Chapter 2

Exercise 2.1

Consider the constraint

$$-2x_1 + x_2 \leq 2$$

(a) Express this constraint as a closed-half space of the form $H_{\leq} = \{x \in R^n \mid \alpha^T x \leq \beta\}$ i.e. determine α and β.

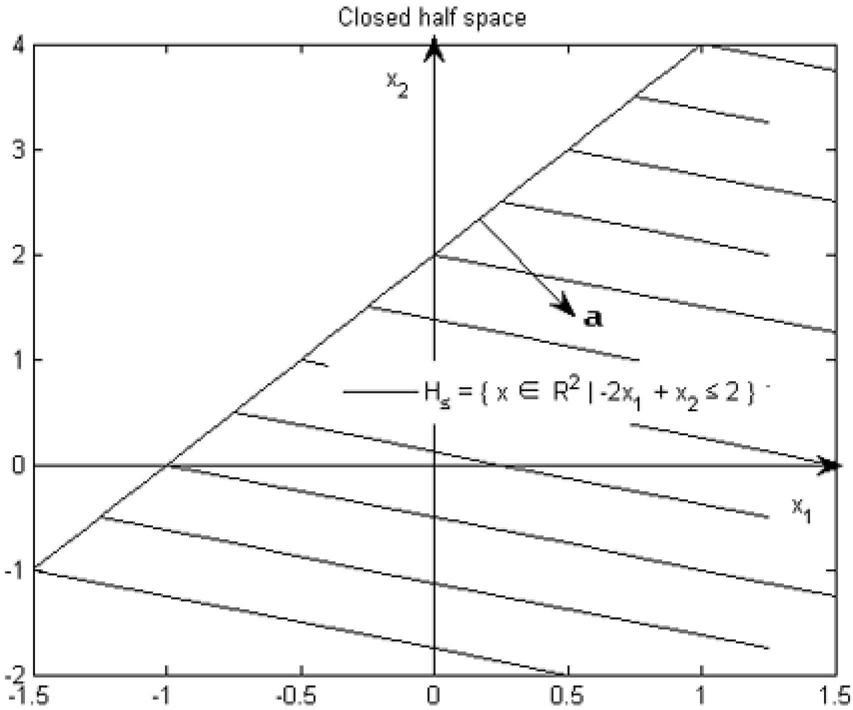
(b) Sketch the closed half-space in (a) showing any vector that is normal to the hyperplane that is contained in $H_{\leq}$.

(c) Show that the closed half-space in (a) is a convex set.

Solution:

(a) $\alpha = [-2 \quad 1]^T$, $\beta = 2$.

(b) It is easy to see that any vector **a** that is a normal to the hyperplane $-2x_1 + x_2 = 2$ is contained in $H_{\leq}$.



(c) For any two points $z = [z_1\ z_2]^T$ and $y = [y_1\ y_2]^T$ in $H_{\leq}$, need to show $[-2\ 1](\lambda z + (1 - \lambda)y) \leq 2$ for all $\lambda \in [0,1]$. Now note that $[-2\ 1](\lambda z + (1 - \lambda)y) = [-2\ 1](\lambda[z_1\ z_2]^T + (1 - \lambda)[y_1\ y_2]^T) = \lambda[-2\ 1][z_1\ z_2]^T + (1 - \lambda)\ [-2\ 1][y_1\ y_2]^T \leq \lambda 2 + (1 - \lambda)2 = 2$, thus the half space $H_{\leq} = \{ x \in R^2 \mid -2x_1 + x_2 \leq 2\}$ is convex.

Exercise 2.2

Consider a linear program in standard form

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax = b \\ &\qquad\qquad x \geq 0 \end{aligned}$$

(a) Prove that the feasible set $P = \{ x \in R^n \mid Ax = b, x \geq 0\}$ of the linear program is a convex set directly using the definition of convex set.

(b) Prove that the set of optimal solutions for the linear program in standard form $P^* = \{ x \in R^n \mid x$ is an optimal solution for LP$\}$ is a convex set.

Solution:

(a) For any two points $z \in R^n$ and $y \in R^n$ in feasible set P, need to show (1) $A(\lambda z + (1 - \lambda)y) = b$ and (2) $\lambda z + (1 - \lambda)y \geq 0$ for all $\lambda \in [0,1]$. For (1), $A(\lambda z + (1 - \lambda)y) = \lambda Az + (1 - \lambda)Ay = \lambda b + (1 - \lambda)b = b$, for (2) $\lambda z + (1 - \lambda)y \geq 0$ since $z \geq 0$, $y \geq 0$, $\lambda \geq 0$ and $(1 - \lambda) \geq 0$. Thus the feasible set $P = \{ x \in R^n \mid Ax = b, x \geq 0 \}$ is a convex set.

(b) Since P is a convex set, and $P^* \subseteq P$, by Theorem 2.8, P^* also is a convex set.

Exercise 2.3

Solve the following linear programs graphically by using the sketch of the feasible set and illustrate the hyperplane characterization of optimality when a finite optimal solution(s) exists else illustrate the unboundedness of the linear program using hyperplanes.

(a) minimize $-x_1 - 2x_2$

subject to $-2x_1 + x_2 \leq 2$

$-x_1 + x_2 \leq 3$

$x_1 \leq 2$

$x_1 \geq 0, x_2 \geq 0$

(b) minimize $-x_1 - 2x_2$

subject to $x_1 - 2x_2 \geq 2$

$x_1 + x_2 \leq 4$

$x_1 \geq 0, x_2 \geq 0$

(b) maximize $x_1 + x_2$

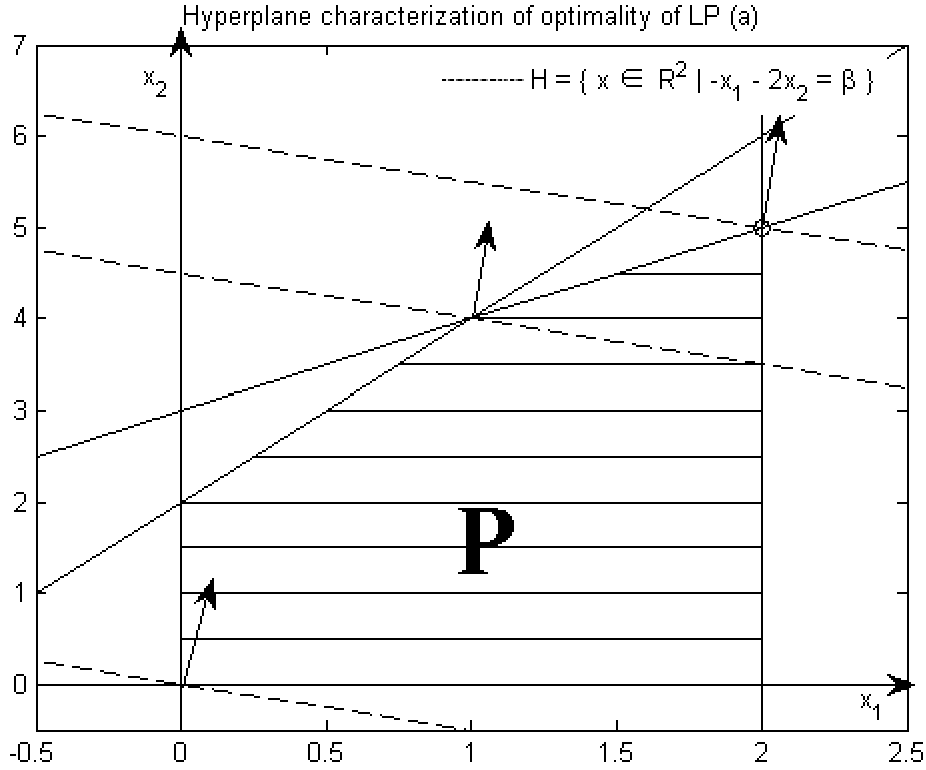
subject to $x_1 - x_2 \geq 1$

$x_1 - 2x_2 \geq 2$

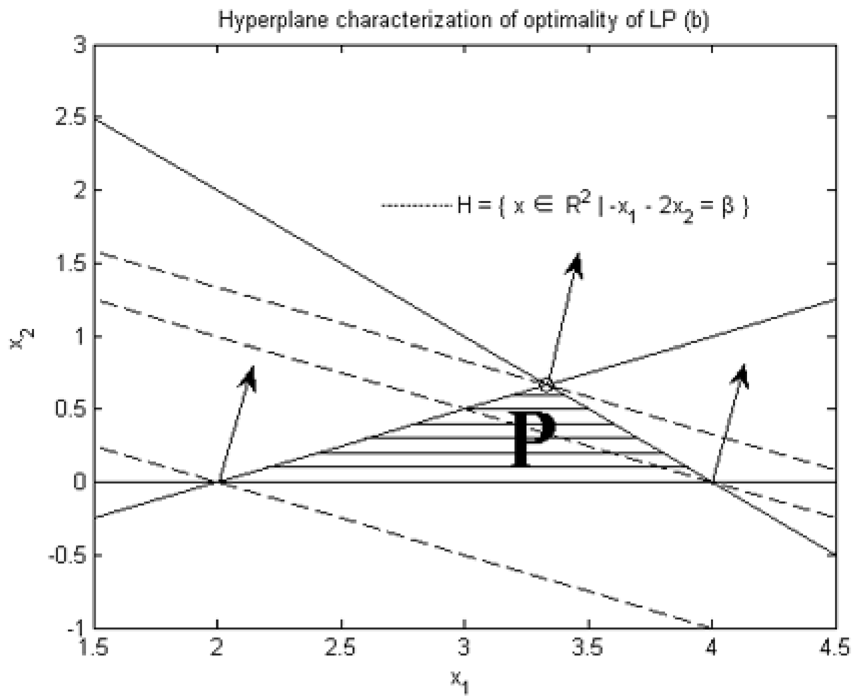
$x_1 \geq 0, x_2 \geq 0$

Solution:

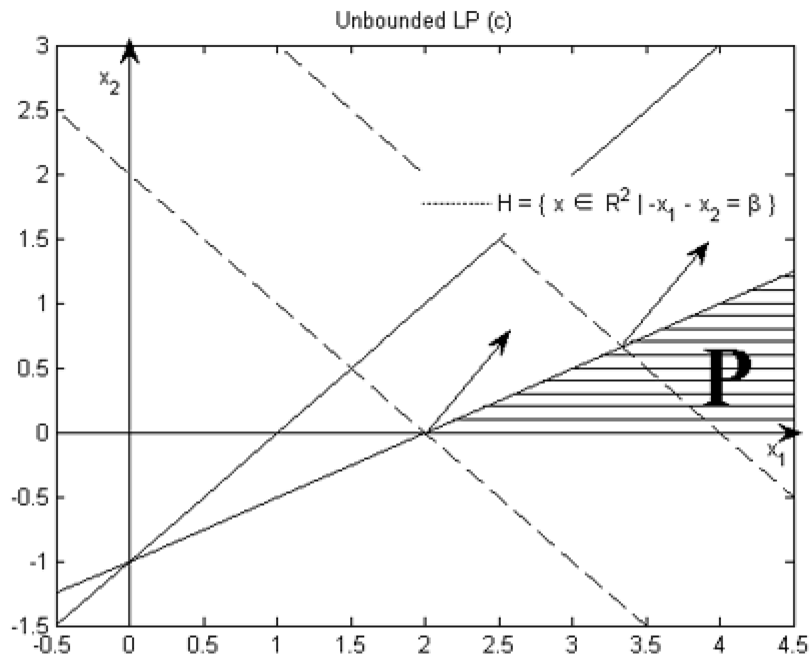
(a) The optimal solution is the point $[2,5]^T$ in the graph. Observe that at the optimal point the feasible set P is completely contained in the closed half space $H_{\leq}$.



(b) The optimal solution is [10/3, 2/3]^T. Observe that at the optimal point the feasible set P is completely contained in the closed half space $H_{\leq}$.



(c) Unbound LP



Exercise 2.4

(a) For the linear program (a) in Exercise 2.3 find all basic feasible solutions by converting the constraints into standard form.

(b) For each linear program in Exercise 2.3 find two linearly independent directions d_1 and d_2 of unboundedness if they exists.

Solution:

(a) First convert into standard form

$$\begin{aligned} \text{minimize} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + x_2 + x_4 = 3 \\ & x_1 + x_5 = 2 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0 \end{aligned}$$

with

$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$. There are $\binom{5}{3} = 10$ possible extreme points. The following table lists all basic feasible solutions only. Other partitions not listed lead to infeasibility or basis B is not invertible.

Partition x = $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$		Basis matrix B	$x_B = B^{-1}b$	x extreme point?
$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	$x_N = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$	yes

$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix}$	$x_N = \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$	yes

(b) LP (c) is unboundness case. One ray is

$r_1 = \left\{x \in R^2 \mid x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{for all } \lambda \geq 0\right\}$ which starts from the point $x^0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ in the direction of

$d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and another ray is $r_2 = \left\{x \in R^2 \mid x = \begin{bmatrix} 3 \\ 0.5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \text{for all } \lambda \geq 0\right\}$ which starts from the point

$x^0 = \begin{bmatrix} 3 \\ 0.5 \end{bmatrix}$ in the direction of $d_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$, one can easily verify that d_1, d_2 are linearly independent directions.

Exercise 2.5

Consider the constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 5 \\ x_1 + x_2 &\leq 4 \\ x_1 \quad &\leq 2 \\ x_1 \geq 0,\ x_2 &\geq 0 \end{aligned}$$

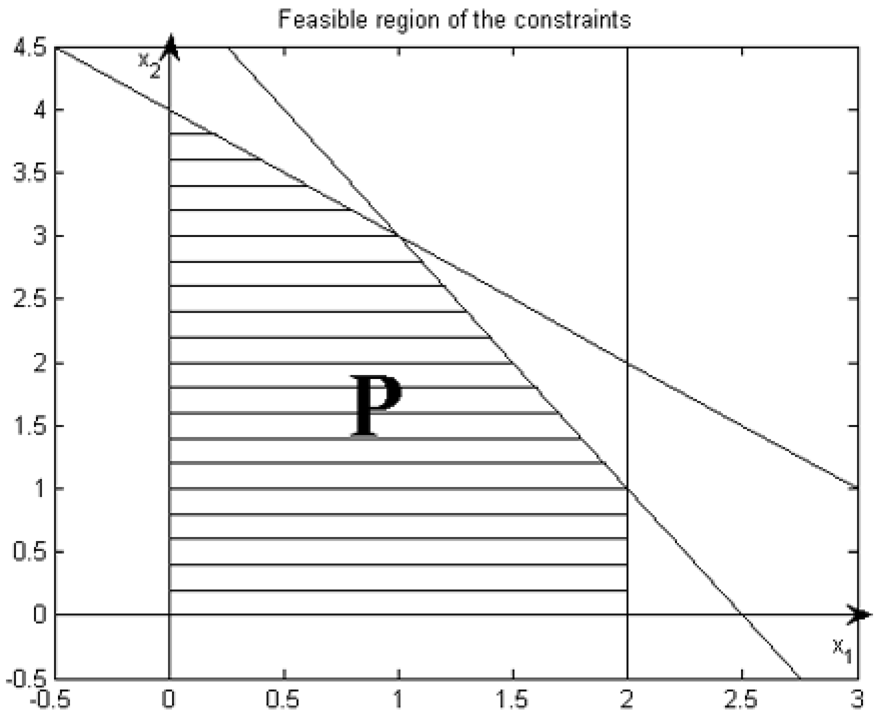
(a) Sketch the feasible region.

(b) Convert the constraints to standard form and find all basic feasible solutions.

(c) Identify the extreme points in the original constraints.

Solution:

(a) the feasible region show as follows:



(b) Converting into the standard form

$$
\begin{aligned}
2x_1 + x_2 + x_3 \quad\quad &= 5 \\
x_1 + x_2 \quad + x_4 \quad &= 4 \\
x_1 \quad\quad\quad + x_5 &= 2 \\
x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 &\geq 0,\ x_5 \geq 0
\end{aligned}
$$

with

$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$. There are $\binom{5}{3} = 10$ possible extreme points. The following table lists all basic feasible solutions.

Partition x = $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$		Basis matrix B	$x_B = B^{-1}b$	x extreme point?
$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$	$x_N = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix}$	$x_N = \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$	yes

$x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$	yes

(c) The extreme points in terms of the original constraint variables $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

Exercise 2.6

Consider the linear program

maximize $x_1 + x_2$

subject to $x_1 - x_2 \geq 1$

$x_1 - 2x_2 \geq 2$

$x_1 \geq 0, x_2 \geq 0$

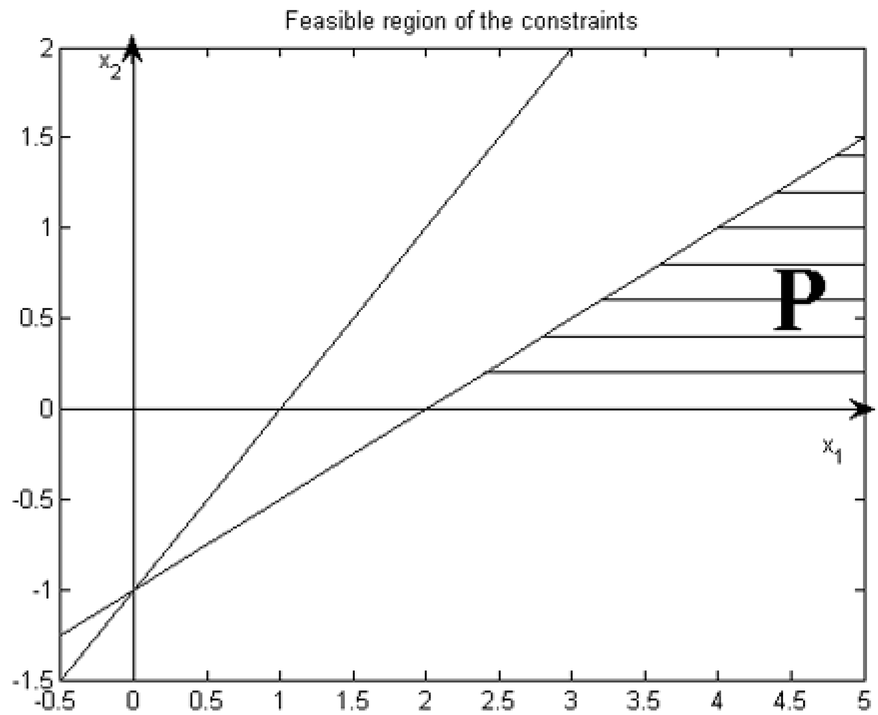
(a) Sketch the feasible region.

(b) Convert the constraints to standard form and find all basic feasible solutions and find two extreme directions d_1 and d_2 (i.e. two linearly independent directions of unboundedness).

(c) Show that the extreme directions d_1 and d_2 from (b) satisfy $Ad = 0$ and $d \geq 0$.

Solution:

(a) the feasible region show as follows:



(b) Converting into standard form

minimize $-x_1 - x_2$

subject to $x_1 - x_2 - x_3 = 1$

$x_1 - 2x_2 - x_4 = 3$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

with

$\mathrm{A} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & -2 & 0 & -1 \end{bmatrix}, \mathrm{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Th following table lists all basic feasible solutions.

Partition x = $\begin{bmatrix} \mathrm{x_B} \\ \mathrm{x_N} \end{bmatrix}$		Basis matrix B	$\mathrm{x_B} = \mathrm{B}^{-1}\mathrm{b}$	x extreme point?
$\mathrm{x_B} = \begin{bmatrix} \mathrm{x_1} \\ \mathrm{x_3} \end{bmatrix}$	$\mathrm{x_N} = \begin{bmatrix} \mathrm{x_2} \\ \mathrm{x_4} \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	yes

Since the feasible region is unbounded, the ray

$\mathrm{r_1} = \left\{ \mathrm{x} \in \mathrm{R}^4 | \, \mathrm{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \text{for all } \lambda \geq 0 \right\}$ from the point $\mathrm{x}^0 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ in the direction of $\mathrm{d_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and the ray $\mathrm{r_2} = \left\{ \mathrm{x} \in \mathrm{R}^4 | \, \mathrm{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \text{for all } \lambda \geq 0 \right\}$ from the point $\mathrm{x}^0 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ in the direction of $\mathrm{d_2} = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix}$, we can verify that $\mathrm{d_1}, \mathrm{d_2}$ are linearly independent directions.

(c) $\mathrm{Ad_1} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathrm{Ad_2} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\mathrm{d_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \geq 0, \mathrm{d_2} = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix} \geq 0.$

Exercise 2.7

Solution:

Let $x^{(0)}$ be a feasible solution for LP, i.e. $Ax^{(0)} = b$. Consider points of the form $x^{(0)} + \alpha d$ for all $\alpha \geq 0$, then $A(x^{(0)} + \alpha d) = Ax^{(0)} + \alpha Ad = Ax^{(0)} = b$ and clearly $x^{(0)} + \alpha d \geq 0$, since $x^{(0)} \geq 0$, and $\alpha \geq 0$. So $x^{(0)} + \alpha d$ is feasible for any $\alpha \geq 0$. (■)

Exercise 2.8

Consider the following system of constraints

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 0$

$x_1 \leq 3$

$x_1 \geq 0, x_2 \geq 0$

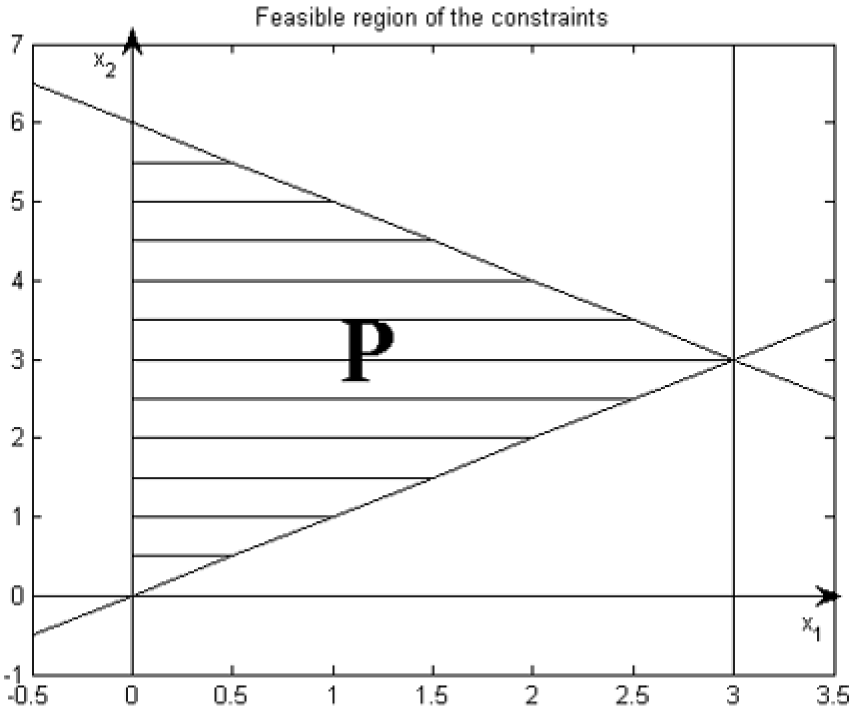
(a) Sketch the feasible region.

(b) Convert to standard form and find all basic feasible solutions.

(c) Is there a one-to-one correspondence between basic feasible solutions and extreme points? If not, which extreme points can be represented by multiple basic feasible solutions?

Solution:

(a) the feasible region show as follows:



(b) Converting into the standard form

$$\begin{aligned} x_1 + x_2 + x_3 \quad\quad &= 6 \\ x_1 - x_2 \quad + x_4 \quad &= 0 \\ x_1 \quad\quad\quad + x_5 &= 3 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0 \end{aligned}$$

with

$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}$. There are $\binom{5}{3} = 10$ possible extreme points. The following table lists all basic feasible solutions.

Partition x = $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$		Basis matrix B	$x_B = B^{-1}b$	x extreme point?
$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	$x_N = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$	$x_N = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$	yes

$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}$	yes

(c) The extreme points in terms of the variables of the original constraints are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left\{ \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. The basic feasible solution $x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}$ corresponds uniquely to the extreme point $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$, the extreme point $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ corresponds to the basic feasible solutions $x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$, $x_B = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$ and $x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}$, and the extreme point $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ corresponds to the basic feasible solutions $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$, $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$ and $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$.

Exercise 2.9

(a) Solve the linear program in Exercise 2.3 (a) by generating all basic feasible solutions.

(b) Solve the linear program in Exercise 2.3 (b) by generating all basic feasible solutions. Also, illustrate Exercise 2.2 (b). i.e. show the set of optimal solutions is convex.

Solution:

(a) Convert into standard form

$$\begin{aligned} \text{minimize} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 \qquad = 2 \\ & -x_1 + x_2 \qquad + x_4 = 3 \end{aligned}$$

$x_1 \qquad\qquad + x_5 = 2$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$

with

$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$. There are $\binom{5}{3} = 10$ possible extreme points. The following table lists all basic feasible solutions.

Partition x = $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$		Basis matrix B	$x_B = B^{-1}b$	x extreme point?
$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	$x_N = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix}$	$x_N = \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$	yes

$c = [-1 \ -2 \ 0 \ 0 \ 0]^T$, then computing the objective value $c_B^T x_B$, for each basic feasible solution we get the values i.e.

$$\left\{ [-1 \ -2 \ 0] \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, [-1 \ -2 \ 0] \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, [-1 \ 0 \ 0] \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}, [-2 \ 0 \ 0] \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, [0 \ 0 \ 0] \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \right\} = \{-12, \ -9, \ -2, \ -4, \ 0\}$$

, thus the optimal objective is -12 corresponding to the optimal basic feasible solution

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}.$$

(b) Convert into standard form

minimize $\ -x_1 - 2x_2$

subject to $\ x_1 - 2x_2 - x_3 = 2$

$x_1 + x_2 \qquad + x_4 = 4$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

with

$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. There are $\binom{4}{2} = 6$ possible extreme points. The following table lists all basic feasible solutions.

Partition x = $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$		Basis matrix B	$x_B = B^{-1}b$	x extreme point?
$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 10/3 \\ 2/3 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$	$x_N = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$	$x_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	yes

$c = [-1\ -2\ \ 0\ \ 0]^T$, then computing the objective value $c_B^T x_B$, for each basic feasible solution we get $\left\{[-1\ -2\,]\begin{bmatrix} 10/3 \\ 2/3 \end{bmatrix}, [-1\ \ 0]\begin{bmatrix} 4 \\ 2 \end{bmatrix}, [-1\ \ 0]\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\} = \left\{-\frac{14}{3},\ -4,\ -2,\right\}$, thus the optimal objective value is $-\frac{14}{3}$ corresponding to the optimal basic feasible solution $x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 2/3 \end{bmatrix}$.

Exercise 2.10

Consider the following polyhedron

$P = \{ (x_1, x_2, x_3, x_4) \in R^4 \mid x_1 - x_2 - 2x_3 \le 1, -3x_1 - x_3 + 2x_4 \le 1, x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0\}$

Find all extreme points and extreme directions of P and represent the point $x = [2\ \ 1\ \ 1\ \ 1]^T$ as a convex combination of the extreme points plus a non-negative combination of extreme directions.

Solution:

(a) Converting into the standard form

$$\begin{aligned} x_1 - x_2 - 2x_3 \quad\quad +x_5 \quad &= 1 \\ -3x_1 \quad - x_3 + 2x_4 \quad +x_6 &= 1 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0&, x_6 \ge 0 \end{aligned}$$

with

$A = \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & 0 \\ -3 & 0 & -1 & 2 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. There are $\binom{6}{2} = 15$ possible extreme points. The following table lists all basic feasible solutions.

Partition x = $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$		Basis matrix B	$x_B = B^{-1}b$	x extreme point?
$x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$	$x_N = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \\ x_6 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_1 \\ x_6 \end{bmatrix}$	$x_N = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$	yes
$x_B = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_6 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$	yes

$x_B = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix}$	$x_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	yes

Then we have 4 extreme points to P i.e. $x^1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, $x^2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $x^3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}$, $x^4 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Construct direction set D = {d∈R^n | Ad ≤ 0, d ≥ 0, e^Td = 1}. A direction of D is an extreme direction of P if and only if d is an extreme point of D when D is a polyhedral set. Then

$$\begin{aligned} d_1 - d_2 - 2d_3 \quad\quad\quad &\le 0 \\ -3d_1 \quad\quad - d_3 + 2d_4 &\le 0 \\ d_1 + d_2 + d_3 + d_4 \quad &= 1 \\ d_1 \ge 0,\ d_2 \ge 0,\ d_3 \ge 0,\ d_4 \ge 0 \end{aligned}$$

Converting into standard form

$$\begin{aligned} d_1 - d_2 - 2d_3 \quad\quad + d_5 \quad &= 0 \\ -3d_1 \quad\quad - d_3 + 2d_4 \quad + d_6 &= 0 \\ d_1 + d_2 + d_3 + d_4 \quad\quad &= 1 \\ d_1 \ge 0,\ d_2 \ge 0,\ d_3 \ge 0,\ d_4 \ge 0,\ d_5 \ge 0,\ d_6 \ge 0 \end{aligned}$$

with

$A = \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & 0 \\ -3 & 0 & -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. There are $\binom{6}{3} = 20$ possible extreme points.

The following table lists all basic feasible solutions.

Partition d = $\begin{bmatrix} d_B \\ d_N \end{bmatrix}$		Basis matrix B	$d_B = B^{-1}b$	d extreme point?
$d_B = \begin{bmatrix} d_1 \\ d_2 \\ d_4 \end{bmatrix}$	$d_N = \begin{bmatrix} d_3 \\ d_5 \\ d_6 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 \\ -3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0.28571429 \\ 0.28571429 \\ 0.42857143 \end{bmatrix}$	yes
$d_B = \begin{bmatrix} d_1 \\ d_2 \\ d_5 \end{bmatrix}$	$d_N = \begin{bmatrix} d_3 \\ d_4 \\ d_6 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	yes
$d_B = \begin{bmatrix} d_1 \\ d_2 \\ d_6 \end{bmatrix}$	$d_N = \begin{bmatrix} d_3 \\ d_4 \\ d_5 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 \\ -3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ 0.5 \\ 1.5 \end{bmatrix}$	yes
$d_B = \begin{bmatrix} d_1 \\ d_3 \\ d_4 \end{bmatrix}$	$d_N = \begin{bmatrix} d_2 \\ d_5 \\ d_6 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 0 \\ -3 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0.30769231 \\ 0.15384615 \\ 0.53846154 \end{bmatrix}$	yes
$d_B = \begin{bmatrix} d_1 \\ d_3 \\ d_6 \end{bmatrix}$	$d_N = \begin{bmatrix} d_2 \\ d_4 \\ d_5 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 0 \\ -3 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.66666667 \\ 0.33333333 \\ 2.33333333 \end{bmatrix}$	yes

$d_B = \begin{bmatrix} d_2 \\ d_3 \\ d_5 \end{bmatrix}$	$d_N = \begin{bmatrix} d_1 \\ d_4 \\ d_6 \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	yes
$d_B = \begin{bmatrix} d_2 \\ d_4 \\ d_5 \end{bmatrix}$	$d_N = \begin{bmatrix} d_1 \\ d_3 \\ d_6 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	yes
$d_B = \begin{bmatrix} d_2 \\ d_5 \\ d_6 \end{bmatrix}$	$d_N = \begin{bmatrix} d_1 \\ d_3 \\ d_4 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	yes
$d_B = \begin{bmatrix} d_3 \\ d_4 \\ d_5 \end{bmatrix}$	$d_N = \begin{bmatrix} d_1 \\ d_2 \\ d_6 \end{bmatrix}$	$\begin{bmatrix} -2 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.66666667 \\ 0.33333333 \\ 1.33333333 \end{bmatrix}$	yes
$d_B = \begin{bmatrix} d_3 \\ d_5 \\ d_6 \end{bmatrix}$	$d_N = \begin{bmatrix} d_1 \\ d_2 \\ d_4 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$	yes

Then we have 7 extreme directions to P i.e. $d^1 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0.28571429 \\ 0.28571429 \\ 0 \\ 0.42857143 \end{bmatrix}$, $d^2 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$,

$d^3 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$, $d^4 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0.30769231 \\ 0 \\ 0.15384615 \\ 0.53846154 \end{bmatrix}$, $d^5 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0.66666667 \\ 0 \\ 0.33333333 \\ 0 \end{bmatrix}$, $d^6 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} =$

$\begin{bmatrix} 0 \\ 0 \\ 0.66666667 \\ 0.33333333 \end{bmatrix}$, $d^7 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Now for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ by the Resolution Theorem,

$$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mu_1 \begin{bmatrix} 0.28571429 \\ 0.28571429 \\ 0 \\ 0.42857143 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mu_3 \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$$
$$+ \mu_4 \begin{bmatrix} 0.30769231 \\ 0 \\ 0.15384615 \\ 0.53846154 \end{bmatrix} + \mu_5 \begin{bmatrix} 0.66666667 \\ 0 \\ 0.33333333 \\ 0 \end{bmatrix} + \mu_6 \begin{bmatrix} 0 \\ 0 \\ 0.66666667 \\ 0.33333333 \end{bmatrix} + \mu_7 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$$
$$\lambda_i \geq 0, i = 1, \ldots, 4$$
$$\mu_i \geq 0, i = 1, \ldots, 7$$

Then, solve the system of equations to get

$$\lambda = \begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{bmatrix} = \begin{bmatrix}0.07308039\\ 0.45453457\\ 0.04019476\\ 0.43219028\end{bmatrix},\ \mu = \begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\\ \mu_5\\ \mu_6\\ \mu_7\end{bmatrix} = \begin{bmatrix}0.85706436\\ 0.40855584\\ 0.69313724\\ 0.74249143\\ 0.97872297\\ 0.19987637\\ 0.42627865\end{bmatrix}.$$

Exercise 2.11

Solution:

Suppose that x^* is an optimal solution. Select $\varepsilon > 0$ such that the set $B(x^*, \varepsilon) = \{x \mid ||x-x^*|| < \varepsilon\}$ is completely contained in P i.e. $B(x^*, \varepsilon) \subset P$. Let $\tilde{x} = x^* + \frac{c}{\|c\|}\frac{\varepsilon}{2} \in B(x^*, \varepsilon) \subset P$. then

$$c^T\tilde{x} = c^Tx^* + \|c\|\frac{\varepsilon}{2} > c^Tx^* \text{ since } c \neq 0.$$

So x^* is not an optimal solution, a contradiction. (□).

Exercise 2.12

Solution:

A constraint is deleted means the feasible set is larger.

If the problem is a maximization, then the optimal objective value may increase, if the problem is a minimization then the objective may decrease.

Exercise 2.13

Solution:

(=>)

Suppose P is bounded and that it has an extreme direction, then for any $\overline{x} \in P$ and $\overline{x} + \alpha d \in P$ for all $\alpha \geq 0$. In particular we can make $\|\overline{x} + \alpha d\|$ arbitrarily large by choosing α arbitrarily large. Contradiction so there is no extreme direction if P is bounded.

(<=)

Now suppose P has no extreme direction, then by Resolution theorem any $x \in P$ can be represented as $x = \sum_{i=1}^{K}\lambda_i v_i$ where $\lambda_i \geq 0$ and $\sum_{i=1}^{K}\lambda_i = 1$ and v_i is an extreme point. So $\|x\| = \left\|\sum_{i=1}^{K}\lambda_i v_i\right\| = \sum_{i=1}^{K}\|\lambda_i v_i\|$ which is bounded.

Exercise 2.14

Solution:

(a) Suppose $x^* \in P$ is a vertex, then there is a vector c such that $c^Tx^* < c^Tx$ for all $x \in P$, and $x^* \neq x$. Now let x_1 and $x_2 \in P$, and $x^* \neq x_1$ and $x^* \neq x_2$. Then for any $0 \leq \lambda \leq 1$, $c^Tx^* < c^T(\lambda x_1 + (1 - \lambda)x_2)$ since $c^Tx^* < c^Tx_1$ and $c^Tx^* < c^Tx_2$, thus $x^* \neq \lambda x_1 + (1-\lambda)x_2$ and so x^* can't be represented as a convex combination of two other elements in P, therefore x^* is an extreme point. (■)

(b) Let x^* be a basic feasible solution and let $I = \{ i \mid \alpha_i^T x^1 = b_i \}$ be the index set of constraints of P at equality. Now let $c = \sum_{i \in I} \alpha_i$, then $c^T x^* = \sum_{i \in I} \alpha_i^T x^* = \sum_{i \in I} b_i$, now for any feasible $x \in P$ we have $\alpha_i^T x \geq b_i$, so $c^T x = \sum_{i \in I} \alpha_i^T x \geq \sum_{i \in I} b_i$, so x^* is a unique minimize of $c^T x$ over P and thus a vertex. (■)

Exercise 2.15

Write MATLAB code that takes a linear program in standard form and solves for the optimal solution by generating all possible basic feasible solutions. Assume that the linear program has a finite optimal solution.

Solution: the MATLAB code written as follows:

```
function [non_sigular, sigular, non_singu_infeasi] =
enumerate_extre_ps(A, b)
% enumerate_extre_ps returns classify the different set
%   for a given matric A and RHS b of a linear system
%
% Inputs:
%   A is the matrix from of coefficients of the linear system
%   b is a vector of the linear system
%
% Outputs:
%   non_sigular is a struct includes the non-singular basis and the
%     corresponding extreme point and subscript set of variables.
%   sigular is a struct includes the singular basis and
%     the subscript set of variables.
%   non_singu_infeasi is a struct includes the non-singular basis, and
subscript
%     set of variables, however the extreme point is infeasible.

Non_singuSet=[]; SinguSet=[]; InfeasiSet=[];
non_sigular=[]; sigular=[];
non_singu_infeasi=[];
[m, n] = size(A); %number of basic variables and number of total variables
Combi = combntns(1:n, m); % the total possible combinations
epsi=1/10^22;
for a = 1:size(Combi,1)
    B = A(:, Combi(a,:));
    sigu(a) = det(B); % decide B if it is nonsingular
    if abs(sigu(a)) >= epsi
        if min(B\b) < 0 % if any infeasible solution
            InfeasiSet = cat(1, InfeasiSet, Combi(a,:));
        else
```

```
            Non_singuSet = cat(1, Non_singuSet, Combi(a,:));
        end
    else
        SinguSet =cat(1, SinguSet, Combi(a,:));
    end
end
for a = 1:size(Non_singuSet,1)
    non_sigular{a}.set = Non_singuSet(a,:);
    non_sigular{a}.B = A(:, Non_singuSet(a,:));
    non_sigular{a}.extremPoint = non_sigular{a}.B\b;
end
for a = 1:size(SinguSet,1)
    sigular{a}.set = SinguSet(a,:);
    sigular{a}.B = A(:, SinguSet(a,:));
end
for a = 1:size(InfeasiSet,1)
    non_singu_infeasi{a}.set = InfeasiSet(a,:);
    non_singu_infeasi{a}.B = A(:, InfeasiSet(a,:));
    non_singu_infeasi{a}.extremPoint = non_singu_infeasi{a}.B\b;
end
```