2

FLOW KINEMATICS

Problem 2.1

The following graph was drawn using EXCEL.

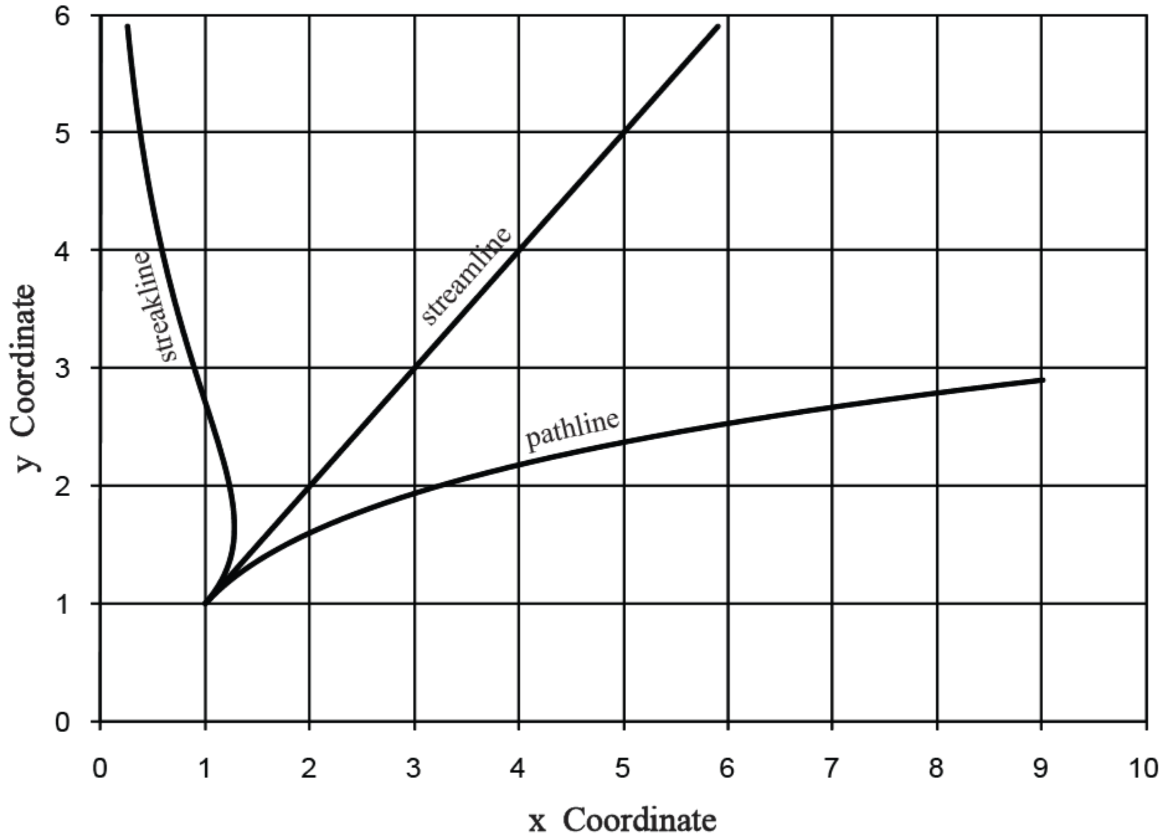


Figure Q.2.1: Streamline, pathline, and streakline for the flow field $u = x(1+2t)$, $v = w = 0$

Problem 2.2

(a)

$$\frac{dy}{dx} = \frac{v}{u} = 1 + t$$

$$\therefore y = (1+t)\,x + C$$

$$\text{But} \quad x = y = 1 \quad \text{when} \quad t = 0 \quad \Rightarrow \quad C = 0$$

Hence at $t = 0$ the equation of the streamline is:

$$\boxed{y = x}$$

(b)

$$\frac{dx}{dt} = u = \frac{1}{1+t} \qquad \frac{dy}{dt} = v = 1$$

$$\therefore x = \log(1+t) + C_1 \qquad \therefore y = t + C_2$$

The condition that $x = y = 1$ when $t = 0$ requires that $C_1 = C_2 = 1$, so that:

$$x = 1 + \log(1+t) \qquad y = 1 + t$$

Eliminating t between these two equations shows that the equation of the pathline is:

$$\boxed{y = e^{x-1}}$$

(c) Here, the equations obtained in (b) above are required to satisfy the condition $x = y = 1$ when $t = \tau$. This leads to the values $C_1 = 1 - \log(1+\tau)$ and $C_2 = 1 - \tau$. Hence the parametric equations of the streakline are:

$$x = \log(1+t) + 1 - \log(1+\tau) \qquad y = t + 1 - \tau$$

At time $t = 0$ these equations become:

$$x = 1 - \log(1+\tau) \qquad y = 1 - \tau$$

Eliminating the parameter τ between these two equations yields the following equation for the streakline at $t = 0$:

$$\boxed{y = 2 - e^{1-x}}$$

<u>Problem 2.3</u>

(a)

$$u = x(1+t) \qquad v = 1 \qquad w = 0$$

$$\frac{dx}{ds} = x(1+t) \Rightarrow x = C_1 e^{(1+t)s} = C_1 e^s \text{ at } t = 0$$

$$\frac{dy}{ds} = 1 \Rightarrow y = s + C_2$$

But $x = y = 1$ when $s = 0$ so that $C_1 = C_2 = 1$. Hence:

$$x = e^s \text{ and } y = s + 1$$

$$\boxed{\therefore\ x = e^{y-1}}$$

(b) $$\frac{dx}{dt} = x(1+t) \Rightarrow x = C_1 e^{t(1+t/2)}$$
$$\frac{dy}{dt} = 1 \Rightarrow y = t + C_2$$

But $x = y = 1$ when $t = 0$ so that $C_1 = C_2 = 1$. Hence:

$$x = e^{t(1+t/2)} \text{ and } y = t+1$$

$$\boxed{\therefore\ x = e^{(y^2-1)/2}}$$

(c) As in (b): $x = C_1 e^{t(1+t/2)}$ and $y = t + C_2$

But $x = y = 1$ when $t = \tau$ so that $C_1 = e^{-\tau(1+\tau/2)}$ and $C_2 = 1-\tau$. Hence:

$$x = e^{t(1+t/2)-\tau(1+\tau/2)} = e^{-\tau(1+\tau/2)} \text{ for } t = 0$$

and $$y = 1 + t - \tau = 1 - \tau \text{ for } t = 0$$

$$\boxed{\therefore\ x = e^{-(1-y)(3-y)/2}}$$

(d) From the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = \frac{D\rho}{Dt} + \rho \nabla \bullet \mathbf{u} = 0$$

Hence: $$\frac{D\rho}{Dt} = -\rho \nabla \bullet \mathbf{u} = -\rho(1+t) \quad \text{since } \nabla \bullet \mathbf{u} = (1+t) \text{ here.}$$

i.e. $$\frac{D}{Dt}(\log \rho) = -(1+t)$$

Therefore $$\log \rho = -t(1+t/2) + \log C$$

So that $$\rho = C e^{-t(1+t/2)}$$

But $\rho = \rho_0$ when $t = 0$ so that $C = \rho_0$. Hence:

$$\boxed{\rho = \rho_0 e^{-t(1+t/2)}} \quad \text{along the streamline}$$

Problem 2.4

The equations that define the streamlines are as follows:

$$\frac{dx_i}{u_i} = ds \qquad or \qquad \frac{dx_i}{x_i} = \frac{ds}{(1+t)}$$

$$\therefore \log x_i = \frac{s}{(1+t)} + \log C_i$$

$$\text{or } x_i = C_i e^{s/(1+t)}$$

Let $x_i = x_{i0}$ when $s = 0$. Then:

$$x_i = x_{i0}\ e^{s/(1+t)}$$

So that: $$\boxed{\frac{x}{x_0} = \frac{y}{y_0} = \frac{z}{z_0}}$$ for any time t.

The equations that define the pathlines are as follows:

$$\frac{dx_i}{dt} = u_i \qquad or \qquad \frac{dx_i}{dt} = \frac{x_i}{(1+t)}$$

$$\therefore \log x_i = \log(1+t) + \log C_i$$

$$\text{or} \quad x_i = C_i\,(1+t)$$

Let $x_i = x_{i0}$ when $s = 0$. Then:

$$x_i = x_{i0}\,(1+t) \qquad \text{or} \qquad \frac{x_i}{x_{i0}} = (1+t)$$

So that: $$\boxed{\frac{x}{x_0} = \frac{y}{y_0} = \frac{z}{z_0}}$$ as per the streamlines.

<u>Problem 2.5</u>

(a) $$u = 16\,x^2 + y \qquad v = 10 \qquad w = y\,z^2$$

On $0 \le x \le 10,\ y = 0$: $$\int_0^{10} u\,dx = \int_0^{10} 16\,x^2 dx = \frac{16{,}000}{3}$$

On $0 \le y \le 5,\ x = 10$: $$\int_0^5 v\,dy = \int_0^5 10\,dy = 50$$

On $10 \ge x \ge 0,\ y = 5$: $$\int_{10}^0 u\,dx = \int_{10}^0 (16\,x^2 + 5)\,dx = -\frac{16{,}000}{3} - 50$$

On $5 \ge y \le 0,\ x = 0$: $$\int_5^0 v\,dy = \int_5^0 10\,dy = -50$$

Then, adding the components around the counter-clockwise path gives:

$$\boxed{\Gamma = -50}$$

(b) For the area specified, $\mathbf{n} = \mathbf{e}_z$ and $\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -1$.

$$\therefore \int_A \boldsymbol{\omega} \bullet \mathbf{n}\,dA = \int_0^{10} dx \int_0^5 (-1)\,dy$$

$$\boxed{\int_A \boldsymbol{\omega}\bullet\mathbf{n}\,dA = -50}$$

This is the same result that was obtained in (a), so that Eq. (2.5) is verified for this flow.

Problem 2.6

$$u = \frac{x}{x^2+y^2} \qquad \text{and} \qquad v = \frac{y}{x^2+y^2}$$

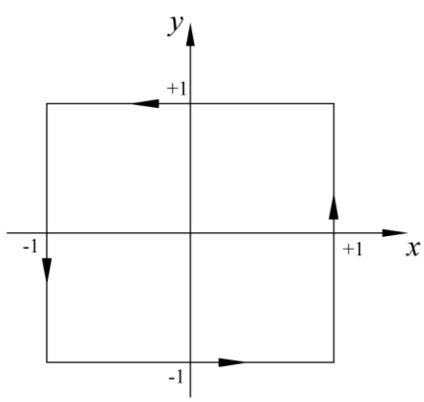
$$\therefore \Gamma = \int_{-1}^{+1} u(x,-1)\,dx + \int_{-1}^{+1} v(+1,y)\,dy + \int_{+1}^{-1} u(x,+1)\,dx + \int_{+1}^{-1} v(-1,y)\,dy$$
$$= \int_{-1}^{+1} \frac{-x\,dx}{x^2+1} + \int_{-1}^{+1} \frac{y\,dy}{1+y^2} + \int_{+1}^{-1} \frac{-x\,dx}{x^2+1} + \int_{+1}^{-1} \frac{y\,dy}{1+y^2}$$

In the foregoing equation, we note that there are two pairs of offsetting integrals. Hence:

$$\boxed{\Gamma = 0}$$

Problem 2.7

(a)



$$u = 9x^2 + 2y \quad v = 10x \quad w = -2yz^2$$

$$\int_{-1}^{+1} u(x,-1)\,dx = \int_{-1}^{+1} (9x^2-2)\,dx = [3x^3-2x]_{-1}^{+1} = 2$$

$$\int_{-1}^{+1} v(+1,y)\,dy = \int_{-1}^{+1} 10\,dy = [10y]_{-1}^{+1} = 20$$

$$\int_{+1}^{-1} u(x,+1)\,dx = \int_{+1}^{-1} (9x^2+2)\,dx = [3x^3+2x]_{+1}^{-1} = -10$$

$$\int_{+1}^{-1} v(-1,y)\,dy = \int_{+1}^{-1} (-10)\,dy = [-10y]_{+1}^{-1} = 20$$

Hence: $$\boxed{\Gamma = \oint \mathbf{u}\bullet d\mathbf{l} = 2+20-10+20 = 32}$$

(b) $$\boldsymbol{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{e}_z = -2z^2\,\mathbf{e}_x + 8\mathbf{e}_z$$

Hence $$\boxed{\boldsymbol{\omega} = -50\,\mathbf{e}_x + 8\,\mathbf{e}_z}$$ on the plane $z = 5$

(c) For the plane $z = 5$ the unit normal is $\mathbf{n} = \mathbf{e}_z$. Hence:

$$\boxed{\int_A \boldsymbol{\omega} \bullet \mathbf{n}\, dA = \int_{-1}^{+1} dx \int_{-1}^{+1} 8\, dy = 32}$$

This agrees with the result obtained in (a) - as it should, since

$$\Gamma = \oint \mathbf{u} \bullet d\mathbf{l} = \int_A \boldsymbol{\omega} \bullet \mathbf{n}\, dA$$

<u>Problem 2.8</u>

$$u = -\frac{y}{x^2 + y^2} \qquad \text{and} \qquad v = \frac{x}{x^2 + y^2}$$

(a)

$$\Gamma = \int_{-1}^{+1} u(x,-1)\, dx + \int_{-1}^{+1} v(+1,y)\, dy + \int_{+1}^{-1} u(x,+1)\, dx + \int_{+1}^{-1} v(-1,y)\, dy$$

$$= \int_{-1}^{+1} \frac{dx}{x^2+1} + \int_{-1}^{+1} \frac{dy}{1+y^2} + \int_{+1}^{-1} \frac{-x\, dx}{x^2+1} + \int_{+1}^{-1} \frac{-dy}{1+y^2}$$

$$= 4 \int_{-1}^{+1} \frac{d\alpha}{1+\alpha^2} = 4[\tan^{-1}\alpha]_{-1}^{+1}$$

$$\boxed{\Gamma = 2\pi}$$

(b)

$$\omega = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$= \left[\frac{1}{(x^2+y^2)} - \frac{2x^2}{(x^2+y^2)^2} \right] + \left[\frac{1}{(x^2+y^2)} - \frac{2y^2}{(x^2+y^2)^2} \right]$$

$$\boxed{\therefore \omega = 0 \quad \text{provided} \quad x \text{ and } y \neq 0}$$

(c)

$$\frac{\partial u}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\boxed{\therefore \nabla \bullet \mathbf{u} = 0 \quad \text{provided} \quad x \text{ and } y \neq 0}$$

Problem 2.9

$$u = \alpha y; \qquad v = \beta x$$

(a)
$$\Gamma = \oint \mathbf{u} \bullet d\ell = \int_{-1}^{+1} (-\alpha)\, dx + \int_{-1}^{+1} \beta\, dy + \int_{+1}^{-1} \alpha\, dx + \int_{+1}^{-1} (-\beta)\, dy$$
$$= \left[-\alpha x\right]_{-1}^{+1} + \left[\beta y\right]_{-1}^{+1} + \left[\alpha x\right]_{+1}^{-1} + \left[-\beta y\right]_{+1}^{-1}$$
$$= -2\alpha + 2\beta - 2\alpha + 2\beta$$

$$\boxed{\Gamma = -4(\alpha - \beta)}$$

(b)
$$\int_A \omega \bullet n\, dA = \int \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy = \int (\beta - \alpha)\, dx\, dy$$

$$\boxed{\int_A \omega \bullet n\, dA = -4(\alpha - \beta)}$$

(c)
$$\frac{dx}{ds} = \alpha y \quad \text{and} \quad \frac{dy}{ds} = \beta x$$
$$\therefore \frac{dy}{dx} = \frac{\beta x}{\alpha y} \quad \text{or} \quad \alpha y\, dy = \beta x\, dx$$
$$\alpha \frac{y^2}{2} = \beta \frac{x^2}{2} - \frac{c^2}{2}$$

$$\boxed{\beta x^2 - \alpha y^2 = c^2}$$

(d) $\alpha = -1$ and $\beta = +1 \;\Rightarrow\; x^2 + y^2 = c^2$ where $u = -y$ and $v = x$.

But x = 1 when y = 0 so that c^2 = 1. Therefore;

$$\boxed{x^2 + y^2 = 1}$$

(e) $\alpha = \beta = 1 \;\Rightarrow\; x^2 - y^2 = c^2$ where $u = y$ and $v = x$.

But x = 0 when y = 0 so that c = 0. Therefore;

$$\boxed{y = \pm x}$$

Problem 2.10

The vorticity vector will be in the z direction and its magnitude will be:

$$\omega(R,\theta) = \frac{1}{R}\frac{\partial}{\partial R}(R u_\theta) - \frac{1}{R}\frac{\partial u_\theta}{\partial \theta}$$

(a)

$$u_R = 0 \quad \text{and} \quad u_\theta = \omega R$$

$$\frac{\partial}{\partial R}(R u_\theta) = \frac{\partial}{\partial R}(\omega R^2) = 2\omega R$$

$$\text{and} \quad \frac{\partial u_R}{\partial \theta} = 0$$

$$\boxed{\therefore \omega(R,\theta) = 2\omega R}$$

(b)

$$u_R = 0 \quad \text{and} \quad u_\theta = \frac{\Gamma}{2\pi R}$$

$$\frac{\partial}{\partial R}(R u_\theta) = \frac{\partial}{\partial R}\left(\frac{\Gamma}{2\pi}\right) = 0$$

$$\text{and} \quad \frac{\partial u_R}{\partial \theta} = 0$$

$$\boxed{\therefore \omega(R,\theta) = 0 \quad \text{provided} \quad R \neq 0}$$
