Given} \quad |\mp\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right), \quad \text{we have}$$

$$\begin{aligned}
S_y|\mp\rangle &= \frac{i\hbar}{2\sqrt{2}}\left(|-+\rangle \ - \ |+-\rangle\right)\left(|+\rangle + i|-\rangle\right) \\
&= \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right) \\
&= \frac{\hbar}{2}|\mp\rangle.
\end{aligned}$$

(b)

$$\begin{aligned}
|\mp, t\rangle = e^{-iHt/\hbar} &= e^{i\gamma B S_z t/\hbar}|\mp, 0\rangle \\
&= e^{i\gamma B S_z t/\hbar}\frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{i\gamma Bt/2}|+\rangle + ie^{-i\gamma Bt/2}|-\rangle\right).
\end{aligned}$$

(c) " Method 2" gives the correct probability as an absolute squared overlap of the two states. We have

$$\begin{aligned}
\langle\mp, t|\mp\rangle &= \frac{1}{2}\left(\langle+|\,e^{-i\gamma Bt/2} - i\langle-|\,e^{i\gamma Bt/2}\right)\left(|+\rangle + i|-\rangle\right) \\
&= \frac{1}{2}\left(e^{-i\gamma Bt/2} + e^{i\gamma Bt/2}\right) \\
&= \cos\left(\frac{\gamma Bt}{2}\right) \\
\Rightarrow I(B) &= I_0\cos^2\left(\frac{\gamma Bt}{2}\right).
\end{aligned}$$