Chapter 2

Sets and Logic

2.4 Try This! Problems on Sets and Logic

1. 4; $\{\emptyset, \{0\}, \{cat\}, \{\{dog\}\}, \{\{2.1,6\}\}, \{0, cat\}, \{0, \{dog\}\}, \{0, \{2.1,6\}\}, \{cat, \{dog\}\}, \{cat, \{2.1,6\}\}, \{\{dog\}, \{2.1,6\}\}, \{0, cat, \{dog\}\}, \{0, cat, \{2.1,6\}\}, \{0, \{dog\}, \{2.1,6\}\}, \{cat, \{dog\}, \{2.1,6\}\}, \{0, cat, \{dog\}, \{2.1,6\}\}\}$. ($2^4 = 16$.)

2. First logic-language it. For all times t, and for all people p, p can be fooled at time t. (There are certainly other possible interpretations.) The negation will be, *There exists a time t and a person p such that p cannot be fooled at time t.*

3. For example, $-16, 8, 12, \ldots$. These are the even integers.

4. (a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$; $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

(b)

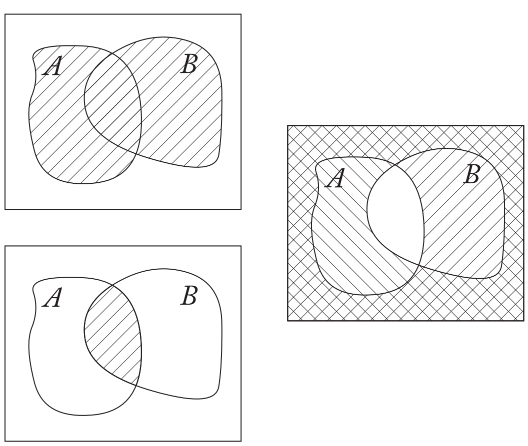
P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

P	Q	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(c)



(d) Let $x \in \overline{A \cup B}$. Then $x \notin (A \cup B)$, so $x \notin A$ and $x \notin B$, so $x \in \overline{A} \cap \overline{B}$. Similarly, let $x \in \overline{A} \cap \overline{B}$. Then $x \notin A$ and $x \notin B$, so $x \notin (A \cup B)$. Therefore, $x \in \overline{A \cup B}$. Now let $x \in \overline{A \cap B}$. Then $x \notin (A \cap B)$, so $x \notin A$ or $x \notin B$, so $x \in \overline{A} \cup \overline{B}$. Similarly let $x \in \overline{A} \cup \overline{B}$. Then $x \notin A$ or $x \notin B$, so $x \notin (A \cap B)$. Therefore $x \in \overline{A \cap B}$.

(e) Sure. $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$ and $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$.

(f) Yes; $\neg(P_1 \vee \cdots \vee P_n)$ is logically equivalent to $\neg P_1 \wedge \cdots \wedge \neg P_n$, and $\neg(P_1 \wedge \cdots \wedge P_n)$ is logically equivalent to $\neg P_1 \vee \cdots \vee \neg P_n$.

5. (a) $\{-10,-9,-8,-7,-6,-4,-2,0,2,4,6,7,8,9,10\}$.

(b) $\{-10,-9,-8,-7,7,8,9,10\}$.

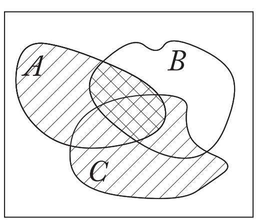
(c) A.

6. Check by truth table to see that the answer is "yes":

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

P	Q	$\neg Q$	$P \wedge \neg Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

7. $\bigcup_{i=1}^{n} A_i = A_n$; $\bigcap_{i=0}^{n} A_i = \{0\}$.

8. 

9. No; $n+5$ has a different value for each n, so such an m cannot have a single value.

2.6 Try This! A Tricky Conundrum

1. Answers will vary.

2. NA.

3. (a) Let *learn about sets* be S, let *learn about logic* be L, and let *go on to the next chapter* be C.

(b) $C \Rightarrow (S \vee L)$.

(c) $\neg S \wedge \neg C$; $(C \Rightarrow (S \vee L)) \wedge (\neg S \wedge \neg C) \Rightarrow \neg L$.

(d)

C	S	L	$S \vee L$	$C \Rightarrow (S \vee L)$	$\neg S \wedge \neg C$	$(C \Rightarrow (S \vee L)) \wedge (\neg S \wedge \neg C)$	$(C \Rightarrow (S \vee L)) \wedge (\neg S \wedge \neg C) \Rightarrow \neg L$
T	T	T	T	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	T	F	F	T
T	F	F	F	F	F	F	T
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	F
F	F	F	F	T	T	T	T

4. Answers will vary.

5. (a) Use the contrapositive. If n is odd, then $n = 2k+1$ and $n^3+6n^2-2n = (2k+1)^3 + 6(2k+1)^2 - 2(2k+1) = 2(4k^3+6k^2+3k+3(2k+1)^2-(2k+1))+1$, which is odd.
   (b) Use the contrapositive. Suppose $x < 0$. Then $x^5+7x^3+5x < 0$ and $x^4+x^2+8 > 0$, so $x^4+x^2+8 > x^5+7x^3+5x$.
   (c) Suppose for the sake of contradiction that it can be tiled with dominoes. Then it has an even number of squares. However, 65 is odd; contradiction.
   (d) Tile the right-most column with $\frac{n-1}{2}$ vertical dominoes. Then tile the left-most $n-1$ columns with $\frac{n-1}{2}$ columns of horizontal dominoes.

2.8 Bonus: Truth Tellers

Puzzle1: The simplest way to do this problem is by reasoning. Examine the three cases where Rachel, Tess, or Nicol is the liar. If Rachel is the liar, Tess and Nicol are truthful. However, Nicol and Tess contradict each other because Tess claims she was out of town whereas Nicol says she saw Tess in town. Thus, Rachel is not the liar. If Tess is the liar, Rachel and Nicol are truthful. Rachel's and Nicol's statements are not really related, so this situation is consistent. If Nicol is the liar, then Rachel and Tess are truthful. However, Rachel and Tess contradict each other because Rachel says Tess knows Amy whereas Tess says she doesn't know Amy. Therefore, Tess is the liar.

Puzzle2:

M	D	M true	D false	(M says) $\wedge$ (D says)
T	T	T	F	F
T	F	$\cancel{T}$ F	T	F
F	T	F	$\cancel{F}$ T	F
F	F	$\cancel{F}$ T	$\cancel{T}$ F	F

There is no way that the parental statements can be consistent.

Project:

1.

D	berries are safe	D says
T	T	T
T	F	F
F	T	F
F	F	T

If duck says *yes*, that could result from the berries being safe or unsafe; the same is true if the duck says *no*.

2. This will give no information about berries . . . .

3.

D	berries are safe	$D \wedge$ berries are safe
T	T	T
T	F	F
F	T	~~F~~ T
F	F	~~F~~ T

If duck says *yes*, that could result from the berries being safe or unsafe.

4.

D	berries are safe	$D \vee$ berries are safe
T	T	T
T	F	T
F	T	~~T~~ F
F	F	~~F~~ T

If duck says *yes*, that could result from the berries being safe or unsafe.

5.

D	berries are safe	$D \Rightarrow$ berries are safe
T	T	T
T	F	F
F	T	~~T~~ F
F	F	~~T~~ F

If duck says *no*, that could result from the berries being safe or unsafe. (If the duck says *yes*, we do learn that the berries are safe. However, we can't depend on that answer.)

6.

D	berries are safe	$D \Leftarrow$ berries are safe
T	T	T
T	F	T
F	T	~~F~~ T
F	F	~~T~~ F

If duck says *yes*, that could result from the berries being safe or unsafe.

7. Again, this tells us nothing about berries.

We need to design a question that will produce answers that are consistent with the safety of the berries. Here are four questions that work:

- Would a duck who answers oppositely from you say that the berries are edible?
- Do you tell the truth if and only if the berries are edible?
- Is it true that (if the berries are safe, then you tell the truth) and (if the berries are safe, then you lie)?

If I asked you whether the berries are safe, what would you say?

We give a truth table for this last question:

D	berries are safe	D says
T	T	T
T	F	F
F	T	~~F~~ T
F	F	~~T~~ F

2.9 Problems about Sets and Logic

1. $\{2\}$.

2. The question is asking about non-urgent care. It excludes dental care and hospital care and urgent care. The question could have been worded, "In the last 12 months, how often did you get a prompt appointment for non-urgent health care at a doctor's office or clinic?"

3. Here they are: (doctor or health provider), (discuss or provide), (methods and strategies), (smoking or using). All three of the ors are meant mathematically, but the and is used to mean union instead of intersection.

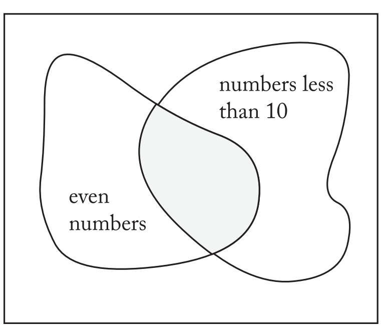
4. These are the negative integers $\{-9, \ldots, -1\}$, so the cardinality of the set is 9.

5.

P	Q	$\neg P \vee Q$	$P \wedge (\neg P \vee Q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

6. $\{2^n \mid n \in \mathbb{W}\}$ or $\{n \mid n = 2^k, k \in \mathbb{W}\}$.

7.



8. (a) $\{(2,2),(2,3),(3,2),(3,3)\}$.
   (b) $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(2,4),(3,4),(4,4),(4,3),(4,2)\}$.
   (c) $\{(1,1),(2,1),(3,1)\}$.

9.

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

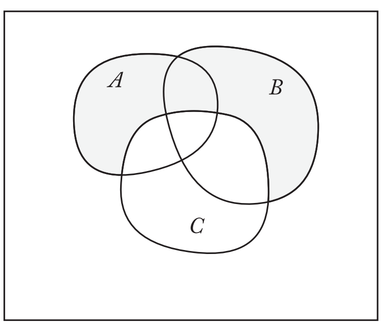
10.

P	Q	$\neg P$	$\neg Q$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

11. Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$.

12. This is total nonsense. The moon is not made of green cheese. Even if Moscow had existed as a city in the time of Aristotle, he certainly was not and is not its President. Every fiber of your being is probably screaming "False, False, FALSE!" But think again. Because the moon is not made of green cheese, it doesn't matter *what* the statement is claiming this implies; the implication holds anyway, because a false statement implies, well, everything. See the appropriate truth table.

13.



14. Let A, C be disjoint. For example, let $A = \{1,2\}, C = \{3\}, B = \{4\}, D = \{5\}$. Then $(A \times B) \cup (C \times D) = \{(1,4),(2,4),(3,5)\}$ but $(A \cup C) \times (B \cup D) = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$.

15. Use the contrapositive. Let $B = \emptyset$. Then $A \setminus B = A \setminus \emptyset = A$, which is not equal to $\emptyset$ unless $A = \emptyset$.

16. Use a truth table and trace backwards. We will have one value for R but two possibilities each for P, Q, S, so we have eight rows.

P	Q	S	R	$P \Rightarrow Q$	$R \wedge S$	$(P \Rightarrow Q) \Leftrightarrow (R \wedge S)$
T	T	T	F	T	F	F
T	T	F	F	T	F	F
T	F	T	F	F	F	T
T	F	F	F	F	F	T
F	T	T	F	T	F	F
F	T	F	F	T	F	F
F	F	T	F	T	F	F
F	F	F	F	T	F	F

We see that $(P \Rightarrow Q) \Leftrightarrow (R \wedge S)$ is true when P, S are true and Q is false, or when P is true and Q, S are false. Thus, we conclude that P is true and Q is false.

17. In math-logic language, this expression is $(c > 5 \wedge b = a) \vee c \geq 5$. The crucial point is to notice that $(c \geq 5)$ is logically equivalent to $(c > 5) \vee (c = 5)$. So let $c > 5$ be P, let $b = a$ be Q, and let $c = 5$ be R.

P	Q	R	$P \wedge Q$	$P \vee R$	$(P \wedge Q) \vee (P \vee R)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	F	F

Now notice that $P \vee R$ would have the same truth table. Thus, we can rewrite this as $c \geq 5$.

18. For every k that is a multiple of 3, there is a subset of the natural numbers with cardinality k. It's not true because k could be negative, but no set has negative cardinality. The negation is $\exists k \in 3\mathbb{Z}$, such that $\forall S \subseteq \mathbb{N}, |S| \neq k$. In English, that says there is a k evenly divisible by 3 such that no subset of $\mathbb{N}$ has cardinality k. And that's true, because we can take $k = -63$.

19. This is a biconditional, so do it in two parts.

    $(\Rightarrow)$ Direct proof. Suppose n is odd, so $n = 2k+1$. Then $n^2 = (2k+1)^2 = 2(2k^2+2k)+1$, which is odd.

    $(\Leftarrow)$ Use the contrapositive of "n^2 odd implies n odd." That is, prove that n even implies n^2 even, as follows: if n is even, then $n = 2k$, so $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even.

20. Here goes some double-inclusion provin': Consider $x \in \{3k \mid k \in \mathbb{Z}\}$; because $k \in \mathbb{Z}$, then $3k \in \mathbb{Z}$ so $x \in \mathbb{Z}$. Similarly, we see that for $y \in \{3k+1 \mid k \in \mathbb{Z}\}$, $y \in \mathbb{Z}$ and for $z \in \{3k+2 \mid k \in \mathbb{Z}\}$, $z \in \mathbb{Z}$. Now consider $\ell \in \mathbb{Z}$. If ℓ is divisible by 3, then $\ell \in \{3k \mid k \in \mathbb{Z}\}$. Otherwise, divide ℓ by 3; the remainder must be 1 or 2 (as if it were 0, ℓ would be divisible by 3). If the remainder is 1, then $\ell \in \{3k+1 \mid k \in \mathbb{Z}\}$, and if the remainder is 2, then $\ell \in \{3k+2 \mid k \in \mathbb{Z}\}$. Therefore, $\ell \in \{3k \mid k \in \mathbb{Z}\} \cup \{3k+1 \mid k \in \mathbb{Z}\} \cup \{3k+2 \mid k \in \mathbb{Z}\}$.

21. Suppose there are only finitely many primes. List these k primes, $p_1, p_2, \ldots, p_k$. Multiply them all together to get a number larger than any of the primes, and add 1. It's not divisible by any known prime (as otherwise 1 would also be divisible by that prime), so it is prime itself. This is a contradiction, because there are now $k+1$ primes.

22. The binary representation of a natural number is a sum of powers of two, so notice that n is divisible by 4 exactly when (binary representation of $n) = 2^2($ sum of some powers of 2). In other words, the binary representation must end in 00. The contrapositive of the original statement is $n \in \mathbb{N}$ *is divisible by 4 if and only if the binary representation of* n *ends in* 00.

23.

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

24. Example 1.5.4, Example 1.5.5.

25. In logic language, the statement says, *If you own a boat, then you wash it after each use.* If you don't own a boat, then you wash your non-boat every time you use it. The truth table for *implies* says that if the hypothesis is false, then the statement as a whole is true, which is the case here.