# *Chapter 2*

**1.** Let $A$ and $B$ correspond to having a pool and air-conditioning, respectively. Then $P(A \cup B) = P(A) + P(B) - P(AB) = P(A) + P(B) - P(A)P(B) = 0.2 + 0.6 - 0.2 \cdot 0.6 = 0.68$.

**2.** $P(A^c B) = P(B) - P(AB) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A^c)P(B)$. The other cases are considered similarly.

**3.** (a) If $\omega_1$ and $\omega_2$ do not coincide, then $P([\omega_1] \cap [\omega_2]) = P(\emptyset) = 0$. Hence, if $P([\omega_1]) \neq 0$ and the same is true for $[\omega_2]$, then the events are dependent. However, if $P([\omega_i]) = 0$ for at least one $i = 1, 2$, then the events are independent by definition.

(b) If both events have positive probabilities, then they are dependent. However, if for example, $P(A) = 0$, then $P(AB) \leq P(A) = 0$, and the events $A, B$ are independent by definition.

(c) The events are independent, since as was proved in Exercise **1**.24, $P(A_1 A_2) = 1$, and hence, $P(A_1 A_2) = P(A_1)P(A_2)$.

**4.** Our intuition says that the events are dependent because under the condition that among two cards chosen there is, say, a king, the chances of having an ace decrease. Formally, $P(A_1) = P(A_2) = \left(\binom{4}{2} + \binom{4}{1}\binom{48}{1}\right) \Big/ \binom{52}{2}$. (To have at least one king we should select either two kings out of four, or one king and one card from 48 cards which are not kings.) Similarly, $P(A_1 A_2) = \binom{4}{1}\binom{4}{1} \Big/ \binom{52}{2}$. We see that $P(A_1 A_2) \neq P(A_1)P(A_2)$.

**5.** Clearly, $P(A) = 1/2$ and $P(B) = P(X \text{ and } Y \text{ are both even}) + P(X \text{ and } Y \text{ are both odd}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1/2$. On the other hand, $P(AB) = P(X \text{ and } Y \text{ are both even}) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$. Thus, $A$ and $B$ are independent.

Furthermore, clearly, $P(\overline{A}) = P(X = 4) = 1/6$ and $P(\overline{A}\,\overline{B}) = P(X = 4, Y = 4) = 1/36$. On the other hand, $P(\overline{B}) = \frac{9}{36} = \frac{1}{4}$, as is easy to compute, for example, counting all outcomes for which $X + Y$ is divided by 4. Hence, $\overline{A}, \overline{B}$ are dependent.

**6.** Straight calculations lead to $P(A) = 14/15$ , $P(B) = 11/15$, $P(AB) = 10/15$. Hence, as is easy to see, $A$ and $B$ are dependent.

However, we did not have to provide calculations. If $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $A = \{\omega_1, \omega_2, \omega_3\}$, and $B = \{\omega_2, \omega_3, \omega_4\}$, and $p_1$ and $p_4$ the elementary probabilities of the outcomes $\omega_1$ and $\omega_4$ respectively, then $P(A) = 1 - p_4$, $P(B) = 1 - p_1$ and $P(AB) = 1 - p_1 - p_4$. For the independency of $A$ and $B$, we should have

$$(1 - p_1 - p_4) = (1 - p_1)(1 - p_4),$$

which is equivalent to $p_1 p_4 = 0$. Thus, at least one of probabilities, $p_1$ or $p_4$, should equal zero.

**7.** (a) There are four outcomes, and hence $P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{2}$, and $P(A_1 A_2) = \frac{1}{4}$.

(b) Similarly, if there are $n$ trials, and $A_i = \{\text{the } i\text{th trial is successful}\}$, then $P(A_i) = \frac{2^{n-1}}{2^n} = \frac{1}{2}$. On the other hand, for any sample $i_1, ..., i_k$, we have $P(A_{i_1} \cdots A_{i_k}) = \frac{2^{n-k}}{2^n} = \frac{1}{2^k}$.

**8.** We have $P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{2}$, $P(A_3) = \frac{1}{2}$. The events $A_1, A_2$ are independent. Furthermore, $A_1A_3 = A_1A_2$, and $A_2A_3 = A_1A_2$, either. This implies that $P(A_1A_3) = P(A_2A_3) = P(A_1A_2) = P(A_1)P(A_2) = \frac{1}{4}$, and all three events are pairwise independent. However, $A_1A_2A_3 = A_1A_2$, and hence, $P(A_1A_2A_3) = P(A_1A_2) = \frac{1}{4} \neq P(A_1)P(A_2)P(A_3)$.

**9.** Since $n = 3$, relation (1.1.4) leads to (1.1.2) for $k = 2$, and to (1.1.3) for $k = 3$.

**10.** For Fig.6a, we have $1-(1-p_1p_2p_3)(1-p_4) = p_4 + p_1p_2p_3 - p_1p_2p_3p_4$.

**11.** The probability of passing both tests is

$$\left(\sum_{i=5}^{10}\binom{10}{i}p_1^i(1-p_1)^{10-i}\right)\left(\sum_{i=10}^{20}\binom{20}{k}p_2^k(1-p_2)^{20-k}\right).$$

**12.** If $p = q = \frac{1}{2}$, then the r.-h.s. of (1.2.2) becomes $\binom{n}{k}2^{-n}$.

**13.** The probability in hand is $1-(0.91)^{15} - 15\cdot 0.09(0.91)^{14} \approx 0.396$.

**14.** $\binom{5}{3}\left(\frac{1}{6^4}\right)^3\left(1-\frac{1}{6^4}\right)^2 = 10\cdot 6^{-12}(1-6^{-4})^2$.

**15.** This exercise is relevant to Exercise **1**.43. Let $n$ be the number of tosses, and $S_n$ be the number of heads. If $n = 100$, then $S_n$ takes on an odd number of values and the maximum of $P(S_n = k)$ is attained at $k = 50$. Hence, $P(S_n \le 49) = P(S_n \ge 51) < 0.5$. Hence $P(S_n \le 50) > 0.5$. Excel gives $P(S_{100} \le 50) > 0.5397$.

If $n = 101$, then the maximum of $P(S_n = k)$ is attained at two points: $k = 50$ and $k = 51$, and $P(S_n \le 50) = 0.5$.

**16.** Excel gives $\sum_{i=0}^{55}\binom{100}{i}(0.55)^i(0.45)^{100-i} \approx 0.53867$. The theoretical comments are given in the exercise itself.

**17.** The events $B_{k,n}$ are disjoint because they concern the precise values of the number of successes. The sum of the probabilities equals one because $\cup_{i=0}^{n} B_{k,n} = \Omega$. It also follows from the binomial formula $\sum_{i=0}^{n}\binom{n}{k}p^kq^{n-k} = (p+q)^n = 1^n = 1$.

**18.** (a) $\left(\frac{4}{6}\right)^3\left(\frac{1}{6}\right)^2 \approx 0.00823$.

(b) $\binom{5}{2}\left(\frac{4}{6}\right)^3\left(\frac{1}{6}\right)^2 \approx 0.0823$.

(c) $\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) \approx 00274$.

(d) $\left(\frac{1}{3}\right)^5 \approx 0.004$.

(e) the same as (d): $\left(\frac{1}{3}\right)^5 \approx 0.004$.

(f) The first run of black will happen on the fifth roll with a probability of $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$. The same probabilities for green and red are $\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)$. Hence, the probability of interest is $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 2\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right) \approx 0.039$.

**19.** (a) $\left(\frac{1}{6}\right)^6$. (b) $\frac{6!}{3!3!1!}\left(\frac{1}{6}\right)^6$.

**20.** Let $A = \{bb, bg, gb\}$ and $B = \{bg, gb, gg\}$. Then $P(A\,|\,B) = \frac{P(AB)}{P(B)} = \frac{P(bg,gb)}{P(B)} = \frac{2/4}{3/4} = 2/3$.

**21.** There are 12 outcomes for which the sum is divided by 3. Hence, the probability in hand is $\frac{1/36}{12/36} = \frac{1}{12}$.

**22.** Let $A$ be the event that the first player does not have spades, and $B$ be the analogous event for the second player. Then, $P(A) = \binom{39}{13} \Big/ \binom{52}{13}$, and

$$P(AB) = \frac{\binom{39}{13}\binom{26}{13}}{\binom{52}{13}\binom{39}{13}} = \frac{\binom{26}{13}}{\binom{52}{13}}.$$

Hence, $P(B\,|\,A) = \binom{26}{13} \Big/ \binom{39}{13}$.

Another way of reasoning may be as follows.

Once the first player did not get spades, there are totaly $\binom{39}{13}$ possible hands for the second player, and *all* 13 spades are among these 39. Hence, there are $\binom{26}{13}$ hands without spades for the second player.

**23.** In the first case, the area of $A_1A_2$ constitutes one-forth of the area of $A_2$. Hence, $P(A_1\,|\,A_2) = 1/4$, and $P(A_1) = 1/4$ also. (This confirms that $A_1$ and $A_2$ are independent.) For the second case, $P(A_1A_2) = 0$, and hence, $P(A_1\,|\,A_2) = 0$.

If $A_1 = \{x_1 \geq 0\}$, $A_2 = \{x_2 \geq 0\}$, then in the first case, $P(A_1\,|\,A_2) = 1/2 = P(A_1)$, so $A_1$ and $A_2$ are independent. In the case (b), the area of $A_1A_2$ constitutes a half of the area of $A_2$, and we have $P(A_1\,|\,A_2) = 1/2 = P(A_1)$. So, $A_1$ and $A_2$ are independent.

**24.** If $A \subseteq B$, then $AB = A$, and hence $\frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)}$. If $A \supseteq B$, then $\frac{P(AB)}{P(B)} = \frac{P(B)}{P(B)} = 1$. If $A$ and $B$ are disjoint, then $\frac{P(AB)}{P(B)} = 0$. The last two answers are obvious because, in the first case if $B$ has occurred, then $A$ occurred also, while in the second case, if $B$ has occurred, then $A$ cannot occur.

**25.** (a) The probability in hand is $\frac{1}{2}(1-p_1) + \frac{1}{2}p_2 = \frac{1}{2}(1-p_1+p_2)$. (b) The conditional probability in hand is

$$\frac{\frac{1}{2}(1-p_1)}{\frac{1}{2}(1-p_1)+\frac{1}{2}p_2} = \frac{1-p_1}{1-p_1+p_2}.$$

**26.** For the conditional probability in hand, the corresponding ratio is $\dfrac{\frac{6}{15}}{\frac{6}{15}+\frac{4}{15}+\frac{1}{15}} = \dfrac{6}{11}$.

**27.** (a) Let $B_1$ be the event that a student selected at random knows the correct answer, $B_2 = B_1^c$, and $A$ be the event that the student has answered correctly. We have $P(A) = P(A\,|\,B_1)P(B_1) + P(A\,|\,B_2)P(B_2) = 1 \cdot 0.8 + 0.5 \cdot 0.2 = 0.9$.

(b) $P(B_2\,|\,A) = \frac{P(A\,|\,B_2)P(B_2)}{P(A)} = \frac{0.5\cdot 0.2}{0.9} = \frac{1}{9}$.

**28.** (a) Let $B_1$ be the event that Joan hikes in the first area, $B_2 = B_1^c$, and $A$ be the event that Joan saw a snake. We have $P(A) = P(A\,|\,B_1)P(B_1) + P(A\,|\,B_2)P(B_2) = 0.02 \cdot 0.5 + 0.01 \cdot 0.5 = 0.015$.

(b) The two areas are equally likely to be chosen, and the probability of seeing a snake in the first area is twice as large as the probability for the second area. So, it is reasonable to guess that given that a snake has been seen, the probability that it happened in the first area is $2/3$. Rigorously, $P(B_1 \,|\, A) = \frac{P(A\,|\,B_1)P(B_1)}{P(A)} = \frac{0.02 \cdot 0.5}{0.015} = \frac{2}{3}$.

**29.** (a) Depending on what happened in the first draw, the chances to select a red ball in the second draw may get larger or smaller. By analogy with the classical case $c = 0$, one *may conjecture* that one possibility compensates the other.

(b) By the formula for total probability, the probability in hand is

$$\frac{r+c}{r+b+c} \cdot \frac{r}{r+b} + \frac{r}{r+b+c} \cdot \frac{b}{r+b} = \frac{r}{r+b}\left(\frac{r+c}{r+b+c} + \frac{b}{r+b+c}\right) = \frac{r}{r+b}.$$

(c) Given that the second ball is red, the event that the first ball was red becomes more plausible, and one may conjecture that the larger $c$, to the larger extent it is manifested. Taking into account the result of Problem (b), for the conditional probability in hand we have

$$\frac{\frac{r+c}{r+b+c} \cdot \frac{r}{r+b}}{\frac{r}{r+b}} = \frac{r+c}{r+b+c}.$$

**30.** Given the fifth component works, the probability that the signal will go through is $q_1 = (p_1 + p_2 - p_1 p_2)(p_3 + p_4 - p_3 p_4)$. Given the fifth component does not work, the same probability is $q_2 = 1 - (1 - p_1 p_3)(1 - p_2 p_4) = p_1 p_3 + p_2 p_4 - p_1 p_2 p_3 p_4$. By the formula for total probability, the probability of interest is $p_5 q_1 + (1 - p_5) q_2$.

**31.** There are three possible situations: both will catch a fish, only one will catch a fish, and nobody will catch a fish. Denote the corresponding events by $B_1, B_2, B_3$, respectively; and by $A$ the event that a particular person, say the first, will bring a fish home. By the formula for total probability,

$$\begin{aligned} P(A) &= P(A\,|\,B_1)P(B_1) + P(A\,|\,B_2)P(B_2) + P(A\,|\,B_3)P(B_3) \\ &= 1 \cdot p^2 + \frac{1}{2} 2p(1-p) + 0 \cdot (1-p)^2 = p. \end{aligned}$$

This means that the agreement (being, certainly, friendly) does not make chances to bring a fish home larger. The friends should just arrange a joint dinner with what they caught.

**32.** In an output sequence, digits appear independently regardless of whether they will be distorted or not. Consider a particular position and denote by $B_k$ the event that the digit $k$ will appear in the input sequence, and by $C$ the event that a particular digit, say 0, will appear in the output sequence. Then, $P(C) = P(C\,|\,B_0)P(B_0) + \sum_{k=1}^{9} P(C\,|\,B_k)P(B_k) = \frac{1}{10}p + 9 \cdot \frac{1}{10}(1-p)\frac{1}{9} = \frac{1}{10}$. Thus, in the output sequence, all digits are equally likely also.

Let $A$ be the event that the output sequence is 01, and $B$ is the event that the input sequence is 00. We have $P(A\,|\,B) = [p + (1-p)\frac{1}{9}] \cdot [(1-p)\frac{1}{9}]$. Then

$$P(B\,|\,A) = \frac{P(A\,|\,B)P(B)}{P(A)} = \frac{[p + (1-p)\frac{1}{9}] \cdot [(1-p)\frac{1}{9}] \cdot \frac{1}{10} \cdot \frac{1}{10}}{\frac{1}{10} \cdot \frac{1}{10}} = [p + (1-p)\frac{1}{9}] \cdot (1-p)\frac{1}{9}.$$

**33.** The Bayes formula may be rewritten as $P(B_1\,|\,A) = \dfrac{P(AB_1)}{P(AB_1)+P(AB_2)}$. So, if $P(AB_1) > P(AB_2)$, then $P(B_1\,|\,A) > \frac{1}{2}$, and hence $P(B_2\,|\,A) < \frac{1}{2} < P(B_1\,|\,A)$.

**34.** Let $B_1$ be the event that Ms. K. took a tutoring course, $B_2 = B_1^c$, and $A$ be the event that Ms. K. has passed the exam. We have $P(B_1) = 0.12$, $P(A\,|\,B_1) = 0.9$ while $P(A\,|\,B_2) = 0.7$. We have $P(A) = P(A\,|\,B_1)P(B_1) + P(A\,|\,B_2)P(B_2) = 0.9 \cdot 0.12 + 0.7 \cdot 0.88 = 0.724$, and $P(B_1\,|\,A) = \frac{P(A\,|\,B_1)P(B_1)}{P(A)} = \frac{0.9 \cdot 0.12}{0.724} \approx 0.150$.

**35.** (a) Either by the formula for total probability or directly by the multiplication rule, considering all possible results concerning the first selection, for the desired probability we have

$$\frac{10}{21}\cdot\frac{5}{11}\cdot\frac{17}{30}+\frac{10}{21}\cdot\frac{6}{11}\cdot\frac{16}{30}+\frac{11}{21}\cdot\frac{5}{11}\cdot\frac{16}{30}+\frac{11}{21}\cdot\frac{6}{11}\cdot\frac{15}{30}=\frac{368}{693}.$$

(b) We have already found the probability of the condition. The numerator of the corresponding formula is the probability that all people chosen are females. It is $\frac{11}{21}\frac{6}{11}\frac{15}{30}$. Hence, the conditional probability in question is $\frac{11\cdot 6\cdot 15}{21\cdot 11\cdot 30} = \frac{99}{368}$.

**36.** Let $B_i$ be the event that a randomly chosen citizen lives in province $i$; $i = 1,2,3$. Let $A$ be the event that this person called to the company. We have $P(B_1) = 0.4$, $P(B_2) = 0.35$, $P(B_3) = 0.25$; $P(A\,|\,B_1) = 0.55$, $P(A\,|\,B_2) = 0.3$, $P(A\,|\,B_3) = 0.2$. Furthermore, $P(A) = P(A\,|\,B_1)P(B_1) + P(A\,|\,B_2)P(B_2) + P(A\,|\,B_3)P(B_3) = 0.55 \cdot 0.4 + 0.3 \cdot 0.35 + 0.2 \cdot 0.25 = 0.375$, and $P(B_1\,|\,A) = \frac{P(A\,|\,B_1)P(B_1)}{P(A)} = \frac{0.55\cdot 0.4}{0.375} \approx 0.587$. The fact that $P(B_1\,|\,A) > P(B_1)$ is not surprising: people living in the first province are more interested in gardening that people from the other provinces.

**37.** Let $B_1, B_2, B_3$ be the events that a randomly chosen person saw an ad only in a newspaper, saw it in TV, and did not see an ad at all, respectively. Let $A$ be the event that this person bought the product. We have $P(B_1) = 0.02$, $P(B_2) = 0.1$, $P(B_3) = 0.88$; $P(A\,|\,B_1) = 0.1$, $P(A\,|\,B_2) = 0.3$, $P(A\,|\,B_3) = 1/30$.

(a) We have $P(A) = P(A\,|\,B_1)P(B_1) + P(A\,|\,B_2)P(B_2) + P(A\,|\,B_3)P(B_3) = 0.1 \cdot 0.02 + 0.3 \cdot 0.1 + (1/30) \cdot 0.88 = 0.061\bar{3}$.

(b) $P(A\,|\,B_1)P(B_1) + P(A\,|\,B_2)P(B_2) = 0.1 \cdot 0.02 + 0.3 \cdot 0.1 = 0.032$.

(c) We should expect the conditional probability in hand larger than $P(B_2) = 0.1$ because the percentage of people who buy the product is the largest among those who have seen the TV ad.

(d) $P(B_2\,|\,A) = \frac{P(A\,|\,B_2)P(B_2)}{P(A)} \approx \frac{0.3\cdot 0.1}{0.613} \approx 0.489$.

**38.** The probability $p_9$ is the probability of reaching the level $a$ (before ruin), while *in our case*, $p$ coincides with the probability of reaching the level $a$ immediately, in one step.

**39.** Denote by $A_u(a)$ the event that the process dropped to zero level, and it happened before the capital reached the level $a$. That is, this the same event $A_u$ but we have indicated the dependency on $a$. Clearly, if $A_u(a)$ occurred, then for any $b > a$, the event $A_u(b)$ has also occurred. This means that the ruin probability, as a function of $a$, is increasing. In particular, this implies that the probability in (2.4.7) is larger than that in (2.4.5).

This may be also shown directly. For $r \neq 1$,

$$P(A_u(a)) = \frac{r^u - r^a}{1 - r^a} = 1 - \frac{1 - r^u}{1 - r^a}.$$

It is easy to see that the last function is increasing in $a$ regardless of whether $r$ is larger or smaller than 1.

We derived (2.4.7) for the case $p > 1/2$, which is equivalent to $r < 1$. In this case, the function above converges to $r^u$ as $a \to \infty$.

**40.** (a) Since $\frac{p}{1-p} = 3$, the parameter $r = \frac{1}{3}$, and for the capital $u$ needed, we have an equation $\left(\frac{1}{3}\right)^u < 0.01$. A solution is $u > \frac{\ln 100}{\ln 3} \approx 4.19$. Since $u$ is assumed to be an integer, we come to $u \geq 5$.

(b) The probability $q_1$ is the probability of reaching zero, starting from one, and $q$ coincides with the probability of reaching zero, starting from one, in one step. So, if $q \neq 1$, then $q_1$ must be larger than $q$. Certainly, this follows from (2.4.7) also: $q_1 = r = \frac{1-p}{p} \geq 1 - p = q$.

(c) Let \$500 be a unit of money. Mark will be ruined if his capital becomes negative. The probability of dropping to $-1$ starting from 0, equals the probability of dropping to zero starting from one. So, the probability of interest is $1 - q_1$, and $q_1 = r$. If $p = 0.9$, then $r = 1/9$ and the desired probability is 8/9. To find $p$ for which $q_1 \leq 0.05$, we solve the equation $\frac{1-p}{p} \leq 0.05$, which leads to $p \geq \frac{1}{1.05}$.

**41.** (a) (i) Following construction of Section 2.7, it is easy to see that if we lost at the first time, the corresponding conditional probability is $\frac{1-p_1}{(1-p_1)+(1-p_2)}$. This probability is less than $\frac{1}{2}$ if $p_1 < p_2$.

(ii) We have obtained in Section 2.7 that in the situation under discussion we will switch if $p_1 + p_2 > 1$. It is impossible if $p_1 < 0.5$ because in this case $p_2 < p_1 < 0.5$ also.

(b) Let $A$ be the event that we won at the first time and after that, lost $n$ times. Then $P(A\,|\,B_1) = p_1(1-p_1)^n$ and $P(A\,|\,B_2) = p_2(1-p_2)^n$. Since the prior probabilities $P(B_1) = P(B_2) = \frac{1}{2}$, the posterior probability

$$P(B_1\,|\,A) = \frac{P(A\,|\,B_1)}{P(A\,|\,B_1) + P(A\,|\,B_2)}.$$

This probability is larger than $\frac{1}{2}$ if $P(A\,|\,B_1) > P(A\,|\,B_2)$, which is equivalent to

$$\left(\frac{1-p_1}{1-p_2}\right)^n > \frac{p_2}{p_1}.$$

For our particular case, it is equivalent to $\left(\frac{7}{8}\right)^n > \frac{2}{3}$. It is easy to check that this is true for $n \leq 3$. Thus, at $n = 4$, we will switch to the other handle.

**42.** No. If, for example, the events $A_n$ are independent, then for $P(A_n \text{ occur infinitely often}) = 0$, it is necessary that $\sum_1^\infty P(A_n) < \infty$.

**43.** (a) Since in Example 3-1 we assume the trials to be independent,

$$P(B_n) = P(A_n) \prod_{k=1}^{n-1} P(A_k^c). \tag{2.1}$$

b) By Taylor's expansion $p_n = \frac{1}{n} + o\left(\frac{1}{n}\right)$, and hence, the series $\sum_{n=1}^{\infty} p_n = \infty$. Consequently, by the Borel-Cantelli theorem, $P(A) = 1$. By (M-2.1),

$$\begin{aligned} P(B_n) &= \left(1 - e^{-\frac{1}{n}}\right) \prod_{k=1}^{n-1} e^{-\frac{1}{k}} = (1 - e^{-\frac{1}{n}}) \exp\left\{ -\sum_{k=1}^{n-1} \frac{1}{k} \right\} \\ &= \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right) \exp\{-\ln(n-1) - \gamma + o(1)\} = \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right) \frac{1}{n-1} e^{-\gamma} e^{o(1)} \\ &= e^{-\gamma} \left(\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right). \end{aligned}$$

(c) In this case, $p_n = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$, and hence, the series $\sum_{n=1}^{\infty} p_n < \infty$. Consequently, by the Borel-Cantelli theorem, $P(A) = 0$. Now, the probability that there will be no successes during the first $n$ trials is

$$\prod_{k=1}^{n} P(A_k^c) = \prod_{k=1}^{n} e^{-\frac{1}{k^2}} = \exp\left\{ -\sum_{k=1}^{n} \frac{1}{k^2} \right\} \to e^{-\frac{\pi^2}{6}} \quad \text{as} \quad n \to \infty.$$

**44.** (a) No, by Corollary 2, for independent events $A_n$, the probability $P(A)$ may be equal only to 0 or 1.

(b) If the trials are dependent, $P(A)$ may be equal to any number between zero and one. Let us give an example where $P(A) = 0.5$. Consider a box that contains two coins: one is regular, and the other is "crooked" and, being flipped, comes up tails with probability one. Let us imagine that we select a coin at random and toss it infinitely many times. Then, with probability 0.5, there will be infinitely many heads, and with probability 0.5, there will be no heads at all. Hence, in this case, $P(A) = 0.5$.