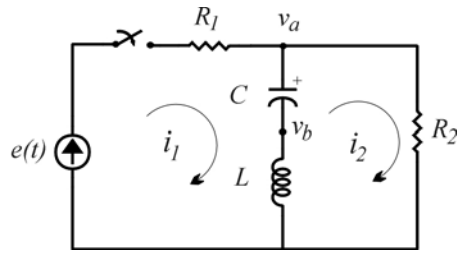
CHAPTER 5



5.1(a) Loop $e = [R_1 + \frac{1}{CD} + LD]i_1 - [\frac{1}{CD} + LD]i_2$

$$0 = [R_2 + LD + \frac{1}{CD}]i_2 - [LD + \frac{1}{CD}]i_1$$

(b) Node $v_a:\quad [\frac{1}{R_1} + CD + \frac{1}{R_2}]v_a - CDv_b = \frac{1}{R_1}e$

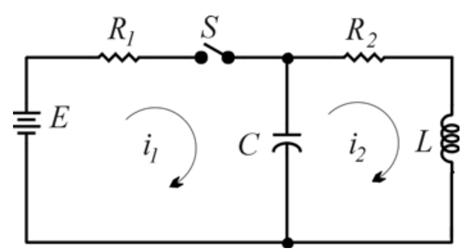
$$v_b:\quad [\frac{1}{LD} + CD]v_b - CDv_a = 0$$

(c) State $x_1 = v_c = v_a - v_b = \frac{i_c}{CD_1}$ $\qquad y_1 = v_a = v_b + v_c = \dot{x}_2 + \dot{x}_1$

$$x_2 = i_L = \frac{v_L}{LD} = i_1 - i_2 = i_c \qquad y_2 = v_b = v_L = L\dot{i}_L = L\dot{x}_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1R_2}{L(R_1+R_2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{R_2}{L(R_1+R_2)} \end{bmatrix} e$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{R_1R_2}{(R_1+R_2)} \\ -1 & -\frac{R_1R_2}{(R_1+R_2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{R_2}{(R_1+R_2)} \\ \frac{R_2}{(R_1+R_2)} \end{bmatrix} e$$



5.2(a) Loop $e = [R_1 + \frac{1}{CD}]i_1 - [\frac{1}{CD}]i_2$

$$0 = [R_2 + LD + \frac{1}{CD}]i_2 - \frac{1}{CD}i_1$$

(b) Node $v_a:\quad [\frac{1}{R_1} + CD + \frac{1}{R_2}]v_a - \frac{1}{R_2}v_b = \frac{1}{R_1}E$

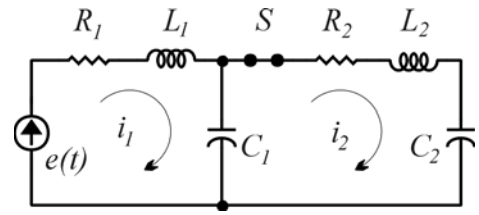
$$v_b:\quad [\frac{1}{R_2} + \frac{1}{LD}]v_b - \frac{1}{LD}v_a = 0$$

(c) State $x_1 = v_c = v_a \qquad x_2 = i_L = i_2 \qquad \dot{x}_2 = \frac{v_b}{L}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C} \\ 0 \end{bmatrix} E$$

5.3 Circuit 1

(a) $e = [R_1 + L_1 D + \frac{1}{C_1 D}] i_1$ when *S* opens $i_2 = 0$



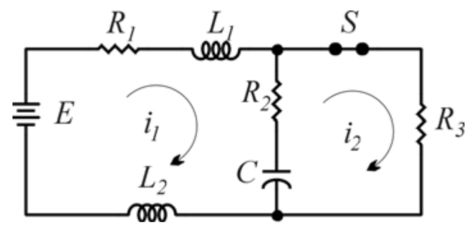
(b) $x_1 = v_{c_1}$ $\qquad x_2 = i_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C_1} \\ -\frac{1}{L_1} & -\frac{R_1}{L_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_1} \end{bmatrix} e$$

Circuit 2

(a) Similar with total resistance and inductance

$$e = [R_1 + R_2 + (L_1 + L_2)D + \frac{1}{C_1 D}] i_1$$

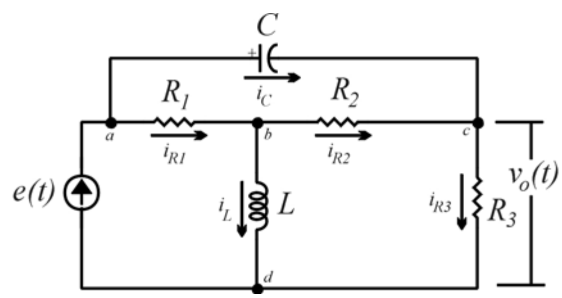


(b) For $x_1 = i_1$ $\qquad x_2 = v_c$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1 + R_2}{L_1 + L_2} & -\frac{1}{L_1 + L_2} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1 + L_2} \\ 0 \end{bmatrix} E$$

5.4 **(a)** Node d is the reference and let $v_o = v_c$.

Node b: $$\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{LD}\right] v_b - \frac{1}{R_1} e - \frac{1}{R_2} v_o = 0$$



Node c: $$-\frac{1}{R_2} v_b - CDe + \left[\frac{1}{R_2} + \frac{1}{R_3} + CD\right] v_o = 0$$

(b) Use clockwise currents i_1 for abda, i_2 for bcdb & i_3 for acba.

$$[R_1 + LD] i_1 - R_1 i_2 - LD i_3 = e$$

$$-R_1 i_1 + \left[R_1 + R_2 + \frac{1}{CD}\right] i_2 - R_2 i_3 = 0$$

$$-LD i_1 - R_2 i_2 + [R_2 + R_3 + LD] i_3 = 0$$

(c) Let $x_1 = i_1, x_2 = v_C = e - v_o, u = e(t)$ then $v_b = L\dot{x}_1, y = v_o = e - x_2$

Node b: $$\left[\frac{1}{R_1} + \frac{1}{R_2}\right] v_b + \frac{1}{LD} v_b - \frac{1}{R_2} v_o = \frac{1}{R_1} e$$

Node c: $\quad i_{R3} = i_1 + i_{R2} = x_1 + \dfrac{u - x_2}{R_1} - C\dot{x}_2$

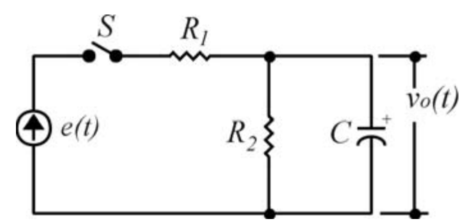
Manipulating gives: $\quad \dot{x}_1 = -\dfrac{R_1 R_2}{L(R_1 + R_2)} x_1 - \dfrac{R_1}{L(R_1 + R_2)} x_2 + \dfrac{1}{L} u$

$$\dot{x}_2 = \frac{R_1}{C(R_1 + R_2)} x_1 - \frac{(R_1 + R_2 + R_3)}{CR_3(R_1 + R_2)} x_2 + \frac{1}{CR_3} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1 R_2}{L(R_1 + R_2)} & -\dfrac{R_1}{L(R_1 + R_2)} \\ \dfrac{R_3}{C(R_2 + R_3)} & -\dfrac{(R_1 + R_2 + R_3)}{CR_3(R_1 + R_2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ \dfrac{1}{CR_3} \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1] u$$

5.5 Let $x = v_c = v_o, u = e(t)$ then $i_{R1} = i_{R2} + i_c = i_{R2} + C\dot{v}_o = \dfrac{v_o}{R_2} + C\dot{v}_o = \dfrac{x}{R_2} + C\dot{x}$

$$[R_1 + R_2] i_{R1} - R_2 i_c = e$$



$$-R_2 i_{R1} + \left[R_2 + \frac{1}{CD} \right] i_C = 0$$

$$R_1 i_{R1} + \left[\frac{1}{CD} \right] i_C = \frac{R_1 v_o}{R_2} + R_1 C\dot{v}_o + v_o = \frac{R_1 x}{R_2} + R_1 C\dot{x} + x = e$$

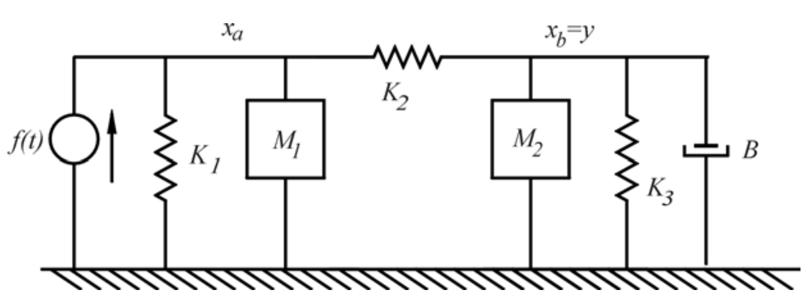
$$\dot{x} = -\frac{(R_1 + R2)}{R_1 R_2 C} x + \frac{1}{R_1 C} e$$

$$y = x$$

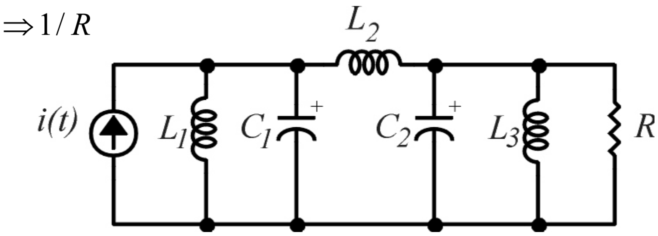
5.6 **(a)** $M_1 D^2 x_a = f(t) - K_1 x_a - K_2(x_a - x_b) \Rightarrow f(t) = (M_1 D^2 + K_1 + K_2) x_a - K_2 x_b$

$$M_2 D^2 x_b = K_2(x_a - x_b) - BDx_b - K_3 x_b \Rightarrow 0 = (M_2 D^2 + BD + K_2 + K_3) x_b - K_2 x_a$$

(b)



5.6 (c) For $f(t) \Rightarrow i(t), M \Rightarrow C, B \Rightarrow 1/L, K \Rightarrow 1/R$



(d) $$x_a = \frac{(M_2D^2 + BD + K_2 + K_3)}{K_2} x_b$$

$$f(t) = (M_1D^2 + K_1 + K_2)\frac{(M_2D^2 + BD + K_2 + K_3)}{K_2} x_b - K_2 x_b$$

$$K_2 f(t) = \{M_1M_2D^4 + M_1BD^3 + [M_1(K_2 + K_3) + M_2(K_1 + K_2)]D^2 + (K_1 + K_2)BD + (K_1 + K_2)(K_2 + K_3)\}y - K_2{}^2 y$$

Given $M_1 = M_2 = 1$ and $K_1 = K_2 = K_3 = K$,

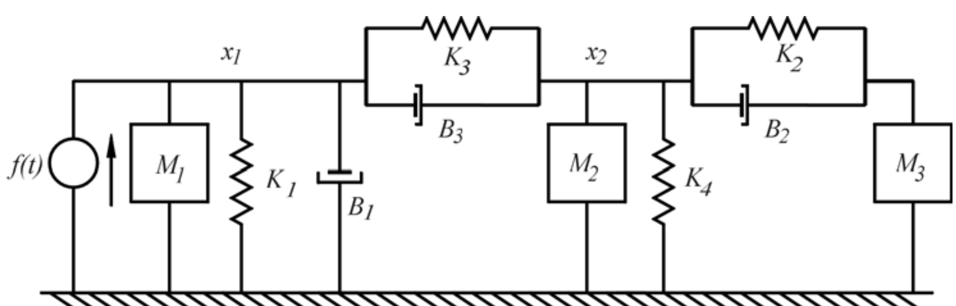
$$Kf(t) = \{D^4 + BD^3 + 4KD^2 + 2KBD + 3K^2\}y$$

$$G(D) = \frac{y}{f(t)} = \frac{K}{D^4 + BD^3 + 4KD^2 + 2KBD + 3K^2}$$

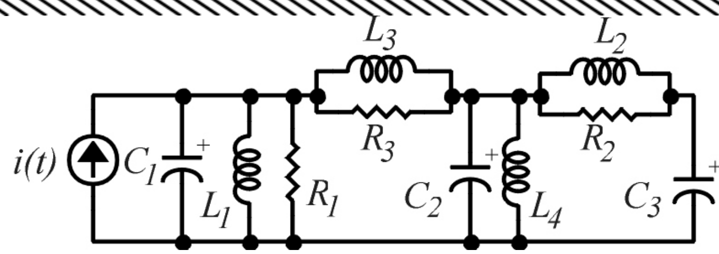
(e) Let $x_1 = x_a, x_2 = \dot{x}_1, x_3 = x_b, x_4 = \dot{x}_3, u = f(t), y = x_b = x_3$

$$\text{A}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -2K & 0 & K & 0 \\ 0 & 0 & 0 & 1 \\ K & 0 & -2K & -B \end{bmatrix} \qquad \text{b}=\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{c}=\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

5.7 (a)



(b)



5.7 (c)

$$M_1D^2x_1 = f(t) - B_1Dx_1 - K_1x_1 - B_3D(x_1 - x_2) - K_3(x_1 - x_2)$$
$$\Rightarrow f(t) = (M_1D^2 + B_1D + B_3D + K_1 + K_3)x_1 - (B_3D + K_3)x_2$$
$$M_2D^2x_2 = B_3D(x_1 - x_2) + K_3(x_1 - x_2) - K_4x_2 - B_2D(x_2 - x_3) - K_2(x_2 - x_3)$$
$$\Rightarrow 0 = \left(M_2D^2 + B_3D + B_2D + K_3 + K_4 + K_2\right)x_2 - (B_3D + K_3)x_1 - (B_2D + K_2)x_3$$
$$M_3D^2x_3 = B_2D(x_2 - x_3) + K_2(x_2 - x_3)$$
$$\Rightarrow 0 = \left(M_3D^2 + B_2D + K_2\right)x_3 - (B_2D + K_2)x_2$$

(d) Let $x_1 = x_1, x_4 = \dot{x}_1, x_2 = x_2, x_5 = \dot{x}_2, x_3 = x_3, x_6 = \dot{x}_3, u = f(t)$

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\dfrac{(K_1+K_3)}{M_1} & \dfrac{K_3}{M_1} & 0 & -\dfrac{(B_1+B_3)}{M_1} & \dfrac{B_3}{M_1} & 0 \\ \dfrac{K_3}{M_2} & -\dfrac{(K_2+K_3+K_4)}{M_2} & \dfrac{K_2}{M_2} & \dfrac{B_3}{M_2} & -\dfrac{(B_2+B_3)}{M_2} & \dfrac{B_2}{M_2} \\ 0 & \dfrac{K_2}{M_3} & -\dfrac{K_2}{M_3} & 0 & \dfrac{B_2}{M_3} & -\dfrac{B_2}{M_3} \end{bmatrix}$$

$$b^T=\begin{bmatrix} 0 & 0 & 0 & \dfrac{1}{M_1} & 0 & 0 \end{bmatrix}$$

5.8 (a) $3\ell f(t) = 3\ell M_2D^2x_c + 2\ell K_2x_b + 2\ell BDx_b + \ell M_1D^2x_a + \ell K_1(x_a - x_d)$

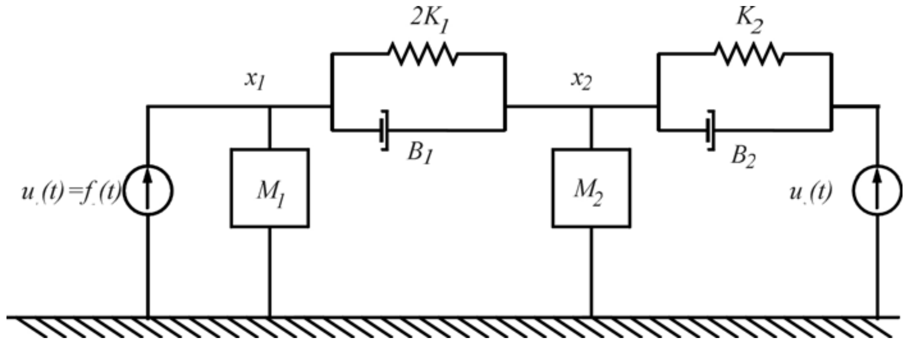
Since $x_b = 2x_a, x_c = 3x_a$

$$3\ell f(t) = 9\ell M_2D^2x_a + 4\ell K_2x_a + 4\ell BDx_a + \ell M_1D^2x_a + \ell K_1(x_a - x_d)$$
$$M_3D^2x_d = K_1(x_a - x_d)$$

(b) Let $x_1 = x_a, x_2 = \dot{x}_1, x_3 = x_d, x_4 = \dot{x}_3, u = f(t)$

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_1+4K_2}{M_1+9M_2} & -\dfrac{4B}{M_1+9M_2} & \dfrac{K_1}{M_1+9M_2} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{K_1}{M_3} & 0 & -\dfrac{K_1}{M_3} & 0 \end{bmatrix} \qquad b=\begin{bmatrix} 0 \\ \dfrac{3}{M_1+9M_2} \\ 0 \\ 0 \end{bmatrix}$$

5.9 (a)



(b)
$$(M_1D^2 + B_1D + 2K_1)x_1 - (B_1D + 2K_1)x_2 = u_1$$
$$-(B_1D + 2K_1)x_1 + (M_2D^2 + B_1D + 2K_1 + K_2)x_2 = (B_2D + K_2)u_2$$

(c) Let $\hat{u}_2 = (B_2D + K_2)u_2$ be the force exerted on the tires by the road. The outputs are $y_1=x_1$ and $y_2=x_2$.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{2K_1}{M_1} & \dfrac{2K_1}{M_1} & -\dfrac{B_1}{M_1} & \dfrac{B_1}{M_1} \\ \dfrac{2K_1}{M_2} & -\dfrac{2K_1 + K_2}{M_2} & \dfrac{B_1}{M_2} & -\dfrac{B_1 + B_2}{M_2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{1}{M_1} & 0 \\ 0 & \dfrac{1}{M_2} \end{bmatrix}\begin{bmatrix} u_1 \\ \hat{u}_2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

(d) Solving the equation from part (b) with $u_1=0$, yields:

$$\frac{x_1}{\hat{u}_2} = \frac{(B_1D + 2K_1)(B_2D + K_2)}{[M_1M_2D^4 + (M_1B_1 + M_1B_2 + M_2B_2)D^3 + (2M_1K_1 + M_1K_2 + B_1B_2 + 2M_2K_1)D^2 + (B_1K_2 + 2B_2K_1)D + 2K_1K_2]}$$

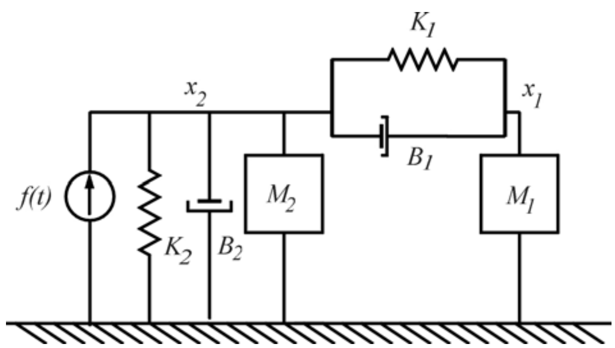
5.10 (a) $[R + LD]i = e, \quad K_i i = f, \quad x_b = \dfrac{b}{a}x_a \Rightarrow f_b = f_a\dfrac{a}{b}$

$$M_2D^2x_b = -K_2x_b - BDx_b + f_b \Rightarrow (M_2D^2 + BD + K_2)x_b = f_b = \frac{a}{b}f_a$$
$$M_1D^2x_a = -K_1x_a - f_a + f \Rightarrow (M_1D^2 + K_1)x_a + f_a = K_i i \Rightarrow (M_1D^2 + K_1)x_a - K_i i = -f_a$$
$$(M_2D^2 + BD + K_2)x_b = \frac{a}{b}f_a = -\frac{a}{b}\left[(M_1D^2 + K_1)x_a - K_i i\right]$$
$$(M_2D^2 + BD + K_2)x_b = -\frac{a}{b}\left[(M_1D^2 + K_1)\frac{a}{b}x_b - K_i i\right] = \frac{aK_i}{b}i - \frac{a^2}{b^2}(M_1D^2 + K_1)x_b$$
$$(M_2 + \frac{a^2}{b^2}M_1)D^2x_b + BDx_b + (K_2 + \frac{a^2}{b^2}K_1)x_b = \frac{aK_i}{b}i$$
$$LDi = -Ri + e,$$

5.10 (b) Let $x_1 = x_b, x_2 = \dot{x}_1, x_3 = i, u = e$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{K_2 + \dfrac{a^2}{b^2}K_1}{M_2 + \dfrac{a^2}{b^2}M_1} & -\dfrac{B}{M_2 + \dfrac{a^2}{b^2}M_1} & \dfrac{aK_i}{b(M_2 + \dfrac{a^2}{b^2}M_1)} \\ 0 & 0 & -\dfrac{R}{L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix} e \qquad \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

5.11 (a)



(b) $$(M_1D^2 + B_1D + K_1)x_1 - (B_1D + K_1)x_2 = 0$$
$$-(B_1D + K_1)x_1 + [M_2D^2 + (B_1 + B_2)D + (K_1 + K_2)]x_2 = f$$

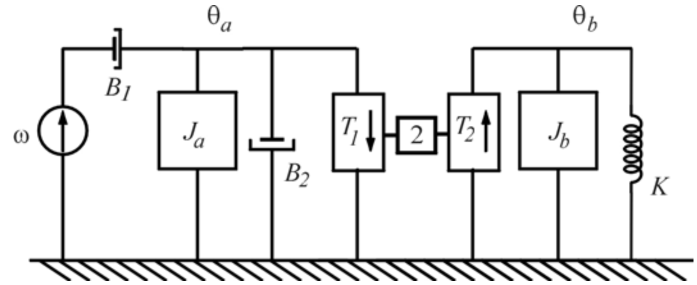
(c) Let $x_1 = x_1, x_2 = x_2, x_3 = \dot{x}_1, x_4 = \dot{x}_2, u = f(t)$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{K_1}{M_1} & \dfrac{K_1}{M_1} & -\dfrac{B_1}{M_1} & \dfrac{B_1}{M_1} \\ \dfrac{K_1}{M_2} & -\dfrac{K_1 + K_2}{M_2} & \dfrac{B_1}{M_2} & -\dfrac{B_1 + B_2}{M_2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{M_2} \end{bmatrix} f \qquad \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

(d)

$$G = \frac{x_1}{f} = \frac{(B_1D + K_1)}{[M_1M_2D^4 + (M_1B_1 + M_1B_2 + M_2B_1)D^3 + (M_1K_1 + M_1K_2 + M_2K_1 + B_1B_2)D^2 + (B_1K_2 + B_2K_1)D + K_1K_2]}$$

5.12 **(a)**



(b) $T_1 = -(J_a D^2 + B_2 D)\theta_a + B_1\omega \qquad T_2 = 2T_1 = (J_b D^2 + K)\theta_b$

$$2T_1 = -2(J_a D^2 + B_2 D)\theta_a + 2B_1\omega = (J_b D^2 + K)\theta_b$$

$$-2(J_a D^2 + B_2 D)2\theta_b + 2B_1\omega = (J_b D^2 + K)\theta_b$$

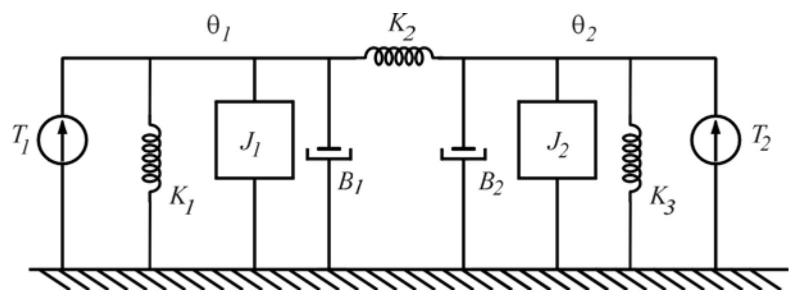
$$(J_b + 2J_a)D^2\theta_b = 2J_b D^2\theta_b = -(4B_2 D + K)\theta_b + 2B_1\omega$$

(c) $$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ \dfrac{-K}{2J_b} & \dfrac{-2B_2}{J_b} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \dfrac{B_1}{J_b} \end{bmatrix}\omega$$

(d) $$(2J_b D^2 + 4B_2 D + K)\theta_b = 2B_1\omega$$

$$\frac{\theta_b}{\omega} = \frac{2B_1}{2J_b D^2 + 4B_2 D + K}$$

5.13 **(a)**



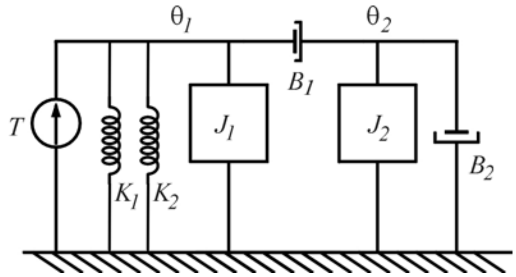
(b) $$T_1 = (J_1 D^2 + B_1 D + K_1 + K_2)\theta_1 - K_2\theta_2$$

$$T_2 = (J_2 D^2 + B_2 D + K_3 + K_2)\theta_2 - K_2\theta_1$$

(c) Let $x_1 = \theta_1, x_2 = \dot{\theta}_1, x_3 = \theta_2, x_4 = \dot{\theta}_2$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_1 + K_2}{J_1} & -\dfrac{B_1}{J_1} & \dfrac{K_2}{J_1} & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{K_2}{J_2} & 0 & -\dfrac{K_3 + K_2}{J_2} & -\dfrac{B_2}{J_2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 \\ \dfrac{1}{J_1} & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{J_2} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

5.14 (a)



(b) $J_1D^2\theta_1 = -B_1D(\theta_1 - \theta_2) - (K_1+K_2)\theta_1 + T \qquad J_2D^2\theta_2 = B_1D(\theta_1 - \theta_2) - B_2D\theta_2$

(c) Let $x_1 = \theta_1, x_2 = \dot{\theta}_1, x_3 = \theta_2, x_4 = \dot{\theta}_2$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_1+K_2}{J_1} & -\dfrac{B_1}{J_1} & 0 & \dfrac{B_1}{J_1} \\ 0 & 0 & 1 & 0 \\ 0 & \dfrac{B_1}{J_2} & 0 & -\dfrac{B_1+B_2}{J_2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \dfrac{1}{J_1} \\ 0 \\ 0 \end{bmatrix}T$$

(d) $(J_1D^2 + B_1D + K_1 + K_2)\theta_1 - T = B_1D\theta_2$

$$\left[J_2D^2 + (B_1+B_2)D\right]\theta_2 = B_1D\theta_1$$

$$\frac{\left[J_2D^2 + (B_1+B_2)D\right]}{B_1D}\theta_2 = \theta_1$$

$$(J_1D^2 + B_1D + K_1 + K_2)\frac{\left[J_2D^2 + (B_1+B_2)D\right]}{B_1D}\theta_2 - T = B_1D\theta_2$$

$$\frac{(J_1D^2 + B_1D + K_1 + K_2)\left[J_2D^2 + (B_1+B_2)D\right] - (B_1D)^2}{B_1D}\theta_2 = T$$

$$\frac{\theta_2}{T} = \frac{B_1}{J_1J_2D^3 + (J_2B_1 + J_1B_1 + J_1B_2)D^2 + (J_2K_1 + J_2K_2 + B_1B_2)D + (K_1+K_2)(B_1+B_2)}$$

5.15 $J_1 D^2\theta_1 = T_1 \qquad T_1 = (\mathrm{n_a})_{2,1} T_{2o} \quad T_{2o} = T_{2i} - J_2 D^2\theta_2 \qquad T_{2i} = (\mathrm{n_a})_{3,2} T_{3o}$

$T_{3o} = T - J_3 D^2\theta_3 \qquad\qquad (\mathrm{n_a})_{2,1} = \theta_2 / \theta_1 \ (\mathrm{n_a})_{3,2} = \theta_3 / \theta_2$

$$J_1 D^2\theta_1 = (\mathrm{n_a})_{2,1}((\mathrm{n_a})_{3,2}(T - J_3 D^2\theta_3) - J_2 D^2\theta_2)$$

$$J_1 D^2\theta_1 = (\mathrm{n_a})_{2,1}((\mathrm{n_a})_{3,2}(T - (\mathrm{n_a})_{2,1}(\mathrm{n_a})_{3,2} J_3 D^2\theta_1) - (\mathrm{n_a})_{2,1} J_2 D^2\theta_1)$$

$$J_1 D^2\theta_1 = (\mathrm{n_a})_{2,1}((\mathrm{n_a})_{3,2}(T - (\mathrm{n_a})_{2,1}(\mathrm{n_a})_{3,2} J_3 D^2\theta_1) - (\mathrm{n_a})_{2,1} J_2 D^2\theta_1)$$

$$\left[J_1 + (\mathrm{n_a})^2{}_{2,1}(\mathrm{n_a})_{3,2} J_2 + (\mathrm{n_a})^2{}_{2,1}(\mathrm{n_a})^2{}_{3,2} J_3\right] D^2\theta_1 = (\mathrm{n_a})_{2,1}(\mathrm{n_a})_{3,2} T$$

$$\left[\frac{1}{(\mathrm{n_a})_{2,1}(\mathrm{n_a})_{3,2}} J_1 + (\mathrm{n_a})_{2,1} J_2 + (\mathrm{n_a})_{2,1}(\mathrm{n_a})_{3,2} J_3\right] D^2\theta_1 = T = J_{eq}\dot{\omega} \qquad \dot{\omega} = \frac{T}{J_{eq}}$$

System 1
(a)

$$J_{eq} = \frac{1}{48} J_1 + 4J_2 + 48J_3$$
$$J_{eq} = 52.02 J_3$$

System 2

$$J_{eq} = \frac{1}{48} J_1 + 8J_2 + 48J_3$$
$$J_{eq} = 56.02 J_3$$

System 1 accelerates faster due to a lower equivalent inertia.

(b)

$$J_{eq} = \frac{1}{48} J_1 + 4J_2 + 48J_3$$
$$J_{eq} = (\frac{100}{48} + 160 + 48) J_3 = 210.08 J_3$$

$$J_{eq} = \frac{1}{48} J_1 + 8J_2 + 48J_3$$
$$J_{eq} = (\frac{100}{48} + 320 + 48) J_3 = 370.08 J_3$$

Again, System 1 accelerates faster due to a lower equivalent inertia.

5.16 **(a)** $T=(JD^2+B_1D)\theta+rf \qquad (MD^2+B_2D+K)x=f \qquad \theta r=x$

$$T=\frac{(JD^2+B_1D)}{r}x+r(MD^2+B_2D+K)x$$

(b)

$$T=\frac{(J+Mr^2)D^2+(B_1+B_2r^2)D+Kr^2}{r}x \Rightarrow \frac{x}{T}=\frac{r}{(J+Mr^2)D^2+(B_1+B_2r^2)D+Kr^2}$$

(c) Let $x_1=x, x_2=\dot{x}_1, u=T$

$$\dot{\mathbf{x}}=\begin{bmatrix} 0 & 1 \\ -\dfrac{Kr^2}{J+Mr^2} & -\dfrac{B_1+B_2r^2}{J+Mr^2} \end{bmatrix}\mathbf{x}+\begin{bmatrix} 0 \\ \dfrac{r}{J+Mr^2} \end{bmatrix}T$$

5.17 **(a)** Let $x_1=\theta_3, x_2=\dot{\theta}_3, x_3=\dot{\theta}_2, y=\theta_3, u=T$

$$\dot{\mathbf{x}}=\begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{K_2}{J_2} & -\dfrac{B_2+B_3}{J_2} & \dfrac{B_3}{J_2} \\ 0 & \dfrac{B_3}{J_1} & -\dfrac{B_1+B_3}{J_1} \end{bmatrix}\mathbf{x}+\begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{J_1} \end{bmatrix}T \qquad \mathbf{y}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\mathbf{x}$$

(b) Let $x_1=\theta_3, x_2=\dot{\theta}_3, x_3=\theta_2, x_4=\dot{\theta}_2, y=\theta_3, u=\theta_1$

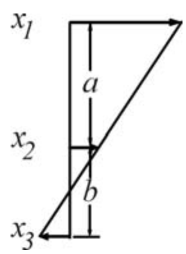
$$\dot{\mathbf{x}}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_2}{J_2} & -\dfrac{B_2+B_3}{J_2} & 0 & \dfrac{B_3}{J_2} \\ 0 & 0 & 1 & 0 \\ 0 & \dfrac{B_3}{J_1} & -\dfrac{K_1}{J_1} & -\dfrac{B_1+B_3}{J_1} \end{bmatrix}\mathbf{x}+\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{K_1}{J_1} \end{bmatrix}\theta_1 \qquad \mathbf{y}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\mathbf{x}$$

5.18 $x_2 = x_2' - x_2'' = \frac{b}{a+b}x_1 - \frac{a}{a+b}x_3$ **(1)** x_2' is due to motion, x_2' and x_2'' are due to x_3

$Dy_1 = C_1 x_2$ **(2)** $x_3 = y_2 = \frac{d}{c+d}y_1$ **(3)**

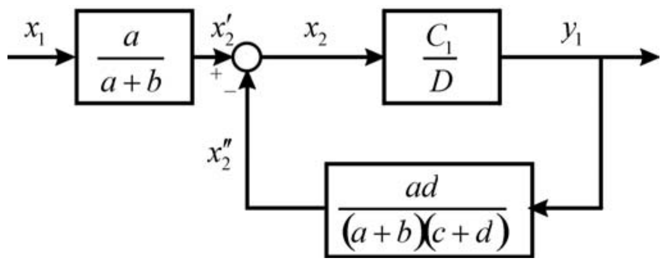
$$\left[\frac{C_1 ad}{(a+b)(c+d)} + D\right]y_1 \approx \frac{C_1 b}{a+b}x_1 \qquad \textbf{(4)}$$

From similar triangles:



$$\frac{x_1 - x_2}{a} = \frac{x_1 + x_3}{a+b} \qquad \textbf{(5)}$$

The block diagram representation is:



(b) If the load has mass and damping, the equation for piston motion is derived in Sec. 5.9. Starting with Eq. (5.107) but using the symbols of this problem gives

$$\left[\frac{VM}{K_B C}\right]D^3 y_1 + \left[\frac{VB}{K_B C} + \frac{M}{C}(L + C_p)\right]D^2 y_1 + \left[C_b + \frac{B}{C}(L + C_p)\right]Dy_1 = C_x x_2 \qquad \textbf{(6)}$$

Combining Eq. (6) and Eq (1) and (3) gives:

$$\left[\frac{VM}{K_B C}\right]D^3 y_1 + \left[\frac{VB}{K_B C} + \frac{M}{C}(L + C_p)\right]D^2 y_1 + \left[C_b + \frac{B}{C}(L + C_p)\right]Dy_1 + \left[\frac{C_x ad}{(a+b)(c+d)}\right]y_1 = \frac{C_x b}{a+b}x_1$$

5.19

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{-K}{M} & \dfrac{-B}{M} & \dfrac{C}{M} \\ 0 & \dfrac{-C_b K_B}{V} & \dfrac{-K_B(L+C_p)}{V} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{K_B C_x}{V} \end{bmatrix} u \qquad \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

5.20 Rewrite Eq. (5.116) in terms of pressure (or head), i.e., P(or h)$=qR$. Thus the pressure drop in the pipes are $P_1 = q_1R_1$, $P_2=q_2R_2$, and $P_3=q_3r_3$. Since

$$q_3 = q_1 + q_2 \quad \textbf{(1)} \qquad P_3 = q_3R_3 = \left[\frac{P_1}{R_1} + \frac{P_2}{R_2}\right]R_3 \quad \textbf{(2)}$$

The pressure at the junction of pipes 1 and 2 is P_3. This pressure is equal to the liquid level in tanks 1 and 2 minus the pressure drop in pipes 1 and 2, respectively. Thus

$$h_1 - P_1 = P_3 \quad \textbf{(3)} \qquad h_2 - P_2 = P_3 \quad \textbf{(4)}$$

Solving for P_1 and P_2 in terms of h_1 and h_2 yields:

$$P_1 = \frac{R_1(R_2+R_3)}{R}h_1 - \frac{R_1R_3}{R}h_2 \quad \textbf{(5)} \qquad P_2 = -\frac{R_2R_3}{R}h_1 + \frac{R_2(R_1+R_3)}{R}h_2 \quad \textbf{(6)}$$

Where $R = R_1R_2 + R_2R_3 + R_1R_3$

Using Eq. (5.116), the change in head in tank 1 when q_A is larger than q_1 is

$$Dh_1 = q_A - q_1 = q_A - \frac{P_1}{R_1} \quad \textbf{(8)} \qquad \text{for tank 2:} \qquad Dh_2 = q_B - q_2 = q_B - \frac{P_2}{R_2} \quad \textbf{(9)}$$

Using Eqs. (5) and (6) in Eqs. (8) and (9) yields the state equations

$$\dot{h}_1 = -\left[\frac{R_2+R_3}{R}\right]h_1 + \frac{R_3}{R}h_2 + q_A \quad \textbf{(10)} \qquad \dot{h}_2 = \frac{R_3}{R}h_1 - \left[\frac{R_1+R_3}{R}\right]h_2 + q_B \quad \textbf{(11)}$$

5.21 $e = R_a i_a + L_a D i_a + e_b = R_a i_a + L_a D i_a + K_b D\theta_m$ **(1)**

$T = K_T i_a = JD^2\theta_m + BD\theta_m$ **(2)** $\qquad i_a = \dfrac{J}{K_T} D^2\theta_m + \dfrac{B}{K_T} D\theta_m$ **(3)**

$$e = \frac{L_a J}{K_T} D^3\theta_m + \frac{R_a J + L_a B}{K_T} D^2\theta_m + \frac{R_a B + K_b K_T}{K_T} D\theta_m \qquad \textbf{(4)}$$

5.22 **(a)** The basic equation of the system is of the form $T = J\theta$. All of the torques in the system are summed about the pivot point P. The force f exerts a torque $f\ell\cos\theta$ about P, the force Mg exerts a torque $-Mg\ell\sin\theta$ about P, and the damper exerts the torque $-B\dot{\theta}$ about P. Therefore the equation of motion is

$$JD^2\theta = M\ell^2 D^2\theta = f\ell\cos\theta - Mg\ell\sin\theta - BD\theta \qquad \textbf{(1)}$$

(b) For small angles θ, Eq. (1) becomes

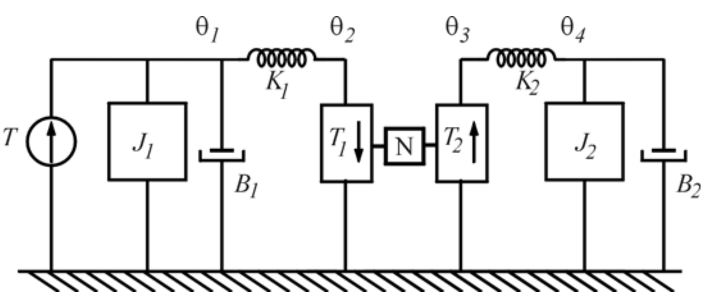
$$M\ell^2 D^2\theta = f\ell - Mg\ell\theta - BD\theta \qquad \textbf{(2)}$$

(c)

$$\ddot{\theta} = \frac{f}{M\ell} - \frac{g}{\ell}\theta - \frac{B}{M\ell^2}\dot{\theta} \qquad \textbf{(3)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{\ell} & -\dfrac{B}{M\ell^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M\ell^2} \end{bmatrix} f \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \textbf{(4)}$$

5.23 (a)



(b) $J_1 D^2\theta_1 = -B_1 D\theta_1 - K_1(\theta_1 - \theta_2) + T \qquad J_2 D^2\theta_4 = K_2(\theta_3 - \theta_4) - B_2 D\theta_4$

$T_1 = K_1(\theta_1 - \theta_2) \qquad T_2 = NT_1\ (\theta_2 = N\theta_3) \qquad T_2 = K_2(\theta_3 - \theta_4)$

5.24 (a) From Prob 5.21, Eq. (2) is modified to

$$T = K_T i_a = JD^2\theta_m + BD\theta_m + K\theta_m \qquad i_a = \frac{J}{K_T}D^2\theta_m + \frac{B}{K_T}D\theta_m + \frac{K}{K_T}\theta_m$$

Solving for i_a and substituting into Eq. (1) yields:

$$e = R_a\left(\frac{J}{K_T}D^2\theta_m + \frac{B}{K_T}D\theta_m + \frac{K}{K_T}\theta_m\right) + L_a D\left(\frac{J}{K_T}D^2\theta_m + \frac{B}{K_T}D\theta_m + \frac{K}{K_T}\theta_m\right) + K_b D\theta_m$$

$$e = \frac{L_a J}{K_T}D^3\theta_m + \left(\frac{R_a J + L_a B}{K_T}\right)D^2\theta_m + \left(\frac{R_a B + L_a K}{K_T} + K_b\right)D\theta_m + \frac{R_a K}{K_T}\theta_m$$

$$K_T v_1 = L_a J D^3\theta_m + (R_a J + L_a B)D^2\theta_m + (R_a B + L_a K + K_b K_T)D\theta_m + R_a K\theta_m$$

(b) Since $\omega_m(t) = D\theta_m(t)$

$$\frac{\theta_m(t)}{v_1(t)} = \frac{K_T}{L_a J D^3 + (R_a J + L_a B)D^2 + (R_a B + L_a K + K_b K_T)D + R_a K}$$

$$\frac{\omega_m(t)}{v_1(t)} = \frac{K_T D}{L_a J D^3 + (R_a J + L_a B)D^2 + (R_a B + L_a K + K_b K_T)D + R_a K}$$