Chapter 2
Intuition about Uncertainty and Risk

2.1. *Suppose that an investment of 1 million dollars leads after one year, to a project worth 3 million dollars with probability 50% and to a complete loss of the million dollar investment the other half the time. This is a high risk, high return project. Now suppose that this project can be turned into 1000 smaller projects, each of which costs \$1000 to begin and each of which returns either \$3000 or \$0 with equal probability. Finally suppose that the success of each of the small projects is independent of the success of the other projects.*

Suppose the investor has two different strategies. Strategy A: invest 1 million dollars to the project worth \$3 million or \$0 with equal probability; Strategy B: invest 1000 independent small projects, each of which costs \$1000 to begin and returns either \$3000 or \$0 with equal probability.

The expected values of the profits of the two investments are the same:

$$E(A) = \left(\frac{1}{2}\right)(\$(3-1)*10^6) + \left(\frac{1}{2}\right)(\$(0-1))*10^6 = \$0.5*10^6$$

$$E(B) = E\left(\sum_{k=1}^{1000} B_k\right) = \sum_{k=1}^{1000} E(B_k) = \$0.5*10^6$$

On the other hand, the variances of the two profits are quite different:

$$\begin{aligned} Var(A) &= E(A^2) - E(A)^2 \\ &= \left(\frac{1}{2}\right)(2^2*10^{12}) + \left(\frac{1}{2}\right)((-1)^2*10^{12}) - 0.25*10^{12} \\ &= 2.25*10^{12} \end{aligned}$$

$$Var(B) = Var\left(\sum_{k=1}^{1000} B_k\right) = \sum_{k=1}^{1000} Var(B_k) = 2.25*10^9$$

The variance of profit A is 1000 times larger than that of profit B, even though they have the same expected value.

Therefore, Strategy B is better than Strategy A.

2.2. *Complete the calculation of the St Petersburg Utility to show:*

$$\sum_{k=1}^{\infty}(k/2^k) = 2$$

First of all, suppose

$$f(x) = \sum_{k=1}^{\infty}\left(\frac{k}{x^k}\right)$$

Then we need only to calculate *f(x)*. In fact,

$$f(x) = x\sum_{k=1}^{\infty}\frac{k}{x^{k+1}} = x\sum_{k=1}^{\infty}\left(\frac{-1}{x^k}\right)' = -x\left(\sum_{k=1}^{\infty}\frac{1}{x^k}\right)'$$
$$= -x\left(\frac{1}{x-1}\right)' = \frac{x}{(x-1)^2}$$

Therefore, we have

$$\sum_{k=1}^{\infty}(k/2^k) = f(2) = 2$$

2.3. *Find under what conditions, for a utility function U(x) which satisfies* $U'(x) > 0$ *and* $U''(x) < 0$, *the expected utility*

$$\sum_{k=1}^{\infty}\left[\frac{U(2^k)}{2^k}\right]$$

is finite.

Using different convergence tests, we can give different conditions. For example:
From Cauchy condensation test theorem, we know that

$$\sum_{k=1}^{\infty}\left[\frac{U(2^k)}{2^k}\right] < \infty$$

if and only if

$$\sum_{k=1}^{\infty}\left[\frac{U(k)}{k^2}\right] < \infty$$

From comparison test, we know it would be true if *U(x)* satisfies the following condition

$$\lim_{x\to\infty}\left[\frac{U(x)}{x^p}\right] = 0$$

for some $p \in (0,1)$. Note: This is a sufficient condition and the utility function given by Bernoulli also satisfies it.

2.4. *Find the value of the St Petersburg game to a player facing a counterparty or "banker" with wealth of* W. *(The notes covered the case where* $W = 1024$*). Using this result, compute the wealth the counterparty must hold for the St. Petersburg game to be worth 100 ducats. Discuss.*

We know the counterparty has the wealth of W ducats. Suppose $n = [\log_2 W]$, which means $2^n \leq W < 2^{n+1}$. Therefore, if heads arises after n flips, then the counterparty is not able to afford the bet, and the game should be over.
The expected value of the bet in this case is

$$E = \sum_{k=1}^{n} 2^k * \frac{1}{2^k} + 2^n * \frac{1}{2^n} = n + 1 = [\log_2 W] + 1$$

If $E = 100$, then $[\log_2 W] = 99$, which means

$$2^{99} \leq W < 2^{100}$$

2.5. *Compute the value of the St Petersburg game if p(Heads) = 0.49 and the counterparty has a) infinite wealth and b) 1024 ducats. Comment on the relative importance of a fair die and a wealthy counterparty to the player of the St Petersburg game.*

A) If p(Head) = 0.51 and the counterparty has infinite wealth, then the expected value of the bet is

$$E_A = \sum_{k=1}^{\infty} 2^k 0.49^{(k-1)} 0.51 = 1.02 \sum_{k=0}^{\infty} 0.98^k = \frac{1.02}{1 - 0.98} = 51$$

B) If p(Head) = 0.51 and the counterparty has 1024 ducats, then the expected value of the bet is

$$E_B = \sum_{k=1}^{10} 2^k 0.49^{(k-1)} 0.51 + 2^{10} 0.49^{10} = 10.1464$$

The expected values are infinite for *a fair die.*