Chapter 2

Solutions for Exercises in Chapter 2

Problem 2.1 Show that for time harmonic plane waves of the form

$$\mathbf{u}(\mathbf{x}, t) = \widehat{\mathbf{u}}\, e^{i\omega(\mathbf{s}\cdot\mathbf{x}-t)} \;. \tag{2.1}$$

the governing equations for the dynamics of an isotropic and homogeneous linear elastic body decouple into the equations

$$\left(\rho - \frac{\mu}{c^2}\right)(\widehat{\mathbf{u}} \times \mathbf{s}) = \mathbf{0} \quad \text{and} \quad \left(\rho - \frac{\lambda + 2\mu}{c^2}\right)(\widehat{\mathbf{u}} \cdot \mathbf{s}) = 0 \;. \tag{2.2}$$

Solution 2.1: The elastodynamic wave equation for an isotropic, homogeneous body is

$$(\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu\, \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathbf{u})^T = \rho\, \ddot{\mathbf{u}} \,. \tag{2.3}$$

Let us solve this problem in rectangular Cartesian coordinates and note that the results translate over to other coordinate systems. For a plane wave solution

$$u_i = \widehat{u}_i e^{i\omega(s_m x_m - t)} \tag{2.4}$$

we have

$$\boldsymbol{\nabla}\mathbf{u} \equiv u_{i,j} = i\omega s_m \delta_{mj} \widehat{u}_i e^{i\omega(s_m x_m - t)} = i\omega s_j u_i \tag{2.5}$$

and

$$(\boldsymbol{\nabla}\mathbf{u})^T \equiv u_{j,i} = i\omega s_i u_j \;. \tag{2.6}$$

Therefore,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathbf{u})^T \equiv u_{j,ii} = i\omega s_i u_{j,i} = (i\omega)^2 s_i s_i u_j = -\omega^2 s_i s_i u_j \equiv -\omega^2 (\mathbf{s} \cdot \mathbf{s})\mathbf{u} \;. \tag{2.7}$$

Also,

$$\boldsymbol{\nabla} \cdot \mathbf{u} \equiv u_{i,i} = i\omega s_i u_i \;. \tag{2.8}$$

Hence,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) \equiv u_{i,ij} = i\omega s_i u_{i,j} = (i\omega)^2 s_i s_j u_i \equiv -\omega^2 (\mathbf{s} \otimes \mathbf{s}) \cdot \mathbf{u} \;. \tag{2.9}$$

Finally

$$\dot{\mathbf{u}} \equiv \dot{u}_i = -i\omega \widehat{u}_i e^{i\omega(s_m x_m - t)} = -i\omega u_i \tag{2.10}$$

and

$$\ddot{\mathbf{u}} \equiv \ddot{u}_i = -i\omega \dot{u}_i = (i\omega)^2 u_i \equiv -\omega^2 \mathbf{u} \;. \tag{2.11}$$

Therefore, if we plug in the plane wave solution into the governing equations we get

$$(\lambda+\mu)\,(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}+\mu\,(\mathbf{s}\cdot\mathbf{s})\,\widehat{\mathbf{u}}=\rho\,\widehat{\mathbf{u}}\,. \tag{2.12}$$

Now

$$[(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}]\times\mathbf{s}\equiv e_{kij}s_is_p\widehat{u}_ps_j\equiv(\mathbf{s}\times\mathbf{s})(\mathbf{s}\cdot\widehat{\mathbf{u}})=\mathbf{0} \tag{2.13}$$

and

$$[(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}]\cdot\mathbf{s}\equiv s_is_j\widehat{u}_js_i\equiv(\mathbf{s}\cdot\mathbf{s})(\mathbf{u}\cdot\mathbf{s})\,. \tag{2.14}$$

Therefore, taking the cross product of the governing equation with $\mathbf{s}$ gives

$$(\lambda+\mu)\,[(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}]\times\mathbf{s}+\mu\,(\mathbf{s}\cdot\mathbf{s})\,(\widehat{\mathbf{u}}\times\mathbf{s})=\rho\,(\widehat{\mathbf{u}}\times\mathbf{s}) \tag{2.15}$$

or,

$$[\rho-\mu\,(\mathbf{s}\cdot\mathbf{s})]\,(\widehat{\mathbf{u}}\times\mathbf{s})=\mathbf{0}\,. \tag{2.16}$$

Using the definition of $\mathbf{s}$ we have $\mathbf{s}\cdot\mathbf{s}=1/c^2$, which leads to the first decoupled equation

$$\left[\rho-\frac{\mu}{c^2}\right](\widehat{\mathbf{u}}\times\mathbf{s})=\mathbf{0}\,. \qquad \square \tag{2.17}$$

Similarly, taking the dot product of the governing equation with $\mathbf{s}$, we have

$$(\lambda+\mu)\,[(\mathbf{s}\otimes\mathbf{s})\cdot\widehat{\mathbf{u}}]\cdot\mathbf{s}+\mu\,(\mathbf{s}\cdot\mathbf{s})\,(\widehat{\mathbf{u}}\cdot\mathbf{s})=\rho\,(\widehat{\mathbf{u}}\cdot\mathbf{s}) \tag{2.18}$$

or,

$$(\lambda+\mu)\,(\mathbf{s}\dot{\mathbf{s}})(\widehat{\mathbf{u}}\cdot\mathbf{s})+\mu\,(\mathbf{s}\cdot\mathbf{s})\,(\widehat{\mathbf{u}}\cdot\mathbf{s})=\rho\,(\widehat{\mathbf{u}}\cdot\mathbf{s}) \tag{2.19}$$

Replacing $\mathbf{s}\cdot\mathbf{s}$ with $1/c^2$ gives us our second decoupled equation

$$\left[\rho-\frac{\lambda+2\mu}{c^2}\right](\widehat{\mathbf{u}}\cdot\mathbf{s})=0\,. \qquad \square \tag{2.20}$$

Problem 2.2 Derive the relations

$$R_{pp} = \frac{4c_s^4\alpha^2\rho^2 - Z_p Z_s(1-2c_s^2\alpha^2)^2}{4c_s^4\alpha^2\rho^2 + Z_p Z_S(1-2c_s^2\alpha^2)^2} \,; \;\; R_{ps} = \frac{4c_s^2\alpha\rho Z_s(1-2c_s^2\alpha^2)}{4c_s^4\alpha^2\rho^2 + Z_p Z_S(1-2c_s^2\alpha^2)^2} \,. \tag{2.21}$$

starting from

$$\begin{aligned} \kappa_p^2 \sin(2\theta_{ip})(1-R_{pp}) &= \kappa_s^2\cos(2\theta_{rs})R_{ps} \\ (\kappa_s^2 - 2\kappa_p^2\sin^2\theta_{ip})(1+R_{pp}) &= \kappa_s^2\sin(2\theta_{rs})R_{ps} \,. \end{aligned} \tag{2.22}$$

Solution 2.2: From the first equation we have

$$2\frac{\kappa_p^2}{\kappa_s^2}\sin\theta_{ip}\cos\theta_{ip}(1-R_{pp}) = (1-2\sin^2\theta_{rs})R_{ps} \tag{2.23}$$

or,

$$2\frac{c_s^2}{c_p^2}\sin\theta_{ip}\cos\theta_{ip}(1-R_{pp}) = (1-2\sin^2\theta_{rs})R_{ps} \tag{2.24}$$

or,

$$2c_s^2\alpha\frac{\rho}{Z_p}(1-R_{pp}) = (1-2c_s^2\alpha^2)R_{ps} \,. \tag{2.25}$$

or,

$$2c_s^2\alpha\rho(1-R_{pp}) = Z_p(1-2c_s^2\alpha^2)R_{ps} \,. \tag{2.26}$$

From the second equation,

$$\left(1 - 2\frac{\kappa_p^2}{\kappa_s^2}\sin^2\theta_{ip}\right)(1+R_{pp}) = 2\sin\theta_{rs}\cos\theta_{rs}R_{ps} \tag{2.27}$$

or,

$$\left(1 - 2\frac{c_s^2}{c_p^2}\sin^2\theta_{ip}\right)(1+R_{pp}) = 2c_s^2\frac{\sin\theta_{rs}}{c_s}\frac{\cos\theta_{rs}}{c_s}R_{ps} \tag{2.28}$$

or,

$$\left(1-2c_s^2\alpha^2\right)(1+R_{pp}) = 2c_s^2\alpha\frac{\rho}{Z_s}R_{ps} \tag{2.29}$$

or,

$$Z_s\left(1-2c_s^2\alpha^2\right)(1+R_{pp}) = 2c_s^2\alpha\rho R_{ps} \,. \tag{2.30}$$

The required relations are obtained in a straightforward manner by solving equations (2.26) and (2.30) for R_{pp} and R_{ps}.

Problem 2.3 Verify the reflection coefficient relations given in the equations for a P-wave incident upond th einterface between two solids using a symbolic computation tool if needed. Solve these equations numerically and plot the magnitude and phase of the reflection and transmission coefficients as a function of the angle of incidence for the interface between two materials with $\rho_1 = 820$, $c_{p1} = 1320$, $c_{s1} = 1.0e-4$, $\rho_2 =1000$, $c_{p2} =1500$, $c_{s2} = 1.0e-4$.

Solution 2.3: A Mathematica script that shows that the calculation is given below.

```
In[1]:= Phr := Rp*Phi
Pht := Tp*Phi
Psr := Rs*Phi
Pst := Ts*Phi
pi = Phi*Exp[I* kp1 *(x1*sti - x2*cti)]
pr = Phr*Exp[I* kp1 *(x1*str + x2*ctr)]
pt = Pht*Exp[I* kp2 *(x1*stt - x2*ctt)]
sr = Psr*Exp[I* ks1 *(x1*strs + x2*ctrs)]
st = Pst*Exp[I* ks2 *(x1*stts - x2*ctts)]
ui1 := D[pi,x1]
ui2 := D[pi,x2]
ur1 := D[pr,x1]
ur2 := D[pr,x2]
ut1 := D[pt,x1]
ut2 := D[pt,x2]
urs1 := D[sr,x2]
urs2 := -D[sr,x1]
uts1 := D[st,x2]
uts2 := -D[st,x1]
s12i := 2*mu1*D[D[pi,x1],x2]
s12r := 2*mu1*D[D[pr,x1],x2]
s12t := 2*mu2*D[D[pt,x1],x2]
s12rs := mu1*(D[D[sr,x2],x2]-D[D[sr,x1],x1])
s12ts := mu2*(D[D[st,x2],x2]-D[D[st,x1],x1])
s22i := lambda1*(D[D[pi,x1],x1]+D[D[pi,x2],x2])+2*mu1*D[D[pi,x2],x2]
s22r := lambda1*(D[D[pr,x1],x1]+D[D[pr,x2],x2])+2*mu1*D[D[pr,x2],x2]
s22t := lambda2*(D[D[pt,x1],x1]+D[D[pt,x2],x2])+2*mu2*D[D[pt,x2],x2]
s22rs := -2*mu1*D[D[sr,x1],x2]
s22ts := -2*mu2*D[D[st,x1],x2]
s12top := FullSimplify[s12i + s12r + s12rs/. x2->0]
s12bot := FullSimplify[s12t + s12ts/. x2->0]
s12diff = Collect[FullSimplify[(s12bot - s12top)/(Phi*Exp[I kp1 sti x1])],
        {Rp, Rs, Tp, Ts}]
s22top := FullSimplify[s22i + s22r + s22rs/. x2->0]
s22bot := FullSimplify[s22t + s22ts/. x2->0]
s22diff = Collect[FullSimplify[(s22bot - s22top)/(Phi*Exp[I kp1 sti x1])],
        {Rp, Rs, Tp, Ts}]
u1top := FullSimplify[ui1 + ur1 + urs1/. x2->0]
u1bot := FullSimplify[ut1 + uts1/. x2->0]
u1diff = Collect[FullSimplify[(u1bot - u1top)/(Phi*Exp[I kp1 sti x1])],
        {Rp, Rs, Tp, Ts}]
u2top := FullSimplify[ui2 + ur2 + urs2/. x2->0]
u2bot := FullSimplify[ut2 + uts2/. x2->0]
u2diff = Collect[FullSimplify[(u2bot - u2top)/(Phi*Exp[I kp1 sti x1])],
        {Rp, Rs, Tp, Ts}]
s12diffa = s12diff /. {ctr -> cti, str -> sti, stt -> kp1*sti/kp2,
        strs -> kp1*sti/ks1, stts -> kp1*sti/ks2}
s22diffa = s22diff /. {ctr -> cti, str -> sti, stt -> kp1*sti/kp2,
        strs -> kp1*sti/ks1, stts -> kp1*sti/ks2}
u1diffa = u1diff /. {ctr -> cti, str -> sti, stt -> kp1*sti/kp2,
        strs -> kp1*sti/ks1, stts -> kp1*sti/ks2}
u2diffa = u2diff /. {ctr -> cti, str -> sti, stt -> kp1*sti/kp2,
        strs -> kp1*sti/ks1, stts -> kp1*sti/ks2}
s12diffb := s12diffa /. {kp1 ->omega/cp1, kp2 ->omega/cp2, ks1->omega/cs1,
        ks2->omega/cs2, mu1->cs1^2 rho1, mu2->cs2^2 rho2,
        lambda1->rho1 cp1^2 - 2 mu1, lambda2->rho2 cp2^2 - 2 mu2}
s22diffb := s22diffa /. {kp1 ->omega/cp1, kp2 ->omega/cp2, ks1->omega/cs1,
        ks2->omega/cs2, mu1->cs1^2 rho1, mu2->cs2^2 rho2,
        lambda1->rho1 cp1^2 - 2 mu1, lambda2->rho2 cp2^2 - 2 mu2}
u1diffb := u1diffa /. {kp1 ->omega/cp1, kp2 ->omega/cp2, ks1->omega/cs1,
        ks2->omega/cs2, mu1->cs1^2 rho1, mu2->cs2^2 rho2,
        lambda1->rho1 cp1^2 - 2 mu1, lambda2->rho2 cp2^2 - 2 mu2}
u2diffb := u2diffa /. {kp1 ->omega/cp1, kp2 ->omega/cp2, ks1->omega/cs1,
        ks2->omega/cs2, mu1->cs1^2 rho1, mu2->cs2^2 rho2,
        lambda1->rho1 cp1^2 - 2 mu1, lambda2->rho2 cp2^2 - 2 mu2}
s12diffc = s12diffb /. {sti -> alpha*cp1, strs -> alpha*cs1, stt -> alpha*cp2,
        stts -> alpha*cs2, cti -> rho1*cp1/Zp1, ctt -> rho2*cp2/Zp2,
        ctrs -> rho1*cs1/Zs1, ctts -> rho2*cs2/Zs2}
s22diffc = s22diffb /. { sti -> alpha*cp1, strs -> alpha*cs1, stt -> alpha*cp2,
        stts -> alpha*cs2, cti -> rho1*cp1/Zp1, ctt -> rho2*cp2/Zp2,
        ctrs -> rho1*cs1/Zs1, ctts -> rho2*cs2/Zs2}
u1diffc = u1diffb /. {sti -> alpha*cp1, strs -> alpha*cs1, stt -> alpha*cp2,
        stts -> alpha*cs2, cti -> rho1*cp1/Zp1, ctt -> rho2*cp2/Zp2,
        ctrs -> rho1*cs1/Zs1, ctts -> rho2*cs2/Zs2}
u2diffc = u2diffb /. {sti -> alpha*cp1, strs -> alpha*cs1, stt -> alpha*cp2,
        stts -> alpha*cs2, cti -> rho1*cp1/Zp1, ctt -> rho2*cp2/Zp2,
        ctrs -> rho1*cs1/Zs1, ctts -> rho2*cs2/Zs2}
s12diffd := -s12diffc /. {mu1->cs1^2 rho1, mu2->cs2^2 rho2}
s22diffd := -s22diffc /. {mu1->cs1^2 rho1, mu2->cs2^2 rho2}
u1diffd := -u1diffc /. {mu1->cs1^2 rho1, mu2->cs2^2 rho2}
u2diffd := -u2diffc /. {mu1->cs1^2 rho1, mu2->cs2^2 rho2}
s12diffe = Collect[FullSimplify[s12diffd*(Zp1*Zp2*Zs1^2*Zs2^2/omega^2)],
        {Rp, Rs, Tp, Ts}, Simplify]
s22diffe = Collect[FullSimplify[s22diffd*(Zp1^2*Zp2^2*Zs1*Zs2/omega^2)],
        {Rp, Rs, Tp, Ts}, Simplify]
u1diffe = Collect[FullSimplify[u1diffd*(Zs1*Zs2/(I*omega))],
        {Rp, Rs, Tp, Ts}, Simplify]
u2diffe = Collect[FullSimplify[u2diffd*(Zp1*Zp2/(I*omega))],
        {Rp, Rs, Tp, Ts}, Simplify]
Out[5]= E^(I kp1 (sti x1-cti x2)) Phi
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Out[6]= E^(I kp1 (str x1+ctr x2)) Phi Rp
Out[7]= E^(I kp2 (stt x1-ctt x2)) Phi Tp
Out[8]= E^(I ks1 (strs x1+ctrs x2)) Phi Rs
Out[9]= E^(I ks2 (stts x1-ctts x2)) Phi Ts
Out[32]= -2 cti kp1^2 mu1 sti+2 ctr E^(-I kp1 sti x1+I kp1 str x1) kp1^2 mu1 Rp str
        +E^(-I kp1 sti x1) Rs (ctrs^2 E^(I ks1 strs x1) ks1^2 mu1
        -E^(I ks1 strs x1) ks1^2 mu1 strs^2)
        +2 ctt E^(-I kp1 sti x1+I kp2 stt x1) kp2^2 mu2 stt Tp
        +E^(-I kp1 sti x1+I ks2 stts x1) ks2^2 mu2 (-ctts^2+stts^2) Ts
Out[35]= kp1^2 (cti^2 (lambda1+2 mu1)+lambda1 sti^2)+
        E^(-I kp1 sti x1+I kp1 str x1) kp1^2 Rp (ctr^2 (lambda1+2 mu1)+lambda1 str^2)
        -2 ctrs E^(-I kp1 sti x1+I ks1 strs x1) ks1^2 mu1 Rs strs
        -E^(-I kp1 sti x1+I kp2 stt x1) kp2^2 (ctt^2 (lambda2+2 mu2)+lambda2 stt^2) Tp
        -2 ctts E^(-I kp1 sti x1+I ks2 stts x1) ks2^2 mu2 stts Ts
Out[38]= -I ctrs E^(-I kp1 sti x1+I ks1 strs x1) ks1 Rs
        -I kp1 sti-I E^(-I kp1 sti x1+I kp1 str x1) kp1 Rp str
        +I E^(-I kp1 sti x1+I kp2 stt x1) kp2 stt Tp
        -I ctts E^(-I kp1 sti x1+I ks2 stts x1) ks2 Ts
Out[41]= I cti kp1-I ctr E^(-I kp1 sti x1+I kp1 str x1) kp1 Rp
        +I E^(-I kp1 sti x1+I ks1 strs x1) ks1 Rs strs
        -I ctt E^(-I kp1 sti x1+I kp2 stt x1) kp2 Tp
        -I E^(-I kp1 sti x1+I ks2 stts x1) ks2 stts Ts
Out[42]= -2 cti kp1^2 mu1 sti+2 cti kp1^2 mu1 Rp sti
        +E^(-I kp1 sti x1) Rs (ctrs^2 E^(I kp1 sti x1) ks1^2 mu1
        -E^(I kp1 sti x1) kp1^2 mu1 sti^2)
        +2 ctt kp1 kp2 mu2 sti Tp+ks2^2 mu2 (-ctts^2+(kp1^2 sti^2)/ks2^2) Ts
Out[43]= -2 ctrs kp1 ks1 mu1 Rs sti+kp1^2 (cti^2 (lambda1+2 mu1)+lambda1 sti^2)
        +kp1^2 Rp (cti^2 (lambda1+2 mu1)+lambda1 sti^2)
        -kp2^2 (ctt^2 (lambda2+2 mu2)+(kp1^2 lambda2 sti^2)/kp2^2) Tp
        -2 ctts kp1 ks2 mu2 sti Ts
Out[44]= -I ctrs ks1 Rs-I kp1 sti-I kp1 Rp sti+I kp1 sti Tp-I ctts ks2 Ts
Out[45]= I cti kp1-I cti kp1 Rp+I kp1 Rs sti-I ctt kp2 Tp-I kp1 sti Ts
Out[50]= -((2 alpha cs1^2 omega^2 rho1^2)/Zp1)+(2 alpha cs1^2 omega^2 rho1^2 Rp)/Zp1
        +(2 alpha cs2^2 omega^2 rho2^2 Tp)/Zp2
        +E^(-I alpha omega x1) Rs (-alpha^2 cs1^2 E^(I alpha omega x1) omega^2 rho1
        +(cs1^2 E^(I alpha omega x1) omega^2 rho1^3)/Zs1^2)
        +omega^2 rho2 Ts (alpha^2 cs2^2-(cs2^2 rho2^2)/Zs2^2)
Out[51]= (omega^2 (alpha^2 cp1^2 (-2 mu1+cp1^2 rho1)
        +(cp1^2 rho1^2 (-2 mu1+cp1^2 rho1+2 cs1^2 rho1))/Zp1^2))/cp1^2
        +(1/(cp1^2))omega^2 Rp (alpha^2 cp1^2 (-2 mu1+cp1^2 rho1)
        +(cp1^2 rho1^2 (-2 mu1+cp1^2 rho1+2 cs1^2 rho1))/Zp1^2)
        -(1/(cp2^2))omega^2 Tp (alpha^2 cp2^2 (-2 mu2+cp2^2 rho2)
        +(cp2^2 rho2^2 (-2 mu2+cp2^2 rho2+2 cs2^2 rho2))/Zp2^2)
        -(2 alpha cs1^2 omega^2 rho1^2 Rs)/Zs1-(2 alpha cs2^2 omega^2 rho2^2 Ts)/Zs2
Out[52]= -I alpha omega-I alpha omega Rp+I alpha omega Tp
        -(I omega rho1 Rs)/Zs1-(I omega rho2 Ts)/Zs2
Out[53]= I alpha omega Rs-I alpha omega Ts+(I omega rho1)/Zp1
        -(I omega rho1 Rp)/Zp1-(I omega rho2 Tp)/Zp2
Out[58]= -2 alpha cs2^2 rho2^2 Tp Zp1 Zs1^2 Zs2^2
        +2 alpha cs1^2 rho1^2 Zp2 Zs1^2 Zs2^2-2 alpha cs1^2 rho1^2 Rp Zp2 Zs1^2 Zs2^2
        +cs1^2 rho1 Rs Zp1 Zp2 (-rho1^2+alpha^2 Zs1^2) Zs2^2
        +cs2^2 rho2 Ts Zp1 Zp2 Zs1^2 (rho2^2-alpha^2 Zs2^2)
Out[59]= 2 alpha cs2^2 rho2^2 Ts Zp1^2 Zp2^2 Zs1
        +2 alpha cs1^2 rho1^2 Rs Zp1^2 Zp2^2 Zs2
        +rho1 (2 alpha^2 cs1^2 Zp1^2-cp1^2 (rho1^2+alpha^2 Zp1^2)) Zp2^2 Zs1 Zs2
        +rho1 Rp (2 alpha^2 cs1^2 Zp1^2-cp1^2 (rho1^2+alpha^2 Zp1^2)) Zp2^2 Zs1 Zs2
        +rho2 Tp Zp1^2 (cp2^2 rho2^2+alpha^2 (cp2^2-2 cs2^2) Zp2^2) Zs1 Zs2
Out[60]= rho2 Ts Zs1+rho1 Rs Zs2+alpha Zs1 Zs2+alpha Rp Zs1 Zs2-alpha Tp Zs1 Zs2
Out[61]= rho2 Tp Zp1-rho1 Zp2+rho1 Rp Zp2-alpha Rs Zp1 Zp2+alpha Ts Zp1 Zp2
```
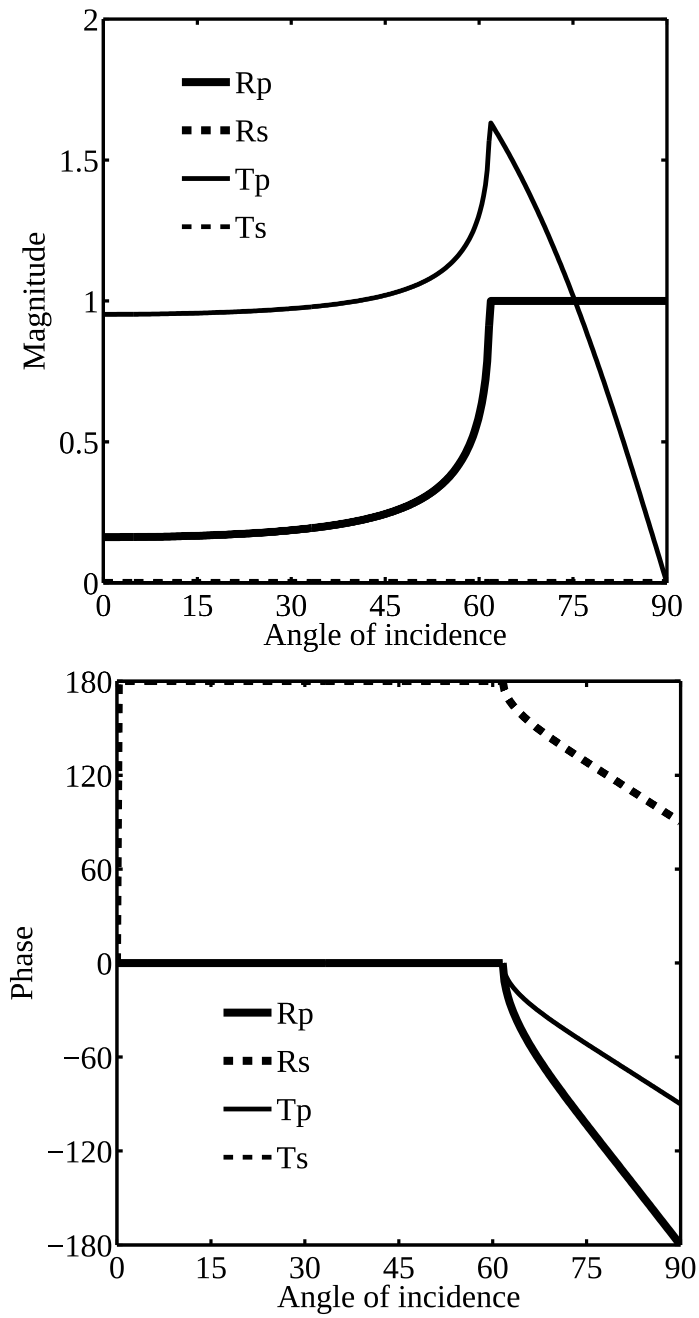

Plots of the magnitude and phase for the reflection and transmission coefficients are given below.

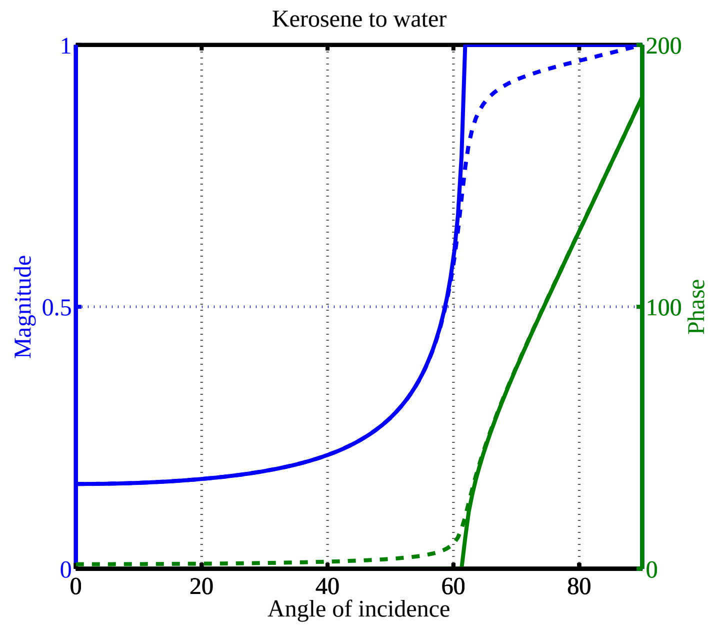


2
1.5
1
0.5
0
Rp
Rs
Tp
Ts
Magnitude
0
15
30
45
60
75
90
Angle of incidence
180
120
60
0
−60
−120
−180
Phase
Rp
Rs
Tp
Ts
0
15
30
45
60
75
90
Angle of incidence

Problem 2.4 Consider a plane acoustic wave propagating from kerosene into water (at room temperature). Kerosene has a density of 820 kg/m^3 and a sound speed of 1320 m/s while water has a density of 1000 kg/m^3 and s sound speed of 1500 m/s. Plot the magnitude and phase of the reflection coefficient (R) as a function of the angle of incidence. Is there any angle at which the entire energy of the wave is transmitted through the interface? At what angle does total internal reflection occur (i.e., the transmission coefficient becomes zero)? What happens as the angle of incidence is increased beyond the angle at which total internal reflection first occurs?

Now consider the case where the materials absorb a small fraction of the energy of the acoustic wave. In that case we can add a damping factor (α) to the refractive index n, i.e, $n \to n(1+i\alpha)$. Plot the magnitude and phase and a function of incidence angle for α = 0.01. Is there total internal reflection in this situation?

Solution 2.4: See the plot below (the phases have been multiplied by -1).



There is no angle at which there is no reflection. Total internal reflection occurs at approximately 62°. At incidence angles greater than 62° $R = 1$ but the phase of the reflected wave changes. There is no total internal reflection for absorbing media and there is a phase lag between incident and reflected waves.

Problem 2.5 We have defined the refractive index for acoustic waves propagating from a medium with phase velocity c_1 into a medium with phase velocity c_2 as $n = c_1/c_2$. If we choose a reference medium, e.g., air with a sound speed of c_0, the we can have an alternative definition of the refractive indexes n_1 and n_2 of the two media given by $n_1 = c_0/c_1$ and $n_2 = c_0/c_2$ in which case $n = n_2/n_1$. We have mentioned earlier that waves cannot propagate in the medium if the phase velocity is imaginary. How then can waves propagate in a medium with a complex refractive index?

Solution 2.5: The phase velocity is related to the wave number ($k = \|\mathbf{k}\|$) by $k = \omega/c$. Therefore, for a medium with phase velocity c_1, we have $k = \omega/c_1 = \omega n_1/c_0$. So we can write a plane wave solution in the form

$$p = p_0 e^{i(\omega n_1 x/c_0 - \omega t)} \; . \tag{2.31}$$

If $n_1 = n_1(1 + i\alpha)$ we have

$$p = p_0 e^{i[\omega n_1 (1+i\alpha) x/c_0 - \omega t]} = p_0 e^{i[\omega n_1 x/c_0 - \omega t] - \omega n_1 \alpha x/c_0} \tag{2.32}$$

or

$$p = p_0 e^{-\omega n_1 \alpha x/c_0} e^{i(\omega n_1 x/c_0 - \omega t)} \; . \tag{2.33}$$

Hence the phase speed remains real and only the amplitude decreases because of the imaginary part of the refractive index.

Problem 2.6 Maxwell's equations for an isotropic material at fixed frequency may be expressed as

$$\boldsymbol{\nabla} \times \mathbf{E} = i\omega\mu\mathbf{H} \; ; \;\; \boldsymbol{\nabla} \cdot \mathbf{H} = 0 \; ; \;\; \boldsymbol{\nabla} \cdot \mathbf{E} = 0 \, . \tag{2.34}$$

Show that for a plane wave electric field $\mathbf{E}(\mathbf{x}) = \mathbf{E}_0 \exp(i\mathbf{k} \cdot \mathbf{x})$ the wavenumber vector is perpendicular to the fields, i.e., $\mathbf{k} \cdot \mathbf{E}_0 = 0$. Then show that this implies that the magnetic field is also a plane wave of the form $\mathbf{H}(\mathbf{x}) = \mathbf{H}_0 \exp(i\mathbf{k} \cdot \mathbf{x})$ where

$$\mathbf{H}_0 = -\frac{1}{\omega\mu}(\mathbf{k} \times \mathbf{E}_0) \quad \text{and} \quad \mathbf{k} \cdot \mathbf{H}_0 = 0 \, . \tag{2.35}$$

Recall also that for fixed frequency

$$\nabla^2 \mathbf{H} + \frac{\omega^2}{c^2}\mathbf{H} = \mathbf{0} \, . \tag{2.36}$$

Show that the above equation implies that for a plane wave

$$(\|\mathbf{k}\|)^2 = \frac{\omega^2}{c^2} \, . \tag{2.37}$$

Solution 2.6: It is convenient to work out this exercise in rectangular Cartesian coordinates.

First, from the relation $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ we have

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\partial}{\partial x_r}\left(E_{0r} e^{ik_m x_m}\right) = iE_{0r}k_m\delta_{mr}e^{i\mathbf{k}\cdot\mathbf{x}} = iE_{0r}k_r e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.38}$$

or,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = i(\mathbf{k} \cdot \mathbf{E}_0)e^{i\mathbf{k}\cot\mathbf{x}} = 0 \, . \tag{2.39}$$

Therefore,

$$\mathbf{k} \cdot \mathbf{E}_0 = 0 \, . \qquad \Box \tag{2.40}$$

Recall that for a vector field $\mathbf{v}(\mathbf{x})$

$$\boldsymbol{\nabla} \times \mathbf{v} = e_{pqr} v_{r,q} \mathbf{e}_p \, . \tag{2.41}$$

So we have

$$\boldsymbol{\nabla} \times \mathbf{E} = e_{pqr}E_{r,q}\mathbf{e}_p = e_{pqr}E_{0r}\frac{\partial}{\partial x_q}\left(e^{ik_m x_m}\right)\mathbf{e}_p = ie_{pqr}E_{0r}k_m\delta_{mq}e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_p = i(e_{pqr}k_qE_{0r}\mathbf{e}_p)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.42}$$

or

$$\boldsymbol{\nabla} \times \mathbf{E} = (\mathbf{k} \times \mathbf{E}_0)e^{i\mathbf{k}\cdot\mathbf{x}} = i\omega\mu\mathbf{H} \, . \tag{2.43}$$

Hence,

$$\mathbf{H} = -\frac{i}{\omega\mu}(\mathbf{k} \times \mathbf{E}_0)e^{i\mathbf{k}\cdot\mathbf{x}} = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.44}$$

where

$$\mathbf{H}_0 := \frac{1}{i\omega\mu}(\mathbf{k} \times \mathbf{E}_0) \, . \tag{2.45}$$

Hence the magnetic field also has the form of a plane wave. This field has to satisfy the relation $\boldsymbol{\nabla} \cdot \mathbf{H} = 0$, i.e,

$$\boldsymbol{\nabla} \cdot \mathbf{H} = \frac{\partial}{\partial x_r}\left(H_{0r}e^{ik_m x_m}\right) = iH_{0r}k_m\delta_{mr}e^{i\mathbf{k}\cdot\mathbf{x}} = iH_{0r}k_r e^{i\mathbf{k}\cdot\mathbf{x}} = i(\mathbf{k} \cdot \mathbf{H}_0)e^{\mathbf{k}\cdot\mathbf{x}} = 0 \, . \tag{2.46}$$

Therefore,

$$\mathbf{k} \cdot \mathbf{H}_0 = 0 \, . \qquad \Box \tag{2.47}$$

The Laplacian of the magnetic field is

$$\begin{aligned}\nabla^2\mathbf{H} &= H_{i,jj}\mathbf{e}_i = \frac{\partial^2 H_i}{\partial x_j \partial x_j}\mathbf{e}_i \\ &= \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}\left(H_{0i}\, e^{ik_m x_m}\right)\right]\mathbf{e}_i = \frac{\partial}{\partial x_j}\left[iH_{0i}k_m\delta_{mj}e^{ik_m x_m}\right]\mathbf{e}_i = \frac{\partial}{\partial x_j}\left[iH_{0i}k_j e^{ik_m x_m}\right]\mathbf{e}_i \\ &= -H_{0i}k_j k_m \delta_{mj} e^{ik_m x_m}\mathbf{e}_i = -H_{0i}k_j k_j e^{ik_m x_m}\mathbf{e}_i = -(\mathbf{k}\cdot\mathbf{k})\mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}}\,.\end{aligned} \tag{2.48}$$

Therefore

$$\nabla^2\mathbf{H} + \frac{\omega^2}{c^2}\,\mathbf{H} = -(\mathbf{k}\cdot\mathbf{k})\mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{\omega^2}{c^2}\mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}} = \mathbf{0} \tag{2.49}$$

which implies that

$$\mathbf{k}\cdot\mathbf{k} = (\|\mathbf{k}\|)^2 = \frac{\omega^2}{c^2}\,. \qquad \Box \tag{2.50}$$

Problem 2.7 Express Fresnel's equations for perpendicular incidence in terms of electromagnetic impedances and then calculate the reflection and transmission coefficients for a medium that is impedance matched with a silicone rubber dielectric material.

Solution 2.7: Fresnel's equations, for a **E** wave polarized perpendicular to the plane of incidence, are

$$R = \frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t} \tag{2.51}$$

and

$$T = \frac{2\frac{n_1}{\mu_1}\cos\theta_i}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}\,. \tag{2.52}$$

For the situation where the incident **E** wave is polarized perpendicular to the plane of incidence, the electrical impedance is defined as

$$Z_i = \frac{1}{\cos\theta_i}\sqrt{\frac{\mu_i}{\varepsilon_i}} = \frac{c_0\mu_i}{n_i\cos\theta_i}\,. \tag{2.53}$$

Since the polarization does not change when we move from medium 1 to medium 2 or vice versa, we can write

$$Z_1 = \frac{c_0\mu_1}{n_1\cos\theta_i} \quad \text{and} \quad Z_2 = \frac{c_0\mu_2}{n_2\cos\theta_t}\,. \tag{2.54}$$

Plugging these into the expressions for R and T gives us

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \text{and} \quad T = \frac{2Z_2}{Z_2 + Z_1} \qquad \square \tag{2.55}$$

Let us assume that medium 2 is a silicone rubber. We want to find a medium 1 that is impedance matched with medium 2, i.e., $Z_1 = Z_2$, $R = 0$, and $T = 1$. Then we must find a medium that satisfies

$$\frac{\mu_1}{n_1\cos\theta_i} = \frac{\mu_2}{n_2\cos\theta_t}\,. \tag{2.56}$$

From Snell's law,

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i \qquad \Longrightarrow \qquad \cos\theta_t = \frac{1}{n_2}\sqrt{n_2^2 - n_1^2\sin^2\theta_i}\,. \tag{2.57}$$

Plugging the first form of Snell's law into (2.56) gives us

$$\frac{\tan\theta_t}{\tan\theta_i} = \frac{\mu_1}{\mu_2}\,. \tag{2.58}$$

Since $\theta_i = \theta_t$ in impedance-matched media, we must have $\mu_1 = \mu_2$. If we plug the second form of Snell's law into (2.56) and rearrange, we have

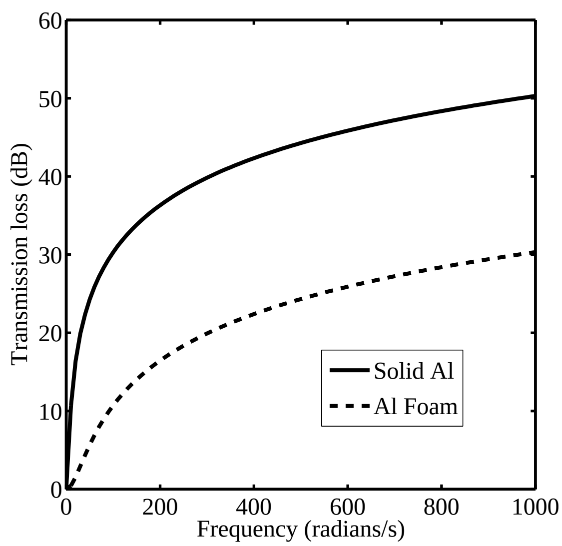
$$\frac{n_2^2}{\mu_2^2} = \frac{n_1^2}{\mu_2^2}\sin^2\theta_i + \frac{n_1^2}{\mu_1^2}\cos^2\theta_i\,. \tag{2.59}$$

Since impedance matching requires that $\mu_1 = \mu_2$, we must have $n_1 = n_2$ (for ordinary materials) and therefore $\varepsilon_1 = \varepsilon_2$. Therefore, we will have to find a material that has exactly the same electrical properties as silicone rubber for impedance matching.

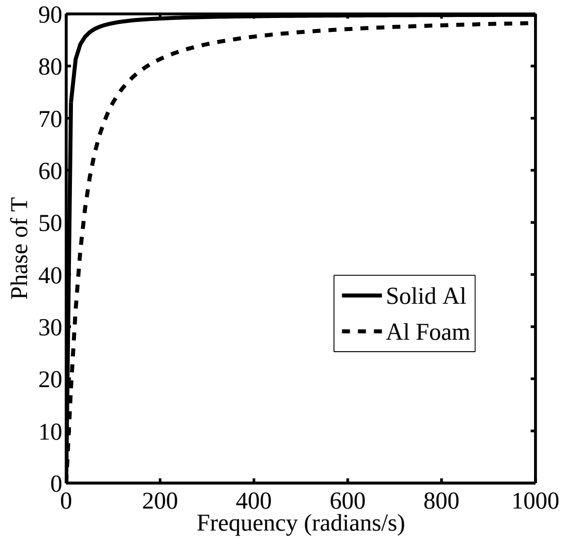
Problem 2.8 Consider a slab of material in an impedance tube. The bulk modulus and density of air on both sides of the slab are 1.42×10^5 Pa and 1.20 kg/m^3, respectively. The Young's modulus (E), Poisson's ratio (ν), and density (ρ) of aluminum are 70 GPa, 0.33, and 2700 kg/m^3, respectively. Assume that the phase velocity in aluminum can be obtained from the relation $c = \sqrt{\kappa/\rho}$ where $\kappa = E/(3(1-2\nu))$ is the bulk modulus.

1. The transmission loss due to the slab is calculated using the relation
$$\text{TL (dB)} = 10\log_{10}\left(\frac{1}{T^2}\right) \tag{2.60}$$
where T is the transmission coefficient. Plot the transmission loss for a 10 cm thick aluminum slab. Compare the transmission loss due to the solid slab with that for a similar slab made of aluminum foam with an aluminum volume fraction (f) of 10%. Assume that the effective foam density is given by $\rho_{\text{eff}} = f\rho_1 + (1-f)\rho_2$ and that the effective foam Young's modulus is given by $E_{\text{eff}} = E(\rho_{\text{eff}}/rho)^2$. The Poisson's ratio of the foam is 0.33. What does the imaginary part of the transmission coefficient indicate? What is the effect of slab density on the transmission loss?
2. Next plot the transmission losses for aluminum and aluminum slabs for a fixed frequency as a function of slab thickness. Assume a frequency of 100 Hz and keep in mind that ω has units of radians/s and not cycles/s. Such a plot is called a mass law plot in acoustics. What would the mass law effect be if ρ_{eff} were a function of frequency and the system had a resonance frequency of 100 Hz?

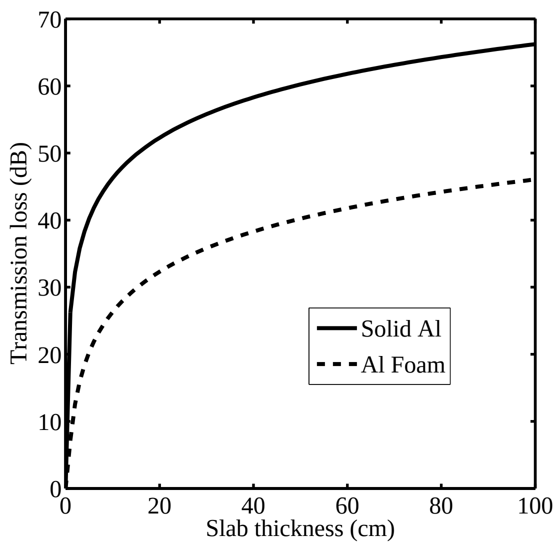
Solution 2.8: 1) Assume normal incidence. The quantities that we need to calculate the transmission coefficient for the foam are E = 0.7 GPa, ρ = 271 kg/m^3. The transmission loss plots is below.



A plot of the phase of the transmission loss is shown below.



2) For a frequency of $\omega = (2\pi)(100)$ radians/s the transmission loss plot is as shown in the plot below.



Problem 2.9 Show that the transmission coefficient for an slab with incident TE-waves can be expressed as

$$T_{13} = \frac{T_{12}T_{23}\, e^{ik_{z2}(d_2-d_1)}}{1 - R_{21}R_{23}\, e^{2ik_{z2}(d_2-d_1)}}. \tag{2.61}$$

Also verify that the series expansion of the above equation is

$$T_{13} = T_{12}T_{23}\, e^{i\theta} + T_{12}T_{23}R_{21}R_{23}\, e^{3i\theta} + T_{12}T_{23}R_{21}^2R_{23}^2\, e^{5i\theta} + \dots . \tag{2.62}$$

Solution 2.9: The process of finding the transmission coefficient is identical to that used to find the generalized reflection coefficient.

As before, we superpose solutions of the form

$$E_y(Z) = E_0 \exp(\pm i k_z Z) \tag{2.63}$$

where $Z = 0$ at the interface. To make sure that the above form can be used in all the layers, we will express all equations in a single coordinate system with $z = -d_1$ at interface $1-2$ and $z = -d_2$ at interface $2-3$.

In medium 1, the electric field consists of a incident part and a reflected part,

$$\begin{aligned} E_{y1}(Z) = E_i + E_r &= E_0 \exp(-ik_{z1}Z) + \widetilde{R}_{12}E_0 \exp(ik_{z1}Z) \\ &= E_0 \exp(-ik_{z1}Z)\left[1 + \widetilde{R}_{12}\exp(2ik_{z1}Z)\right] \end{aligned} \tag{2.64}$$

where $\widetilde{R}_{23}$ is the generalized reflection coefficient at the interface $1-2$. Let us now change the variable so that interface $1-2$ with $Z = 0$ is at $z = -d_1$, i.e, we set $Z = z + d_1$. Then we can write the above equation as

$$\begin{aligned} E_{y1}(z) &= E_0 \exp[-ik_{z1}(z+d_1)]\left[1 + \widetilde{R}_{12}\exp[2ik_{z1}(z+d_1)]\right] \\ &= E_0 \exp(-ik_{z1}d_1)\left[\exp(-ik_{z1}z) + \widetilde{R}_{12}\exp[ik_{z1}(z+2d_1)]\right] \\ &= A_1\left[\exp(-ik_{z1}z) + \widetilde{R}_{12}\exp[ik_{z1}(z+2d_1)]\right] \end{aligned} \tag{2.65}$$

where $A_1 := E_0 \exp(-ik_{z1}d_1)$. Similarly, in medium 2, we have

$$E_{y2}(Z) = E_i + E_r = A\exp(-ik_{z2}Z) + \widetilde{R}_{23}A\exp(ik_{z2}Z) \tag{2.66}$$

where A is the amplitude of the incident wave in medium 2 and $\widetilde{R}_{23}$ is the generalized reflection coefficient at interface $2-3$. A change of variables, $Z = z + d_2$, gives us

$$\begin{aligned} E_{y2}(z) &= A\exp[-ik_{z2}(z+d_2)] + \widetilde{R}_{23}A\exp[ik_{z2}(z+d_2)] \\ &= A\exp(-ik_{z2}d_2)\left[\exp(-ik_{z2}z) + \widetilde{R}_{23}\exp[ik_{z2}(z+2d_2)]\right] \\ &= A_2\left[\exp(-ik_{z2}z) + \widetilde{R}_{23}\exp[ik_{z2}(z+2d_2)]\right] \end{aligned} \tag{2.67}$$

where $A_2 := A\exp(-ik_{z2}d_2)$. There is no reflected wave in medium 3 and we have

$$E_{y3}(Z) = E_i = B\exp(-ik_{z3}Z) \tag{2.68}$$

where B is the amplitude of the transmitted wave in medium 3. With a change of variables $Z = z + d_2$ we have

$$E_{y3}(z) = B\exp[-ik_{z3}(z+d_2)] = A_3\exp(-ik_{z3}z) \tag{2.69}$$

where $A_3 := B\exp(-ik_{z3}d_2)$.

The electric field in the three layers, expressed in a single coordinate system with $z = -d_1$ at interface $1 - 2$ and $z = -d_2$ at interface $2 - 3$, are therefore

$$\begin{aligned} E_{y1}(z) &= A_1 \left[\exp(-ik_{z1}z) + \widetilde{R}_{12}\exp[ik_{z1}(z + 2d_1)]\right] \\ E_{y2}(z) &= A_2 \left[\exp(-ik_{z2}z) + \widetilde{R}_{23}\exp[ik_{z2}(z + 2d_2)]\right] \\ E_{y3}(z) &= A_3 \exp(-ik_{z3}z)\,. \end{aligned} \tag{2.70}$$

Let us now examine the fields above and below interface $1 - 2$ at $z = -d_1$. From the above equations we have

$$\begin{aligned} E_{y1}(-d_1) &= A_1 \left[\exp(ik_{z1}d_1) + \widetilde{R}_{12}\exp[ik_{z1}d_1]\right] \\ E_{y2}(-d_1) &= A_2 \left[\exp(ik_{z2}d_1) + \widetilde{R}_{23}\exp[ik_{z2}(-d_1 + 2d_2)]\right]\,. \end{aligned} \tag{2.71}$$

Similarly, at interface $2 - 3$, we have

$$\begin{aligned} E_{y2}(-d_2) &= A_2 \left[\exp(ik_{z2}d_2) + \widetilde{R}_{23}\exp[ik_{z2}d_2]\right] \\ E_{y3}(-d_2) &= A_3 \exp(ik_{z3}d_2)\,. \end{aligned} \tag{2.72}$$

Consider medium 2 below interface $1 - 2$. Then the transmitted wave from medium 1 has the form

$$E_t = T_{12}A_1\exp(ik_{z1}d_1) \tag{2.73}$$

where T_{12} is the transmission coefficient going from medium 1 to medium 2. The reflected wave from interface $2 - 3$ is also reflected at interface $1 - 2$ and adds to the transmitted wave from medium 1 to medium 2. This reflected wave has the form

$$E_r = R_{21}A_2\widetilde{R}_{23}\exp[ik_{z2}(-d_1 + 2d_2)] \tag{2.74}$$

where R_{21} is the reflection coefficient going from medium 2 to medium 1. These two waves sum to the downgoing wave in medium 2,

$$E_t + E_r = E_i = A_2\exp(ik_{z2}d_1)\,. \tag{2.75}$$

Plugging in the expressions for E_t and E_r,

$$A_2\exp(ik_{z2}d_1) = T_{12}A_1\exp(ik_{z1}d_1) + R_{21}A_2\widetilde{R}_{23}\exp[ik_{z2}(-d_1 + 2d_2)] \tag{2.76}$$

or

$$A_2 = T_{12}A_1\exp[i(k_{z1} - k_{z2})d_1] + R_{21}A_2\widetilde{R}_{23}\exp[2ik_{z2}(-d_1 + d_2)] \tag{2.77}$$

or

$$\frac{A_2}{A_1} = \frac{T_{12}\exp[i(k_{z1} - k_{z2})d_1]}{1 - R_{21}\widetilde{R}_{23}\exp[2ik_{z2}(d_2 - d_1)]}\,. \tag{2.78}$$

Now consider the transmitted wave going from medium 1 to medium 3. In medium 1, the downgoing wave at interface $1 - 2$ is,

$$E_i = A_1\exp(ik_{z1}d_1)\,. \tag{2.79}$$

In medium 3, the downgoing wave at interface $2 - 3$ is

$$E_t = A_3\exp(ik_{z3}d_2)\,. \tag{2.80}$$

If T_{13} is the transmission coefficient for waves going from medium 1 to medium 3, we have

$$E_t = T_{13}E_i \quad \Longrightarrow \quad A_3\exp(ik_{z3}d_2) = T_{13}A_1\exp(ik_{z1}d_1)\,. \tag{2.81}$$

Therefore,

$$\frac{A_3}{A_1} = T_{13}\exp(ik_{z1}d_1 - ik_{z3}d_2)\,. \tag{2.82}$$

Similarly, if we consider only the downgoing wave is from medium 2 to medium 3, we have

$$E_i = A_2\exp(ik_{z2}d_2)\,. \tag{2.83}$$

If T_{23} is the transmission coefficient for waves going from medium 2 to medium 3, we have

$$E_t = T_{23}E_i \quad \Longrightarrow \quad A_3\exp(ik_{z3}d_2) = T_{23}A_2\exp(ik_{z2}d_2)\,. \tag{2.84}$$

Therefore,

$$T_{23}\frac{A_2}{A_3} = \exp(-ik_{z2}d_2 + ik_{z3}d_2)\,. \tag{2.85}$$

Multiply (2.82) and (2.85) to get

$$T_{23}\frac{A_2}{A_1} = T_{13}\exp(ik_{z1}d_1 - ik_{z2}d_2)\,. \tag{2.86}$$

Substitute (2.78) to get

$$T_{13}\exp(ik_{z1}d_1 - ik_{z2}d_2) = \frac{T_{23}T_{12}\exp[i(k_{z1}-k_{z2})d_1]}{1 - R_{21}\widetilde{R}_{23}\exp[2ik_{z2}(d_2-d_1)]} \tag{2.87}$$

or,

$$T_{13} = \frac{T_{23}T_{12}\exp[ik_{z2}(d_2-d_1)]}{1 - R_{21}\widetilde{R}_{23}\exp[2ik_{z2}(d_2-d_1)]} \qquad \square \tag{2.88}$$

If we consider only one reflection at interface $2-3$, we have $\widetilde{R}_{23} = R_{23}$.
To expand in series, we observe that $0 \le 1 - R_{21}R_{23}\exp[2ik_{z2}(d_2-d_1)] < 1$ and recall that

$$\left.\frac{1}{1-f}\right|_{f=a} = \frac{1}{1-a}\sum_{n=0}^{\infty}\left(\frac{f-a}{1-a}\right)^n\,. \tag{2.89}$$

If we expand around $f = 0$, we have

$$\left.\frac{1}{1-f}\right|_{f=0} = 1 + f + f^2 + f^3 + \dots \tag{2.90}$$

If $f = R_{21}R_{23}\exp[2ik_{z2}(d_2-d_1)] = R_{21}R_{23}\exp(2i\theta)$, we can write (2.88) as

$$T_{13} = T_{23}T_{12}\exp(i\theta)\left[1 + R_{21}R_{23}\exp(2i\theta) + R_{21}^2R_{23}^2\exp(4i\theta) + \dots\right]\,. \tag{2.91}$$

Expanded out,

$$T_{13} = T_{23}T_{12}\exp(i\theta) + T_{23}T_{12}R_{21}R_{23}\exp(3i\theta) + T_{23}T_{12}R_{21}^2R_{23}^2\exp(5i\theta) + \dots \qquad \square \tag{2.92}$$