Introduction to Linear Optimization and Extensions with MATLAB

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Chapter 2

Geometry of Linear Programming

Geometry of the Feasible Set

- Consider following primal LP (2.1):

  minimize $-x_1 - 2x_2$

  subject to $x_1 + x_2 \leq 20$

  $$2x_1 + x_2 \leq 30$$

  $$x_1 \geq 0, x_2 \geq 0$$

- Geometrically, the feasible set P is:

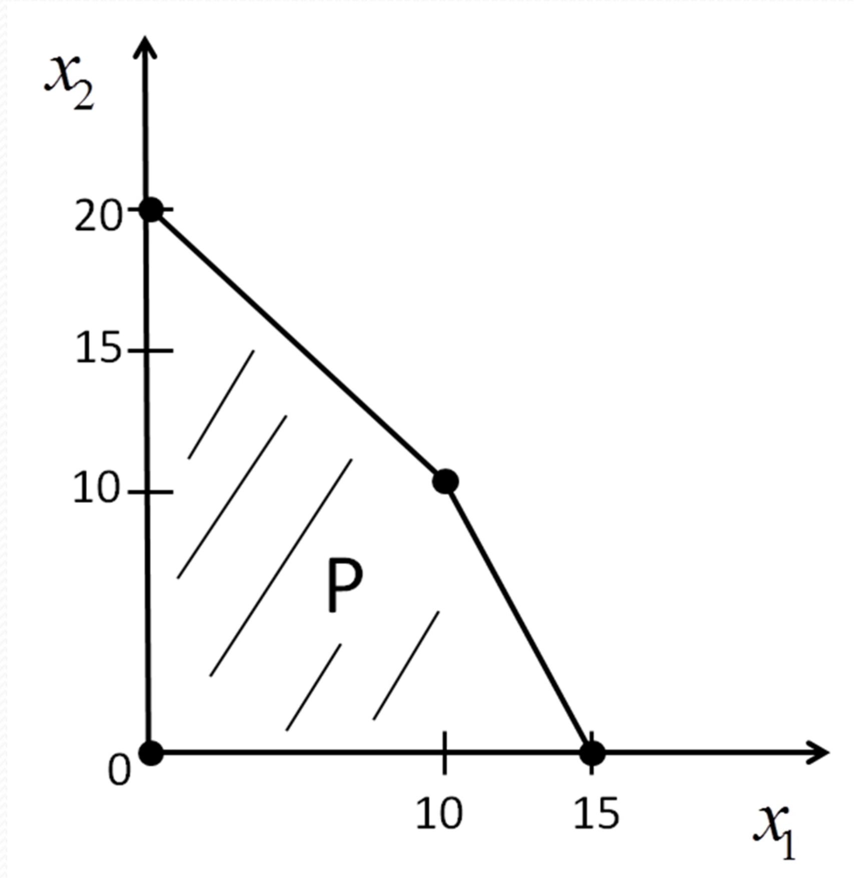


Figure 1: Graph of feasible set of LP (2.1)

Geometry of the Feasible Set

- Definition 2.1:

  A *closed halfspace* is a set of form $H_{\leq} = \{x \in R^n \mid a^T x \leq \beta\}$ or $H_{\geq} = \{x \in R^n \mid \alpha^T x \geq \beta\}$.

- E.g., The constraint $x_1 + x_2 \leq 20$is a closed halfspace $H_{\leq}$ where $\alpha = [1\ 1]^T$, and $\beta = 20$.

- The constraint $x_1 \geq 0$ is a closed halfspace $H_{\geq}$ where $\alpha = [1\ 0]^T$, and $\beta = 0$.

Example: Closed Halfspace

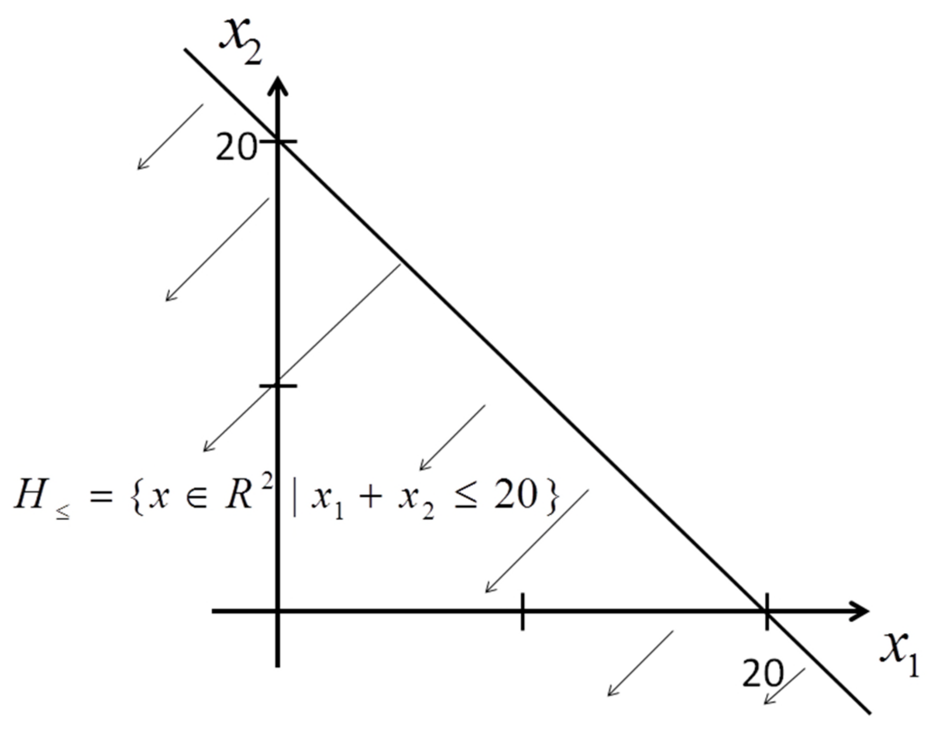


Figure 2: Closed Halfspace $x_1 + x_2 \leq 20$

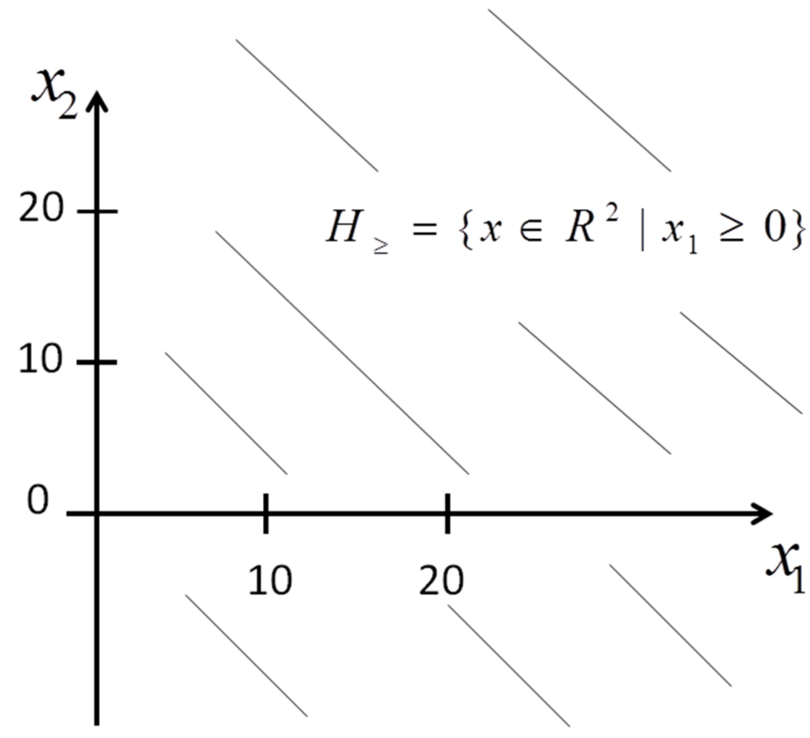


Figure 3: Closed Halfspace $x_1 \geq 0$

Geometry of the Feasible Set

- Definition 2.2:

  A *hyperplane* is a set of the form $H = \{x \in R^n \mid a^T x = \beta\}$ where is a a non-zero vector i.e. $a \neq 0$ and $\beta \in R^1$ is a scalar.

- Geometrically, a *hyperplance H* splits R^n into two halves. E.g. In R^2 a *hyperplance H* is a line that splits the plane into two halves, In R^3 a *hyperplance H* is a plane that splits the space into two halves…

Example: Hyperplane

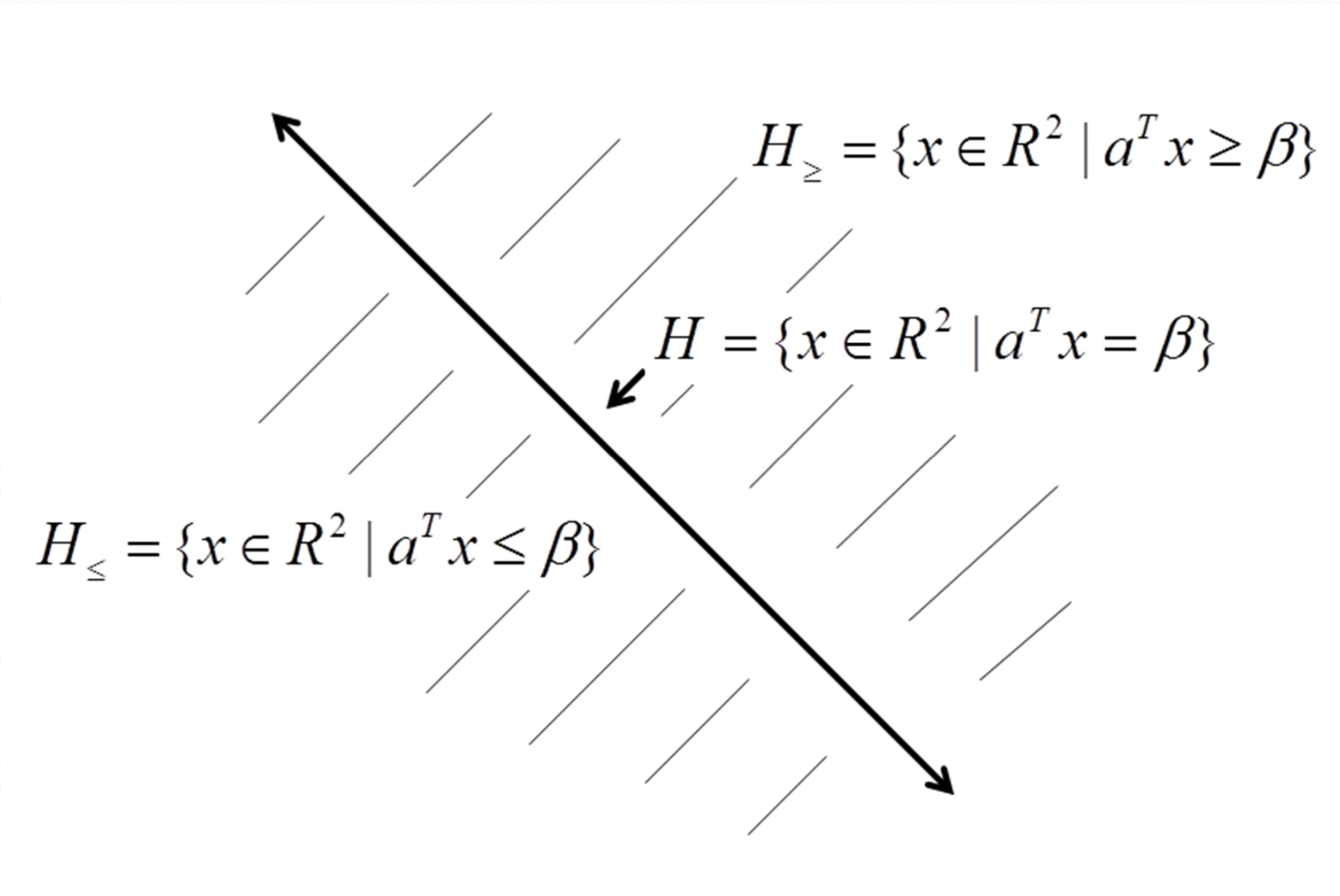


Figure 4: Hyperplane in R^2

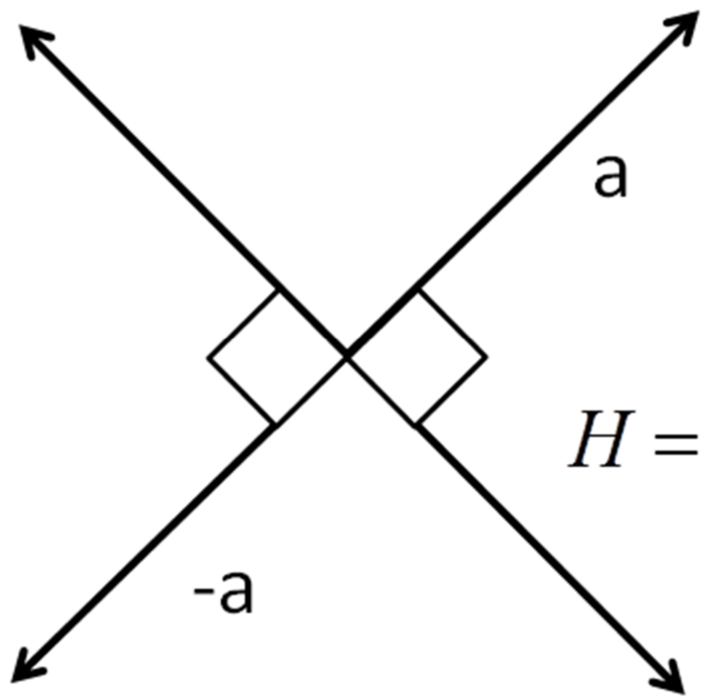
Geometry of the Feasible Set

- The vector *a* in the definition of hyperplane *H* is perpendicular to *H*. *a* is called the *norm vector* of *H*.

- Proof: Let *z*, *y* be in the *H*, then $a^T(z - y) = a^Tz - a^Ty = 0$, then the vector *z* - *y* is parallel to H, thus *a* is perpendicular to *H*.

- The -*a* vector is also perpendicular to *H*, but in the opposite direction to *a*.

Example: Perpendicular to H



$$H = \{x \in R^n \mid a^T x = \beta\}$$

Figure 5: *-a and a are perpenducular to H*

Geometry of the Feasible Set

- Definition 2.3:

  The intersection of a finite number of closed halfspaces is called a *polyhedron* (or *polyhedral set*). A bounded polyhedron is called *polytope*.
- Then the feasible set P of any linear programming is a ployhedral set. The set P of (2.1) is a polytope.
- Take any two points x, y from a closed halfspace $H_{\geq}$ (or $H_{\leq}$), the line segment between x and y in $C \in R^n$ can be expressed as $\lambda x + (1 - \lambda)y$ for $0 \leq \lambda \leq 1$.

Geometry of the Feasible Set

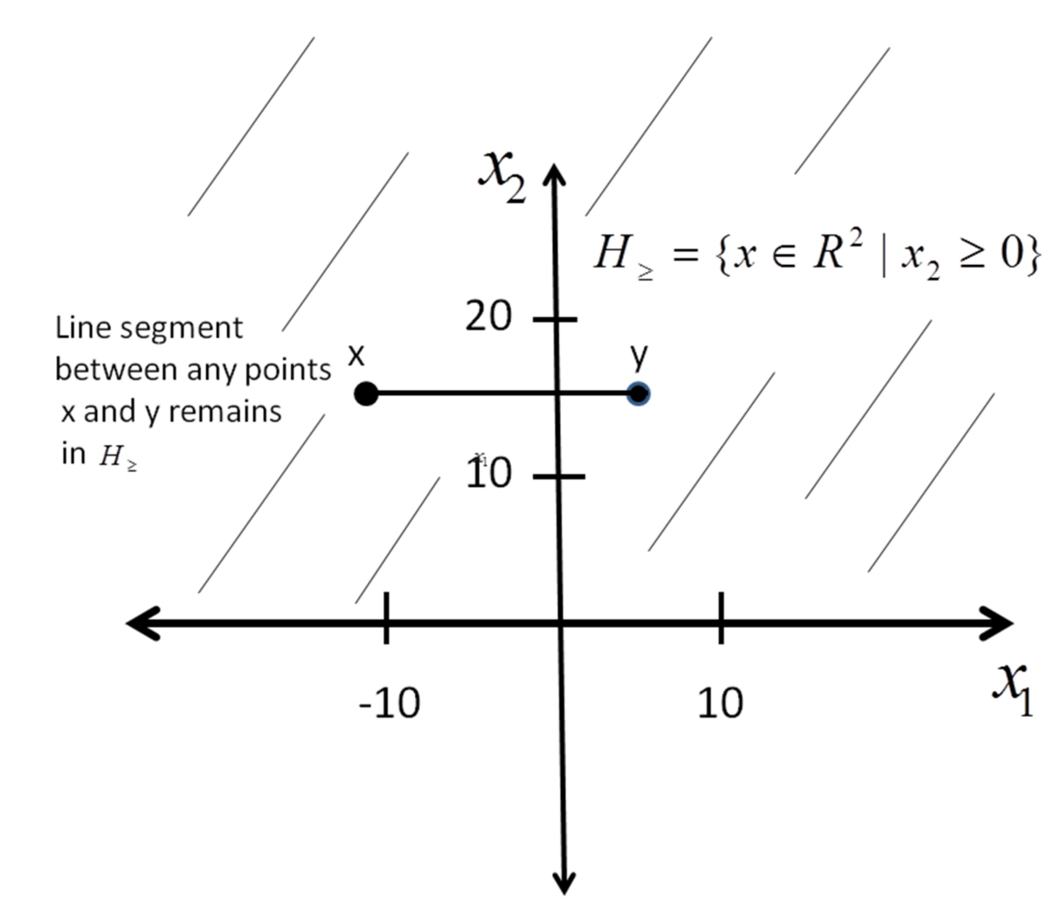


Figure 6: Convexity of the half space $x_2 \geq 0$

Geometry of the Feasible Set

- Definition 2.4:

  A set $C \in R^n$ is said to be convex if for any x and y in C then $\lambda x + (1 - \lambda)y \in C$ for all $\lambda \in [0, 1]$.

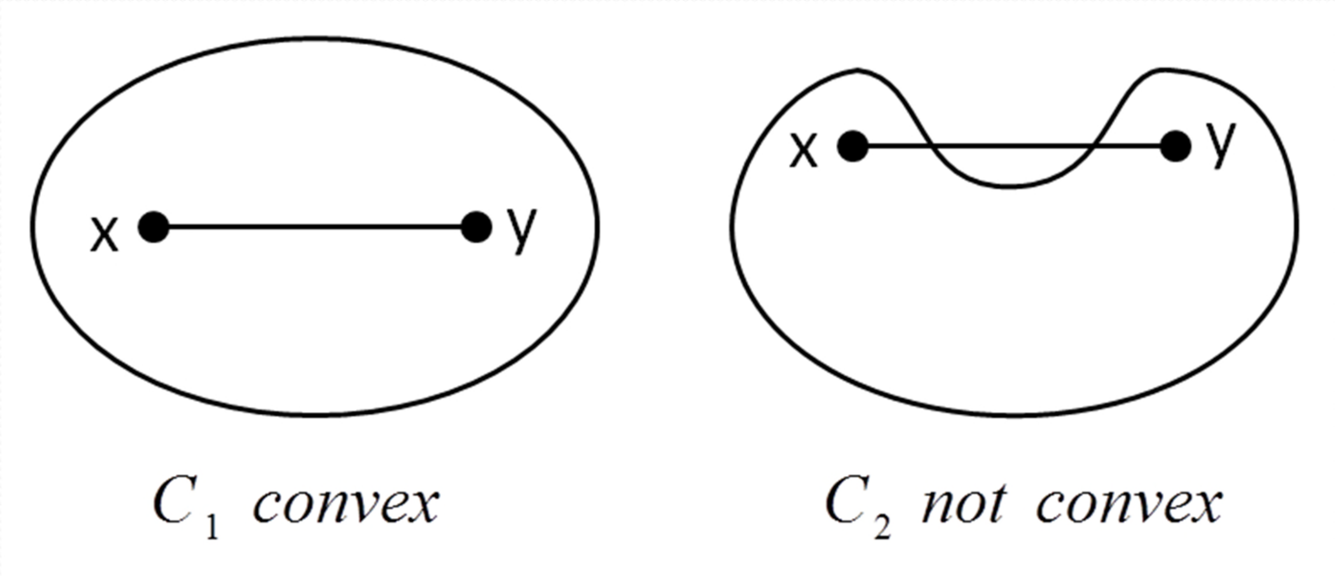


Figure 7: Convexity and non-convexity

Geometry of the Feasible Set

- Theorem 2.1:

  the closed halfspaces $H_{\leq}$ and $H_{\geq}$ are convex sets.

- Proof: Let $z = [z_1\ z_2]^{T}$ and $y = [y_1\ y_2]^{T}$ be any pair of points in $H_{\leq} = \{x \in R^n \mid a^T x \leq \beta\}$. Then consider any point on the line segment between z and y i.e. $\lambda z + (1 - \lambda)y$ for $0 \leq \lambda \leq 1$. Now $a^T(\lambda z + (1 - \lambda)y) = \lambda a^T z + (1 - \lambda)a^T y \leq \lambda\beta + (1 - \lambda)\beta = \beta$ which is in $H_{\leq}$. Thus $H_{\leq}$ is convex. Similar argument can be showed to $H_{\geq}$ is convex.

Geometry of the Feasible Set

- Theorem 2.2:

  The intersection of convex sets are convex.
- Proof: suppose there is an arbitrary collection of convex sets S_i indexed by the set I. Consider the intersection $\cap_{i \in I} S_i$ and let x any y in this intersection. For any $\lambda \in [0, 1]$, $z = \lambda x + (1 - \lambda) y$ is in every set S_i since x and y are in S_i for every $i \in I$ and S_i is a convex set. Thus $\cap_{i \in I} S_i$ is a convex set.

- Corollary 2.1:

  The feasible set of a linear programming is a convex set.

Geometry of Optimal Solutions

- Consider the linear programming (2.1).
- The contours of the objective function $H = \{x \in R^2 \mid -x_1 - 2x_2 = \beta\}$ is a hyperplans.
- The negative of the gradient of the objective function i.e. $-c = [1\ 2]^T$ is perpendicular to all such contours.
- To decrease the objective function in the direction of most rapid descent, the contours of the objective should be moved in the direction of $-c$ while remaining perpendicular to $-c$.
- The optimality appear at the corner point $x^* = [0\ 20]^T$.

Geometry of Optimal Solutions

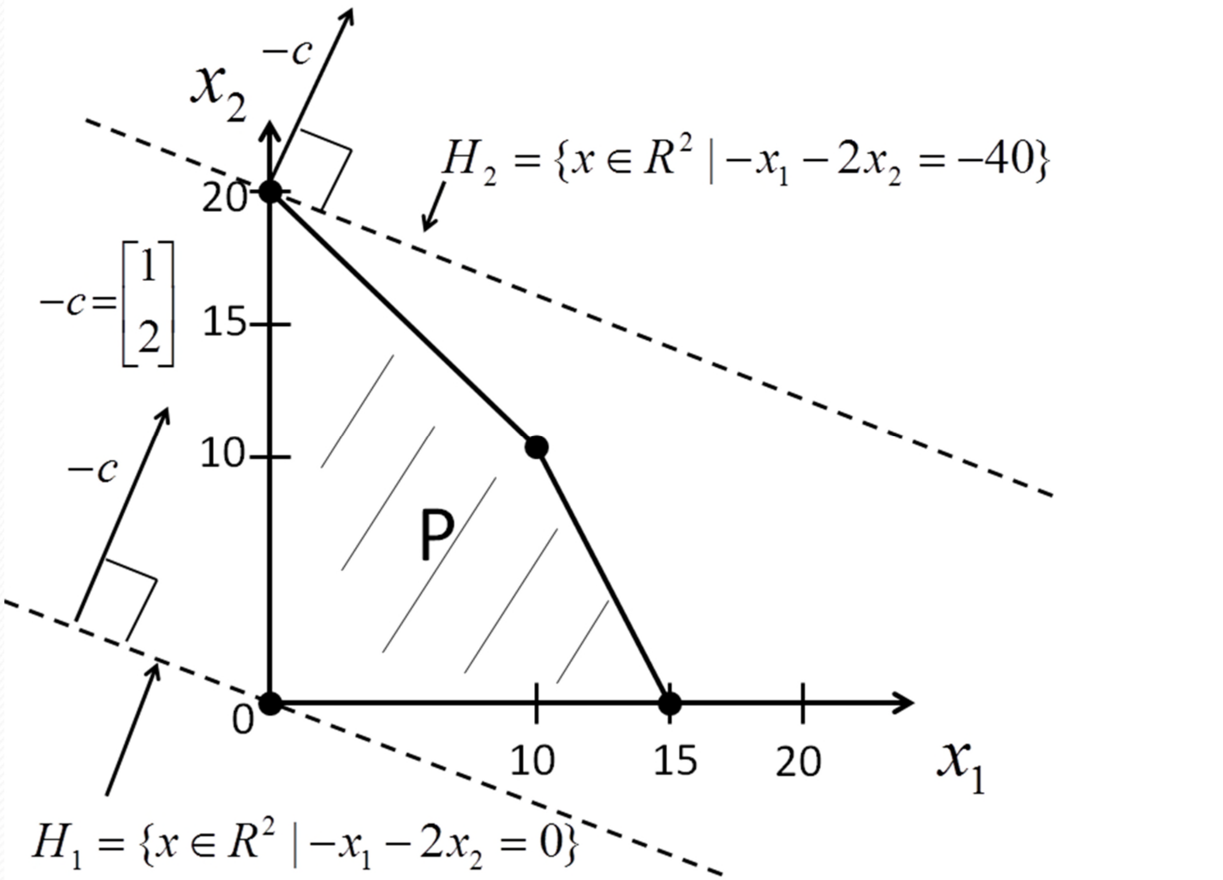


Figure 8: Hyperplane characterization of optimality of LP (2.1)

Geometric Characterization of Optimality

- Let $P \neq \emptyset$ be the feasible set of a linear program and $H = \{x \in R^n \mid -c^T x = \beta\}$. If P $\subset H_{\leq} = \{x \in R^n \mid -c^T x \leq \beta\}$ for some $\beta \in R^1$, then any x in the intersection of P and H is an optimal solution for the linear program.
- Case 1: Unique Intersection

  In the LP (2.1), for $\beta = 40$ the feasible set P is contained in the the half space $H_{\leq} = \{x \in R^2 \mid x_1 + 2x_2 \leq 40\}$ and $x^* = [0\ 20]^T$ is both in P and $H = \{x \in R^2 \mid x_1 + 2x_2 = 40\}$, and is the only such point.
- The optimal is one of 4 “corner points” of feasible set P.

Geometric Characterization of Optimality

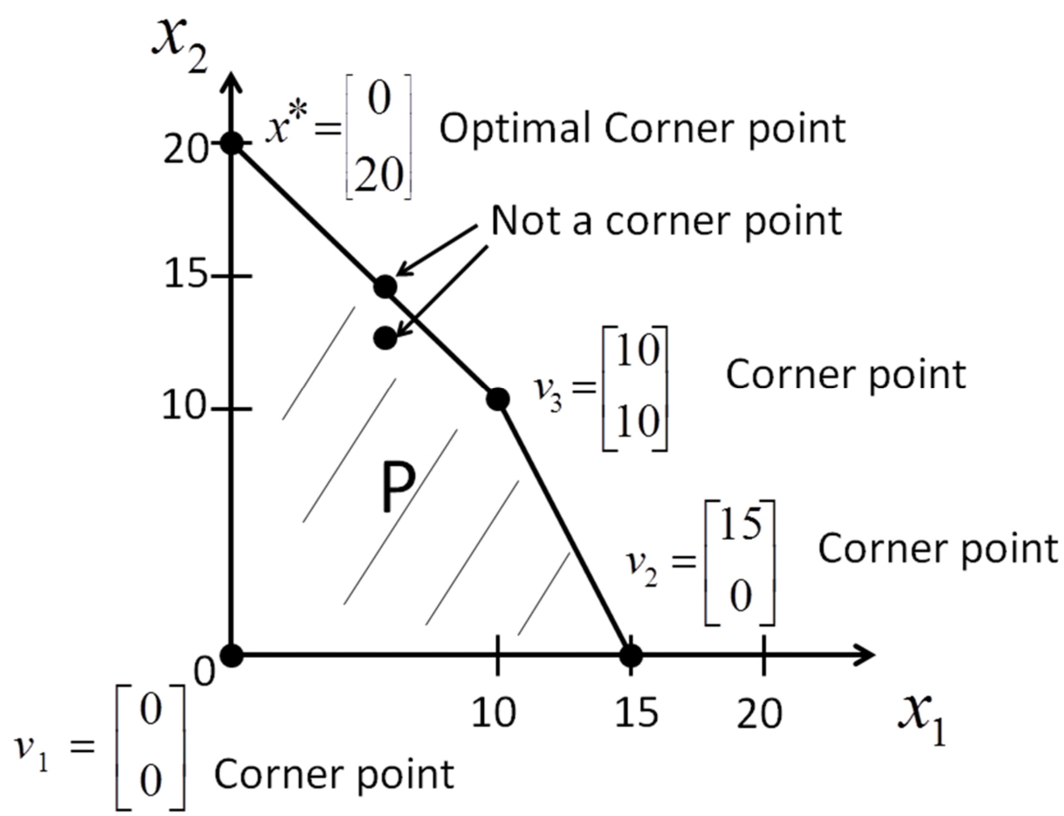


Figure 9: Corner points of feasible set of LP (2.1)

Geometric Characterization of Optimality

- Case 2: Infinite Intersection

  Consider the following LP (2.2):

  minimize $-x_1$

  subject to $x_1 \leq 1$

  $x_2 \leq 1$

  $x_1 \geq 0, x_2 \geq 0$

- The corner points for P are $v_1 = (0\ 0)^T$, $v_2 = (1\ 0)^T$, $v_3 = (0\ 1)^T$, $v_4 = (1\ 1)^T$.
- The line between v_3 and v_4 intersects with $H^* = \{x \in R^2 \mid x_1 = 1\}$, and thus all points on this line segment are optimal solutions, which is infinite.

Geometric Characterization of Optimality

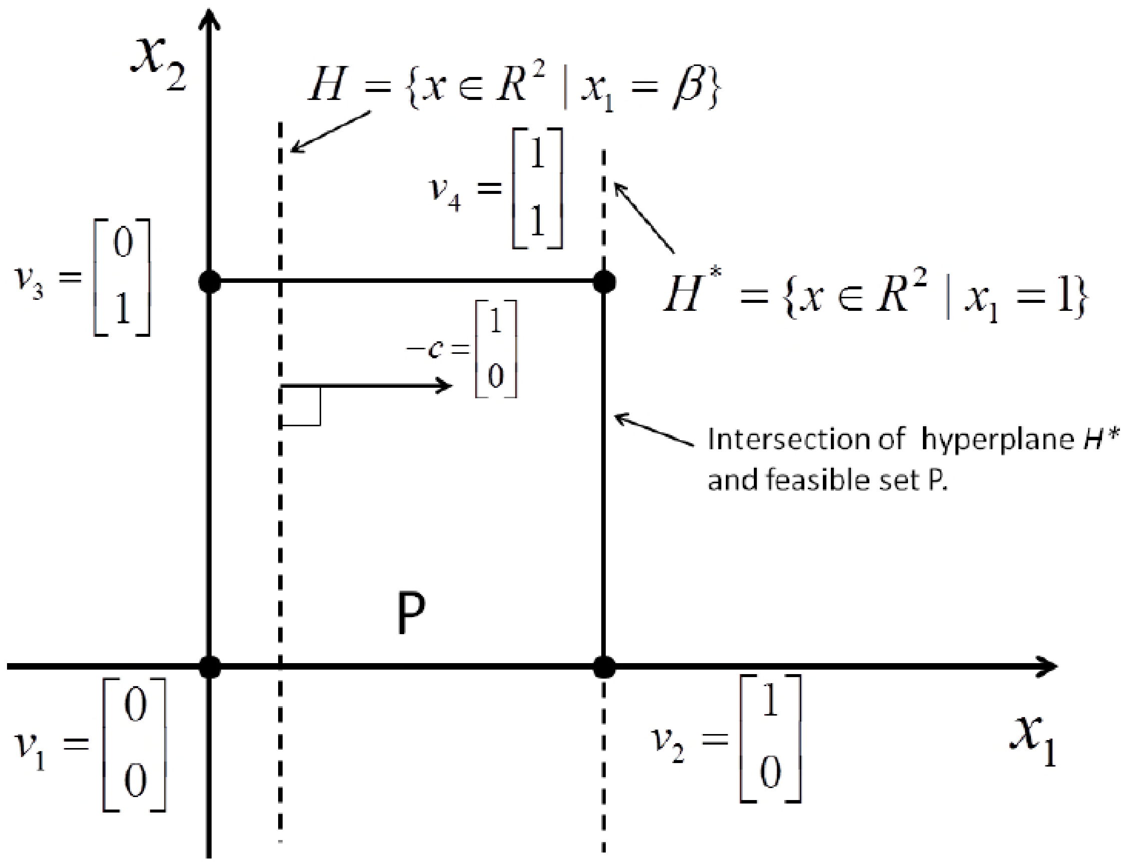


Figure 10: Hyperplane characterization infinite optimal solutions for LP (2.2)

Geometric Characterization of Optimality

- Case 3: Unbounded

  Consider the following LP (2.3):

  minimize $-x_1 - x_2$

  subject to $x_1 + x_2 \geq 1$

  $x_1 \geq 0, x_2 \geq 0$

- We can see that for any positive value of β the hyperplane $H = \{x \in R^2 \mid x_1 + x_2 = \beta\}$ will always intersect the feasible set $P = \{x \in R^2 \mid x_1 + x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$.

Geometric Characterization of Optimality

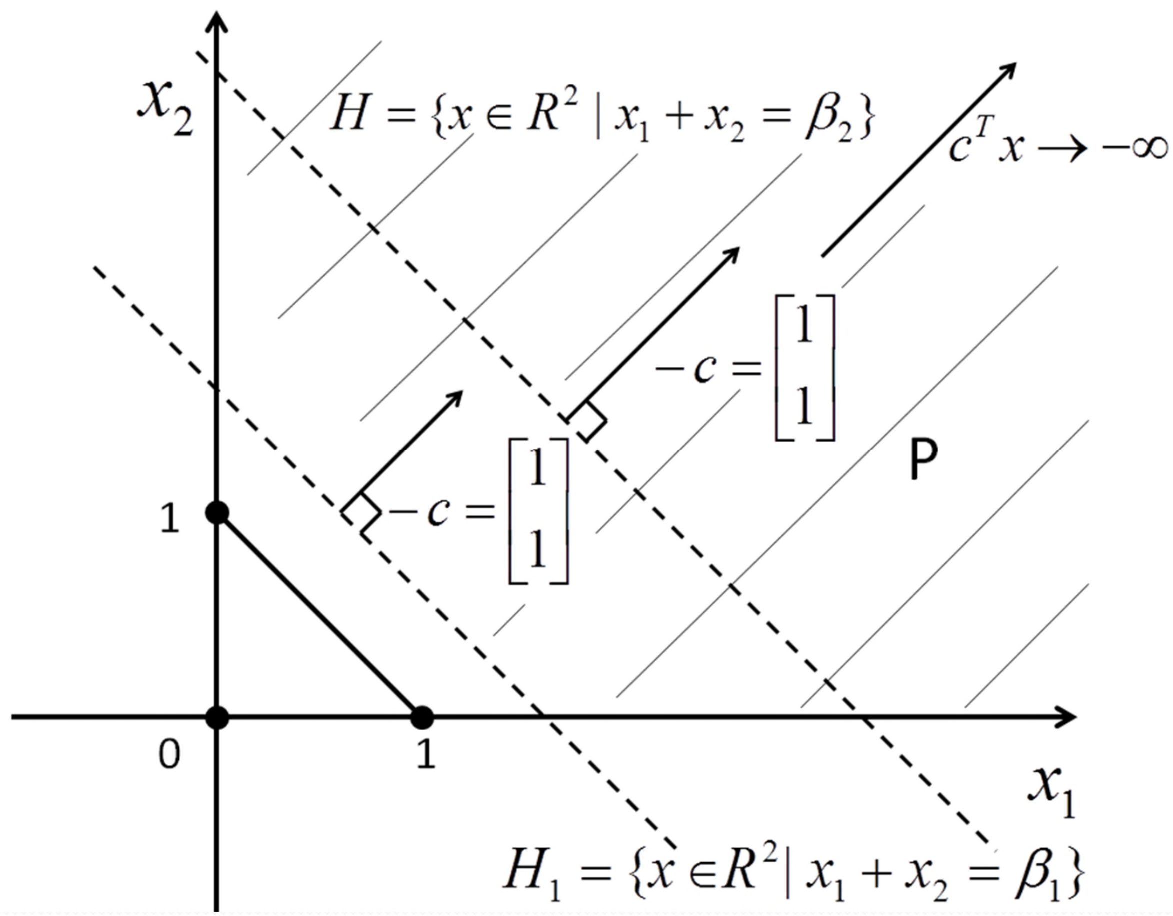


Figure 11: Unbounded LP (2.3)

Extreme Points

- Definition 2.5:

  A *convex combination* of vectors $x_1, x_2, \ldots, x_k$ is a linear combination $\sum_{i=1}^{k} \lambda_i x_i$ of these vectors such that $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 1, \ldots, k$.

- Definition 2.6:

  Let $C \subseteq R^n$ be a convex set and $x \in C$. A point x is an extreme point of C if it cannot be expressed a convex combination of other points in C.

Example: Extreme Points

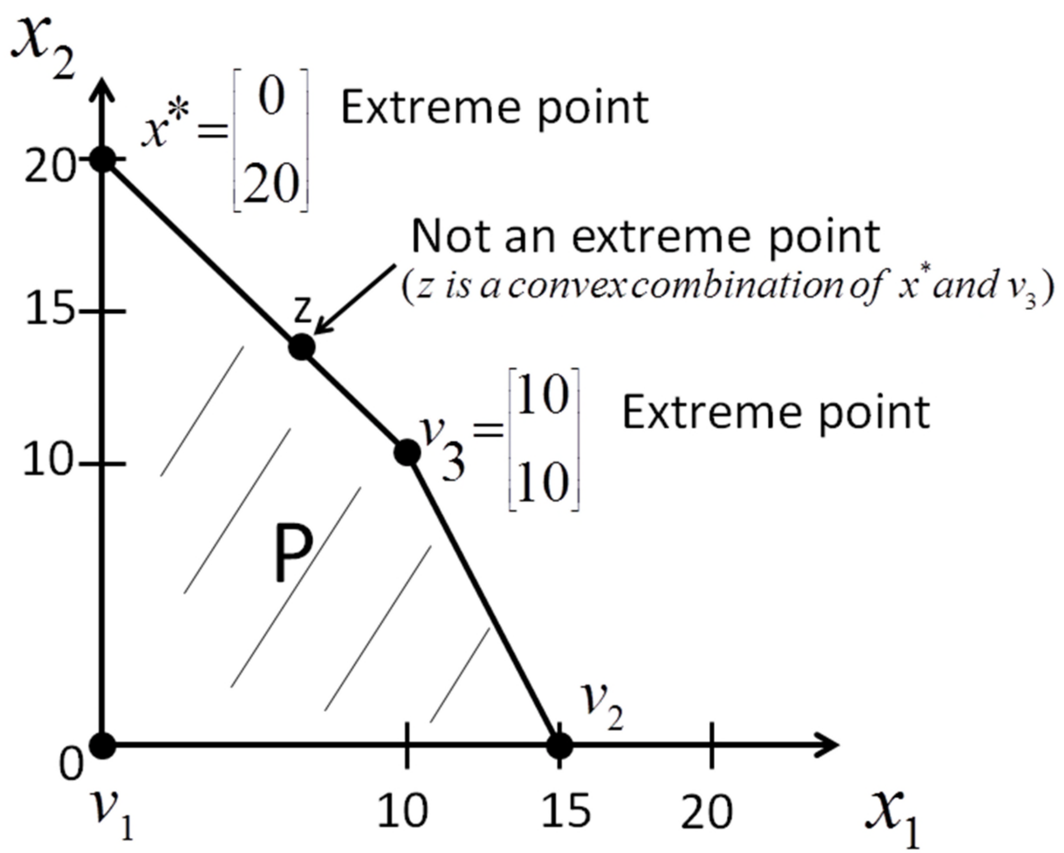


Figure 12: Extreme points of feasible set of LP (2.1)

Example: Extreme Points

- Convert LP (2.1) to standard form:

  minimize $-x_1 - 2x_2$

  subject to $x_1 + x_2 + x_3 = 20$

  $2x_1 + x_2 + x_4 = 30$

  $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

  where corresponding matrix entities are:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 20 \\ 30 \end{bmatrix}, c = \begin{bmatrix} -1 & -2 & 0 & 0 \end{bmatrix}^T$$

- Consider the corner point $v_4 = (x_1\ x_2)^T = (1\ 1)^T$ in (2.1) and $z = (x_1\ x_2\ x_3\ x_4)^T = (1\ 1\ 0\ 0)^T$ in standard form, we can see that the sub-matrix $B = [A_1\ A_2] = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is non-singular.

Example: Extreme Points

- Table 2.1 gives the correspondence between all extreme points and its associated sub-matrix *B*.

corner point		standard form feasible solution				sub-matrix *B*
x_1	x_2	x_1	x_2	x_3	x_4	
0	0	0	0	20	30	$[A_3\ A_4] = [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}]$
15	0	15	0	5	0	$[A_1\ A_3] = [\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}]$
0	20	0	20	0	10	$[A_1\ A_3] = [\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}]$
10	10	10	10	0	0	$[A_1\ A_3] = [\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}]$

Extreme Points

- Theorem 2.3:

  Consider a linear program in standard form where the feasible set $P = \{x \in R^n \mid Ax = b, x \geq 0\}$ is non-empty. A vector $x \in P$ is an extreme point if and only if the column of A corresponding to positive components of x are linearly independent.
- Proof: Suppose that there are k positive components in $x \in P$ and are positioned as the first k components of x i.e. $x = [x_p\ 0]^T$ where $x_p = [x_1\ x_2 \ldots x_k\ \ 0]^T > 0$. Let B the columns of A associated with the components of x_p, then Ax = B = b.

Extreme Points

- Proof of forward direction =>

Assume that $x \in P$ is an extreme point. Now suppose B is singular (i.e. columns of B are linear dependent), then there exists a non-zero vector ω such that $B\omega = 0$.
For sufficiently small $\varepsilon > 0$, $x_p + \varepsilon\omega > 0$, and $x_p - \varepsilon\omega > 0$.
$B(x_p + \varepsilon\omega) = Bx_p + \varepsilon B\omega = b$ and
$B(x_p - \varepsilon\omega) = Bx_p - \varepsilon B\omega = b$. Thus the following two vectors:

$$z^+ = \begin{bmatrix} (x_p + \varepsilon\omega) \\ 0 \end{bmatrix} \text{ and } z^- = \begin{bmatrix} (x_p - \varepsilon\omega) \\ 0 \end{bmatrix}$$

are in the set P since $Az^+ = b$ and $Az^- = b$. However, $.5z^+ + .5z^- = x$ which means x is a convex combination z^+ of and z^- contradicting that it is an extreme point.

Extreme Points

- Proof of reverse direction <=

Suppose that the columns of B are linearly independent and that x is not an extreme point. Then x can be written as the convex combination of two distinct points v_1 and v_2 both in P (and different from x) i.e. $x = [x_p\ 0]^T = \lambda v_1 + (1 - \lambda)v_2$ for some $0 < \lambda < 1$. Now v_1 and v_2 both non-negative since they are in P and λ is positive, so the last n - k components of v_1 and v_2 must be zeros i.e. v_1 and v_2 can be written as

$$v_1 = \begin{bmatrix} v_p^1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} v_p^2 \\ 0 \end{bmatrix}$$

where v_p^1 and v_p^2 are the first components of v_1 and v_2. Thus $B(x - v_1) = Bx - Bv_p^1 = b - b = 0$, but $x_p - v_p^1 \neq 0$ as $x \neq v_1$. So the column of B is linearly dependent is a contradiction

Basic Feasible Solutions

- Definition 2.7:

  A vector $x \in P = \{x \in R^n \mid Ax = b, x \geq 0\}$ is a basic feasible solution (BFS) is there is a partition of the matrix A into an non-singular $m \times m$ square submatrix B and an $m \times (n - m)$ submatrix N such that $x = [x_B\ x_N]^T$ with $x_B \geq 0$ and $x_N = 0$ and $Ax_B = Bx_N = b$. B is called the basis matrix, N is called non-basis (or non-basic) matrix, x_B is the set of basis variables, and x_N is the set of non-basis variables.

- Corollary 2.2:

  A vector $x \in P = \{x \in R^n \mid Ax = b, x \geq 0\}$ is an extreme points if and only if there is some matrix B so that x is a basic feasible solution with B as the basis matrix.

Example: Basic Feasible Solution

- Consider LP (2.1) in standard from:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 30 \end{bmatrix}, \text{ then } x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}, x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

- B is non-singular and so $x_B = B^{-1}b$ and $x_N = 0$ then x is a basic feasible solution.

Generating Basic Feasible Solution

- Corollary 2.3:

  The feasible set $P = \{x \in R^n \mid Ax = b, x \geq 0\}$ has at most $C(n,m) = \frac{n!}{m!(n-m)!}$ extreme points.
- A particular choice of m columns will generate an extreme points if (1) B is non-singular (2) $x_B \geq 0$.
- E.g. Consider the feasible set by constraints (2.4)

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_1 \quad &\leq 1 \\ x_2 &\leq 1 \\ x_1 \geq 0,\ x_2 &\geq 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x_1 + x_2 + x_3 \qquad\qquad &= 1 \\ x_1 \qquad\quad + x_4 \qquad &= 1 \\ x_2 \qquad\qquad + x_5 &= 1 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 &\geq 0 \end{aligned}$$

Generating Basic Feasible Solution

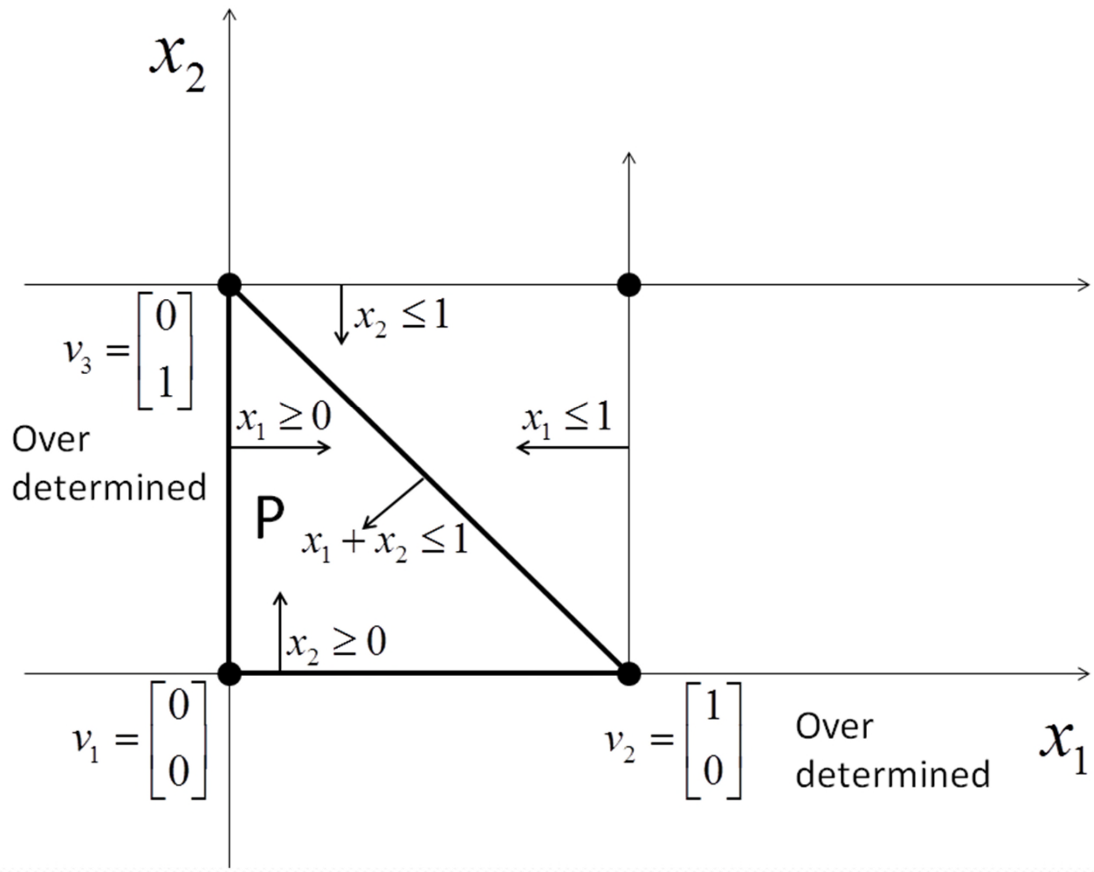


Figure 13: Graph of feasible set (2.4)

Generating Basic Feasible Solution:

- There are $C(5,3) = \frac{5!}{3!(5-3)!} = 10$ possible extreme points.
- Table 2.2 lists those partitions that do no result in BFS either due to infeasibility or non-negativity of basic variables. Table 2.3 lists the BFS partitions.

Partition $[x_B\ x_N]^T$		Basis matrix B	$x_B = B^{-1}b$	x is extreme points?
x_B	x_N			
$(x_3\ x_4\ x_1)^T$	$(x_5\ x_2)^T$	$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$	B is singular	No
$(x_3\ x_2\ x_5)^T$	$(x_1\ x_4)^T$	$\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$	B is singular	No
$(x_1\ x_2\ x_3)^T$	$(x_4\ x_5)^T$	$\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$	x_B is infeasible	No

BFS: Table 2.3

Partition $[x_B\ x_N]^T$		Basis matrix B	$x_B = B^{-1}b$	x is extreme points?
x_B	x_N			
$(x_3\ x_4\ x_5)^T$	$(x_1\ x_2)^T$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$(1\ 1\ 1)^T$	Yes
$(x_1\ x_4\ x_5)^T$	$(x_3\ x_2)^T$	$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$	$(1\ 0\ 1)^T$	Yes
$(x_3\ x_1\ x_5)^T$	$(x_4\ x_2)^T$	$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$	$(0\ 1\ 1)^T$	Yes
$(x_2\ x_4\ x_5)^T$	$(x_1\ x_3)^T$	$\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}$	$(1\ 1\ 0)^T$	Yes
$(x_3\ x_4\ x_2)^T$	$(x_1\ x_5)^T$	$\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$	$(0\ 1\ 1)^T$	Yes
$(x_1\ x_2\ x_4)^T$	$(x_3\ x_5)^T$	$\begin{pmatrix}1&1&0\\1&0&1\\0&1&0\end{pmatrix}$	$(0\ 1\ 1)^T$	Yes
$(x_1\ x_2\ x_5)^T$	$(x_3\ x_4)^T$	$\begin{pmatrix}1&1&0\\1&0&0\\0&1&1\end{pmatrix}$	$(1\ 0\ 1)^T$	Yes

Degeneracy

- Definition 2.8:

  A basic feasible solution $x \in P = \{x \in R^n \mid Ax = b, x \geq 0\}$ is degenerate if at least one of the variables in the basic set x_B is zero. $x \in P$ is said to be non-degenerate if all m of the basic variables are positive.
- E.g. Consider (2.4) and BFS in Table 2.2 and 2.3.
- The BFS in row 1 of Table 2.3 is only corresponds to the extreme point v_1 in Figure 13.
- The BFS in row 2 , 3 of Table 2.3 and row 3 of Table 2.2 are all corresponds to the extreme point v_2 in Figure 13. However, the BFS are degeneracy as v_2 is over determined by the intersection of 3 constraints.

Resolution (Representation) Theorem

- For a feasible set $x \in P = \{x \in R^n \mid Ax = b, x \geq 0\}$, a representation of any $x \in P$ is sought in terms of the extreme points of P and recession directions.
- Case 1: P is bounded e.g. a polytope in Figure 14.

  P has 5 extreme points v_1, v_2, v_3, v_4, v_5.

  In general, any $x \in P$ in a polytope can be represented as a combination of extreme points in P.

Resolution (Representation) Theorem

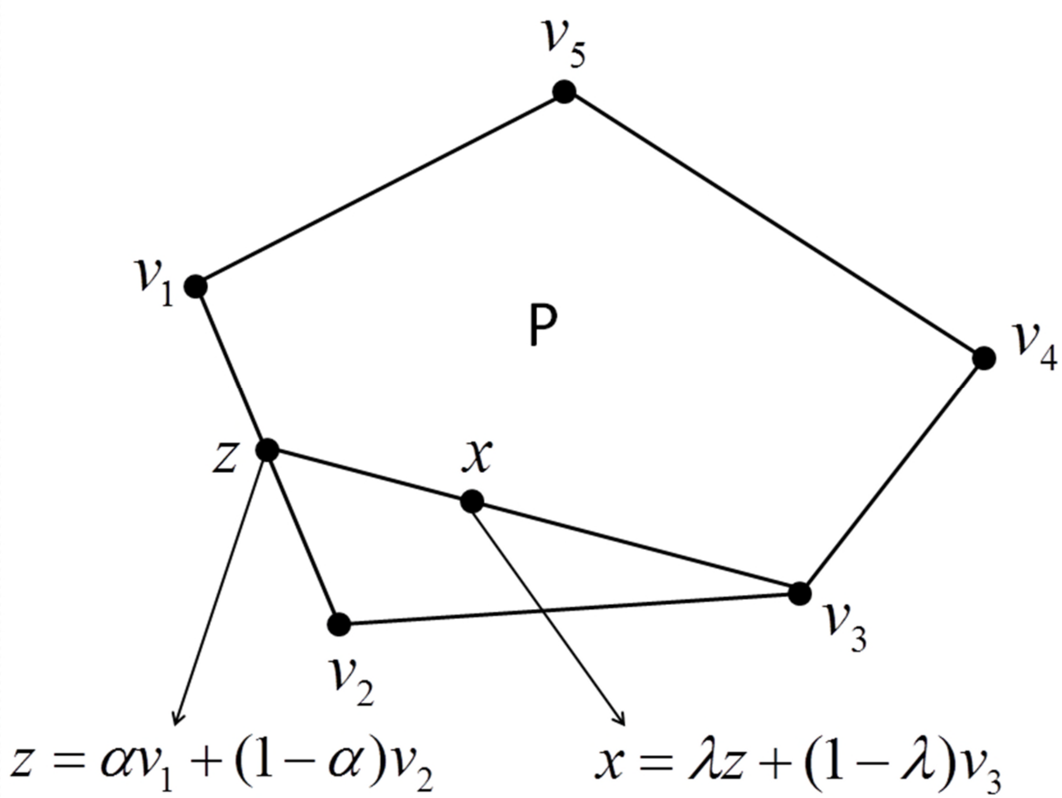


Figure 14: A polytope with 5 extreme points

Resolution (Representation) Theorem

- Case 2: P is unbounded

  Consider the following set of inequalities (2.5)

  $x_2 - x_1 \leq 3$

  $x_1 \geq 0,\ x_2 \geq 0$

- Definition 2.9:

  A *ray* is a set of form $\{x \in R^n \mid x = x_0 + \lambda d, \text{ for } \lambda \geq 0\}$, where x_0 is a given point and d is a non-zero vector called the *direction vector*.

- Definition 2.10:

  Let P be a non-empty feasible set of a LP. A non-zeros direction d is called a *recession direction* if for any $x_0 \in P$ the ray $\{x \in R^n \mid x = x_0 + \lambda d, \text{ for } \lambda \geq 0\} \subset P$.

Resolution (Representation) Theorem

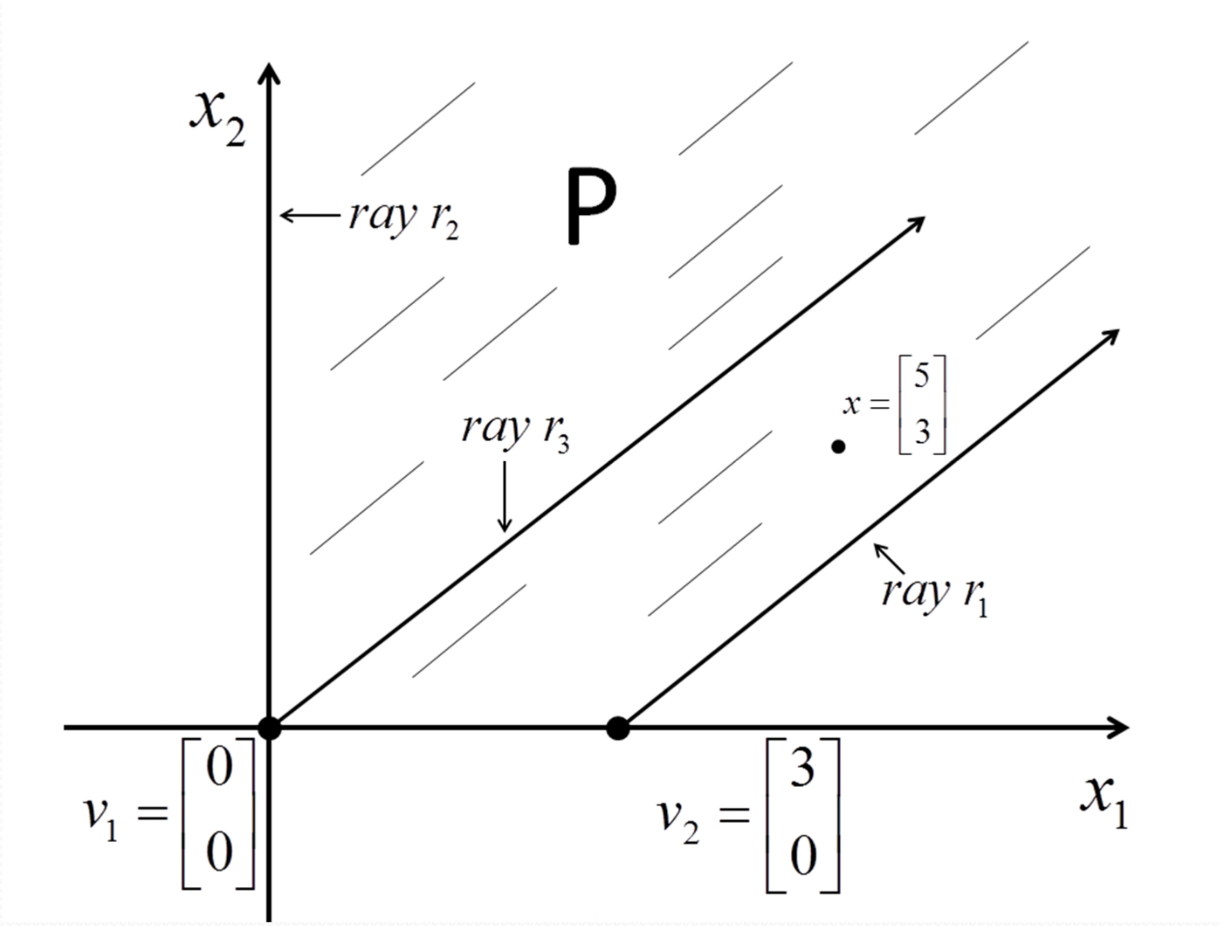


Figure 15: Some rays of feasible set (2.5)

Resolution (Representation) Theorem

- Theorem 2.4 (Resolution Theorem):

  Let $P = \{x \in R^n \mid Ax = b, x \geq 0\}$ be a non-empty set P. Let $v_1, v_2, \ldots, v_k$ be the extreme points of P.
- (Case 1) If P is bounded, then any $x \in P$ can be represented as the convex combination of extreme points

  i.e. $x = \sum_{i=1}^{k} \lambda_i v_i$ for some $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.
- (Case 2) If P is unbounded, then there exists at least one extreme direction. Let $d_1, d_2, \ldots, d_l$ be the extreme direction of P. Then any $x \in P$ can be represented as

  $x = \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{l} \mu_i d_l$ where $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$

  and $\mu_i \geq 0$ for $i = 1, \ldots, l$.

Fundamental Theorem of LP

- Theorem 2.5:

  For a feasible set $P = \{x \in R^n \mid Ax = b, x \geq 0\}$ a non-zero vector d is a recession vector if and only if $Ad = 0$ and $d \geq 0$.
- Corollary 2.4:

  A non-negative linear combination of recession directions of a feasible set P is a recession direction of P.
- Proof: Let $d_1, d_2, \ldots, d_l$ be the recession directions of P and let $d = \sum_{i=1}^{l} \mu_i d_i$ for $\mu_i \geq 0$ for $i = 1, \ldots, l$. Since d_i is a recession direction by Definition 2.10. we have that $Ad_i = A\sum_{i=1}^{l} \mu_i d_i$ $= \mu_i \sum_{i=1}^{l} Ad_i = 0$, also $d_i \geq 0$. So $d = \sum_{i=1}^{l} \mu_i d_i \geq 0$.

  Therefore , by Definition 2.10, d is a recession direction.

Fundamental Theorem of LP

- Theorem 2.6 (Fundamental Theorem of Linear Programming):

Consider an LP in standard form and suppose that *P* is not-empty.

Then, either

the LP is unbounded over P

or

an optimal solution for the LP can be attainted at an extreme point of P.

Fundamental Theorem of LP

Proof: Let $v_1, v_2, \ldots, v_k$ be the extreme points of P and let $d_1, d_2, \ldots, d_l$ be the extreme direction of P. Then by the Resolution Theorem every point $x \in P$ can be expressed as $x = \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{l} \mu_i d_l$ where $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$ and $\mu_i \geq 0$ for $i = 1, \ldots, l$. Without loss generally, let which is a recession direction by Corollary 2.4. There are two cases:

- Case (1) d is such that $c^T d < 0$. In this case, for any $x_o \in P$ the ray $\{x \in R^n \mid x_o + \lambda d$ for $\lambda \in [0, 1]\}$ P, will be such that $c^T x = c^T x_o + \lambda c^T d$ and this can be made to diverge towards $-\infty$ as $\lambda \rightarrow \infty$ since $c^T d < 0$ and $\lambda \geq 0$.

Fundamental Theorem of LP

- Case (2) d is such that $c^T d \geq 0$. So $x = \sum_{i=1}^{k} \lambda_i v_i + d$ where λ_1, $\lambda_2, \ldots, \lambda_k \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$, Now let v_{min} be that extreme point that result in the minimum value of $c^T v_{min}$ over for $i = 1, \ldots, k$. Then for any $x \in P$, $c^T x = c^T(\sum_{i=1}^{k} \lambda_i v_i + d) =$ $c^T(\sum_{i=1}^{k} \lambda_i v_i) + c^T d \geq c^T(\sum_{i=1}^{k} \lambda_i v_i) = \sum_{i=1}^{k} \lambda_i c^T v_i \geq \sum_{i=1}^{k} \lambda_i c^T v_{\min}$ $= c^T v_{\min} \left(\sum_{i=1}^{k} \lambda_i\right) = c^T v_{min}$. Thus the minimum value for the LP is attainted at v_{min} that is an extreme point.