2 INTEGRAL TRANSFORMS

2.1 The Laplace Transform.

2.1.1 Definition and Properties of the Laplace Transform.

1. Let $\mathcal{L}\{f\} = F$.

(a) $f(t) = t$.

$$
\begin{aligned}
F(s) &= \int\limits_0^\infty e^{-st} f(t)\,dt = \int\limits_0^\infty te^{-st}\,dt \\
&= \frac{te^{-st}}{s}\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \\
&= \lim_{M\to\infty} \frac{Me^{-sM}}{s} - 0 + \frac{1}{s}\,\frac{-e^{-st}}{s}\Bigg|_{t=0}^\infty = -\frac{1}{s^2}\lim_{\to\infty}\frac{e^{-sM}}{s} + \frac{1}{s^2} \\
&= \frac{1}{s^2},\ s > 0.
\end{aligned}
$$

(b) $f(t) = e^{3t+1}$.

$$
\begin{aligned}
F(s) &= \int\limits_0^\infty e^{-st} f(t)\,dt = \int\limits_0^\infty e^{3t+1}e^{-st}\,dt = e\int\limits_0^\infty e^{(3-s)t}\,dt \\
&= e\frac{e^{(3-s)t}}{3-s}\Bigg|_0^\infty = e\Big[\lim_{M\to\infty}\frac{e^{(3-s)M}}{3-s} - \frac{1}{3-s}\Big] \\
&= \frac{e}{s-3},\ s > 3.
\end{aligned}
$$

(c) $f(t) = \cos 2t$.

$$
\begin{aligned}
F(s) &= \int\limits_0^\infty e^{-st} f(t)\,dt = \int\limits_0^\infty \cos 2t\, e^{-st}\,dt \\
&= \frac{e^{-st}(-s\cos 2t + 2\sin 2t)}{s^2+4}\Bigg|_0^\infty \\
&= \lim_{M\to\infty}\frac{e^{-sM}(-s\cos 2M + 2\sin 2M)}{s^2+4} + \frac{s}{s^2+4} \\
&= \frac{s}{s^2+4}.
\end{aligned}
$$

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(d) $f(t) = \sin^2 t$.

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \sin^2 t\, e^{-st}\,dt \\
&= \frac{1}{2}\int_0^\infty e^{-st}(1-\cos 2t)\,dt \\
&= \frac{1}{2}\int_0^\infty e^{-st}\,dt - \frac{1}{2}\int_0^\infty e^{-st}\cos 2t dt \\
&= -\frac{e^{-st}}{2s}\Big|_{t=0}^{\infty} - \frac{1}{2}e^{-st}\frac{-s\cos 2t + 2\sin 2t}{s^2+4}\Big|_{t=0}^{\infty} \\
&= \frac{1}{2s} - \frac{s}{2(s^2+4)} \\
&= \frac{2}{s(s^2+4)},\ s>0.
\end{aligned}
$$

(e) $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ 0, & t > 2. \end{cases}$

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 1\, e^{-st}\,dt \\
&= \frac{e^{-st}}{-s}\Big|_{t=0}^{2} = \frac{1-e^{-2s}}{s},\ s>0.
\end{aligned}
$$

(f) $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 3, & t > 2. \end{cases}$

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 t\, e^{-st}\,dt + \int_0^\infty e^{-st} f(t)\,dt \\
&= \left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^2 + \frac{3e^{-st}}{s}\Big|_2^{\infty} \\
&= \frac{-2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} + \frac{3e^{-2s}}{s} \\
&= \frac{e^{-2s}(s-1+e^{2s})}{s^2},\ s>0.
\end{aligned}
$$

(g) $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t > 1. \end{cases}$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_1^\infty e^{-st} f(t)\, dt$$

$$= \frac{e^{-st}}{-s}\Big|_{t=1}^{\infty} = \frac{e^{-s}}{s}, \; s > 0.$$

2. The Shifting property: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$.

(a) $g(t) = e^t \sin 2t$.

Since $(\mathcal{L}\{\sin 2t\})(s) = \dfrac{2}{s^2+4}$ using the shifting property, we have $(\mathcal{L}\{e^t \sin 2t\})(s) = \dfrac{2}{(s-1)^2+4}$.

(b) $g(t) = e^{-t} \cos 2t$.

Since $(\mathcal{L}\{\cos 2t\})(s) = \dfrac{s}{s^2+4}$ using the shifting property, we have $(\mathcal{L}\{e^{-t} \cos 2t\})(s) = \dfrac{s+1}{(s+1)^2+4}$.

(c) $g(t) = e^t \cos 3t$.

Since $(\mathcal{L}\{\cos 3t\})(s) = \dfrac{s}{s^2+9}$ using the shifting property, we have $(\mathcal{L}\{e^t \cos 3t\})(s) = \dfrac{s-1}{(s-1)^2+9}$.

(d) $g(t) = e^{-2t} \cos 4t$.

Since $(\mathcal{L}\{\cos 4t\})(s) = \dfrac{s}{s^2+16}$ using the shifting property, we have $(\mathcal{L}\{e^{-2t} \cos 4t\})(s) = \dfrac{s+2}{(s+2)^2+16}$.

(e) $g(t) = e^{-t} \sin 5t$.

Since $(\mathcal{L}\{\sin 5t\})(s) = \dfrac{5}{s^2+25}$ using the shifting property, we have

$$(\mathcal{L}\{e^{-t}\sin 5t\})(s) = \frac{5}{(s+1)^2+25}.$$

(f) $g(t) = e^t\, t$.

Since $(\mathcal{L}\{t\})(s) = \dfrac{1}{s^2}$ using the shifting property, we have

$$(\mathcal{L}\{e^t\, t\})(s) = \frac{1}{(s-1)^2}.$$

3. (a) $f(t) = -18e^{3t}$.

Using the linear property, we have

$$(\mathcal{L}\{f(t)\}(s)) = (\mathcal{L}\{-18e^{3t}\})(s) = -18(\mathcal{L}\{e^{3t}\})(s) = -18\frac{1}{s-3}.$$

(b) $f(t) = t^2e^{-3t}$.

Since $(\mathcal{L}\{t^2\})(s) = \dfrac{2!}{s^3}$ using the shifting property, we have

$$(\mathcal{L}\{f(t)\}(s)) = (\mathcal{L}\{e^{-3t}t^2\})(s) = \frac{2}{(s+3)^3}.$$

(c) $f(t) = 3t\sin 2t$.

Since $(\mathcal{L}\{\sin 2t\})(s) = \dfrac{2}{s^2+4}$ from Theorem 2.1.5 we have

$$(\mathcal{L}\{f(t)\}(s)) = 3(\mathcal{L}\{t\sin 2t\})(s) = -3\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{12}{(s^2+4)^2}.$$

(d) $f(t) = e^{-2t}\cos 7t$.

Since $(\mathcal{L}\{\cos 7t\})(s) = \dfrac{s}{s^2+49}$ using the shifting property, we have

$$(\mathcal{L}\{f(t)\}(s)) = (\mathcal{L}\{e^{-2t}\cos 7t\})(s) = \frac{s+2}{(s+2)^2+49}.$$

(e) $f(t) = 1 + \cos 2t$.

Using the linear property, we have

$$(\mathcal{L}\{1+\cos 2t\})(s) = (\mathcal{L}\{1\})(s) + (\mathcal{L}\{\cos 2t\})(s) = \frac{1}{s} + \frac{s}{s^2+4}.$$

(f) $f(t) = t^3 \sin 2t$.

Since $(\mathcal{L}\{\sin 2t\})(s) = \dfrac{2}{s^2+4}$ from Theorem 2.1.5 we have

$$(\mathcal{L}\{f(t)\}(s)) = (\mathcal{L}\{t^3 \sin 2t\})(s) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{2}{s^2+4}\right)$$

$$= 4\frac{d^2}{ds^2}\left(\frac{s}{(s^2+4)^2}\right) = 4\frac{d}{ds}\left(\frac{4-3s^2}{(s^2+4)^3}\right)$$

$$= 4\frac{(s^2+4)^3(-6s) - (4-3s^2)3(s^2+)^2 2s}{(s^2+4)^6}$$

$$= 48\frac{s(s^2-4)}{s^2+4)^4}.$$

4. (a) $f(t) = t^{-\frac{1}{2}}$.

$$F(s) = (\mathcal{L}f(t))(s) = \int_0^\infty t^{-\frac{1}{2}} e^{-st}\, dt$$

using the substitution $st = u$, we get

$$F(s) = \sqrt{s}\int_0^\infty u^{-\frac{1}{2}} e^{u} \frac{1}{s}\, du = \frac{1}{\sqrt{s}}\int_0^\infty u^{\frac{1}{2}} e^{-u}\, du$$

$$= \frac{1}{\sqrt{s}}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{\sqrt{s}}.$$

(b) $f(t) = t^{\frac{1}{2}}$

$$F(s) = (\mathcal{L}f(t))(s) = \int_0^\infty t^{\frac{1}{2}} e^{-st}\, dt$$

using the substitution $st = u$, we get

$$F(s) = \frac{1}{\sqrt{s}}\int_0^\infty u^{\frac{1}{2}} e^{-u} \frac{1}{s}\, du = \frac{1}{\sqrt{s^3}}\int_0^\infty u^{\frac{3}{2}-1} e^{-u}\, du = \frac{\Gamma(\frac{3}{2})}{\sqrt{s^3}} = \frac{1}{2}\frac{\Gamma(\frac{1}{2})}{\sqrt{s^3}}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{s^3}}.$$

5. (a) $F(s) = \dfrac{1}{s^6}$.

$f(t) = (\mathcal{L}^{-1}\{F(s)\})(t) = (\mathcal{L}^{-1}\{\dfrac{1}{s^6}\})(t) = \dfrac{t^5}{5!}$

$= \dfrac{t^5}{120}$.

(b) $F(s) = \dfrac{1}{s^2 - 4s + 4}$.

$f(t) = (\mathcal{L}^{-1}\{F(s)\})(t) = (\mathcal{L}^{-1}\{\dfrac{1}{s^2 - 4s + 4}\})(t)$

$= (\mathcal{L}^{-1}\{\dfrac{1}{(s-2)^2}\})(t)$.

By the shifting property we have

$f(t) = e^{2t}t$.

(c) $F(s) = \dfrac{1}{(s^2+9)^2}$.

From Theorem 2.1.5 and the Table for Laplace transforms we have

$(\mathcal{L}\{t\cos 3t\})(s) = -\dfrac{d}{ds}\Big(\dfrac{s}{s^2+9}\Big) = \dfrac{s^2-9}{(s^2+9)^2}$

$(\mathcal{L}\{\sin 3t\})(s) = \dfrac{9}{s^2+9}$.

Consider the decomposition

$$\frac{1}{(s^2+9)^2} = A\frac{s^2-9}{(s^2+9)^2} + B\frac{9}{s^2+9}.$$

From the above decomposition we find $A = -\dfrac{1}{18}$ and $B = \dfrac{1}{54}$.

Therefore

$f(t) = (\mathcal{L}^{-1}\{F(s)\})(t) = \Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{(s^2+9)^2}\Big\}\Big)(t)$

$= -\dfrac{1}{18}\Big(\mathcal{L}^{-1}\Big\{\dfrac{s^2-9}{(s^2+9)^2}\Big\}\Big)(t) + \dfrac{1}{54}\Big(\mathcal{L}^{-1}\Big\{\dfrac{9}{s^2+9}\Big\}\Big)(t)$

$= \dfrac{1}{18}t\cos 3t + \dfrac{1}{54}\sin 3t$.

(d) $F(s) = \dfrac{1}{s^2 - 6s + 9}$.

$f(t) = (\mathcal{L}^{-1}\{F(s)\})(t) = (\mathcal{L}^{-1}\{\dfrac{1}{s^2 - 4s + 4}\})(t)$

$= (\mathcal{L}^{-1}\{\dfrac{1}{(s-3)^2}\})(t)$.

By the shifting property we have

$f(t) = e^{3t}t$.

(e) $F(s) = \dfrac{1}{s^2 + 15s + 56}$.

$f(t) = \Big(\mathcal{L}^{-1}\{F(s)\}\Big)(t) = \Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{s^2 + 15s + 56}\Big\}\Big)(t)$

$= \Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{(s+8)(s+7)}\Big\}\Big)(t)$

$= \Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{s+7)}\Big\}\Big)(t) - \Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{s+8}\Big\}\Big)(t)$.

By the shifting property we have

$f(t) = e^{-7t} - e^{-8t}$.

(f) $F(s) = \dfrac{1}{s^2 + 16s + 36}$.

$f(t) = \Big(\mathcal{L}^{-1}\{F(s)\}\Big)(t) = \Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{s^2 + 16s + 36}\Big\}\Big)(t)$

$= \Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{(s+8+2\sqrt{7})(s+8-2\sqrt{8})}\Big\}\Big)(t)$

$= \dfrac{1}{4\sqrt{7}}\Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{s+8+2\sqrt{7}}\Big\}\Big)(t) - \dfrac{1}{4\sqrt{7}}\Big(\mathcal{L}^{-1}\Big\{\dfrac{1}{s+8-2\sqrt{7}}\Big\}\Big)(t)$.

By the shifting property we have

$f(t) = \dfrac{1}{4\sqrt{7}}\left(e^{-(8+2\sqrt{7})t} - e^{-(8-2\sqrt{7})t}\right)$.

(g) $F(s) = \dfrac{2s - 7}{2s^2 - 14s + 55}$.

We write $F(s)$ as

$$F(s) = \frac{s - \frac{7}{2}}{(s - \frac{7}{2})^2 + 61}.$$

By the shifting property we have

$$f(t) = e^{\frac{7}{2}t} \cos(\sqrt{61}\, t).$$

(h) $F(s) = \dfrac{s+2}{s^2 + 4s + 12}$.

We write $F(s)$ as

$$F(s) = \frac{s+2}{(s+2)^2 + 8}.$$

By the shifting property we have

$$f(t) = e^{-2t} \cos(\sqrt{8}\, t).$$

6. (a) $F(s) = \dfrac{2s-5}{(s-1)^3(s^2+4)}$

Decompose $F(s)$ into partial fractions as follows

$$\frac{2s-5}{(s-1)^3(s^2+4)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{Ds+E}{s^2+4}.$$

By comparison , we find that the coefficients are

$$A = -\frac{17}{125}, B = \frac{16}{25}, C = -\frac{3}{5}, D = \tfrac{17}{125}, E = -\frac{63}{125}.$$

Using shifting property and the Laplace transform of 1, t, t^2,

sin $2t$, and cos $2t$, we have

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{17e^t}{125} + \frac{16e^t t}{25} - \frac{3e^t t^2}{10} + \frac{17\cos\, 2t}{125} - \frac{63\sin 2t}{250}$$

$$= \frac{1}{250}(-34e^t + 160te^t - 75t^2e^t + 34\cos 2t - 63\sin 2t).$$

(b) $F(s) = \dfrac{3s-5}{(s^2(s^2+9)(s^2+1)}$.

Decompose $F(s)$ into partial fractions as follows

$$\frac{3s-5}{(s^2(s^2+9)(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9} + \frac{Es+F}{s^2+1}.$$

By comparison, we find that the coefficients are

$A = \frac{1}{3}$, $B = -\frac{5}{9}$, $C = \frac{1}{24}$, $D = -\frac{5}{72}$, $E = -\frac{3}{8}$, $F = \frac{5}{8}$.

Using shifting property and the Laplace transform of 1, t, $\sin 3t$,

and $\cos 3t$, we have

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{3} - \frac{5t}{9} + \frac{\cos 3t}{24} - \frac{5\sin 3t}{216} - \frac{3\cos t}{8} + \frac{5\sin t}{8}$$
$$= \frac{1}{216}(72 - 120t + 9\cos 3t - 5\sin 3t - 81\cos t + 135\sin t).$$

7. (a) $F(s) = \frac{s}{4-s}$.

   Since $\lim_{s\to\infty} F(s) = -1 \neq 0$, by Corollary 2.1.1 there isn't a piecewise function $f(t)$ such that $\mathcal{L}\{f\} = F$.

   (b) $F(s) = \frac{s^2}{(s-2)^2}$.

   Since $\lim_{s\to\infty} F(s) = 1 \neq 0$, by Corollary 2.1.1 there isn't a piecewise function $f(t)$ such that $\mathcal{L}\{f\} = F$.

   (a) $F(s) = \frac{s^2}{s^2+9}$

   Since $\lim_{s\to\infty} F(s) = 1 \neq 0$, by Corollary 2.1.1 there isn't a piecewise function $f(t)$ such that $\mathcal{L}\{f\} = F$.

8. (a) $f(t) = t^n$, $n \in \mathbb{N}$.

   From McLauren's expansion for e^t we have

   $t^n \leq n!\, e^t$ for every $t \geq 0$.

9. (b) For $n \geq 2$ and any $c \in \mathbb{R}$ the function t^n is not bounded on any interval $[T_0, \infty)$.

10. Consider the function $F(z) = e^{-a\sqrt{z}}$. This function is analytic on the whole complex plane except on the negative real axis (the function has a branch point at the origin and the branch cut along the negative real axis). From the Laplace Inversion Formula (2.1.3) and Cauchy Residue Theorem we have

$$f(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(z) e^{zt}\, dz = \frac{1}{2\pi i} \int_{\gamma_1} F(z) e^{zt}\, dz + \frac{1}{2\pi i} \int_{\gamma_2} e^{-a\sqrt{z}} e^{zt}\, dz,$$

where γ_1 is the upper branch cut oriented from $-\infty$ to 0 and γ_2 is the lower branch cut oriented from 0 to $-\infty$. On γ_1 we have $z = xe^{i\pi} = -x$ and so $\sqrt{z} = i\sqrt{x}$ and $dz = -dx$. On the lower branch cut γ_2 have $z = xe^{-i\pi} = -x$ and so $\sqrt{z} = -i\sqrt{x}$ and $dz = -dx$. Thus

$$\frac{1}{2\pi i} \int_{\gamma_1} e^{-a\sqrt{z}} e^{zt}\, dz = \frac{1}{2\pi i} \int_{\infty}^{0} e^{-a\sqrt{x}} e^{-xt}\, dx = -\frac{1}{2\pi i} \int_{0}^{\infty} e^{-ai\sqrt{x}} e^{-xt}\, dx$$

$$\frac{1}{2\pi i} \int_{\gamma_2} e^{-a\sqrt{z}} e^{zt}\, dz = -\frac{1}{2\pi i} \int_{\infty}^{0} e^{ai\sqrt{x}} e^{-xt}\, dx = \frac{1}{2\pi i} \int_{0}^{\infty} e^{ai\sqrt{x}} e^{-xt}\, dx.$$

Therefore

$$f(t) = \frac{1}{2\pi i} \int_{0}^{\infty} e^{ai\sqrt{x}} e^{-xt}\, dx - \frac{1}{2\pi i} \int_{0}^{\infty} e^{-ai\sqrt{x}} e^{-xt}\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\infty} e^{-xt} \sin(a\sqrt{x})\, dx.$$

Integration by parts the above integral gives

$$f(t) = \frac{1}{2at\pi} \int_{0}^{\infty} \frac{\cos(a\sqrt{x})}{\sqrt{x}}\, e^{-xt}\, dx$$

Let

$$I(a) = \int_{0}^{\infty} \frac{\cos(a\sqrt{x})}{\sqrt{x}}\, e^{-xt}\, dx.$$

If we differentiate the last integral with respect to a and use the integration by parts formula we obtain

$$I'(a) = -\int_{0}^{\infty} \sin(a\sqrt{x})\, e^{-xt}\, dx = \frac{1}{t} \int_{0}^{\infty} \sin(a\sqrt{x})\, d\big(e^{-xt}\big)$$

$$= -\frac{a}{2t} \frac{\cos(a\sqrt{x})}{\sqrt{x}}\, e^{-xt}\, dx = -\frac{a}{2t} I(a).$$

We have the following differential equation

$I'(a) = -\dfrac{a}{2t} I(a).$

A general solution of the above equation is given by

$I(a) = C\, exp\Big(-\frac{a^2}{4t}\Big).$

$C = I(0) = \displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\, e^{-xt}\, dx = \frac{1}{\sqrt{t}} \int_0^\infty e^{-u^2}\, du = \frac{1}{\sqrt{t}}\sqrt{\pi}.$

Therefore

$I(a) = \dfrac{1}{\sqrt{t}}\sqrt{\pi}\; exp\Big(-\frac{a^2}{4t}\Big)$ and so

$f(t) = \dfrac{a}{2\pi\sqrt{t^3}}\; exp\Big(-\frac{a^2}{4t}\Big).$

2.1.2 Step and Impulse Functions.

2. Let $F(s) = \mathcal{L}\{f\}$.

(a) $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2. \end{cases}$

Notice that this function can be written in the following form

$f(t) = H(t-2).$

Using the translation property in Theorem 1.2.10 we have

$F(s) = \dfrac{e^{-2s}}{s}.$

(b) $f(t) = \begin{cases} t, & t < 1 \\ t^2 - 2t + 2, & t \geq 1. \end{cases}$

Notice that this function can be written in the following form

$f(t) = t + (t^2 - 3t + 2)H(t-1) = t + (t-1)^2 H(t-1)$

$-(t-1)H(t-1).$

Using the translation property in Theorem 1.2.10 we have

$F(s) = \dfrac{1}{s^2} + \dfrac{e^{-s}}{s^3} - \dfrac{e^{-s}}{s^2}.$

(c) $f(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1. \end{cases}$

Notice that this function can be written in the following form

$f(t) = H(t-1).$

Using the translation property in Theorem 1.2.10 we have

$F(s) = \dfrac{e^{-s}}{s}.$

(d) $f(t) = \begin{cases} 0, & t < \pi \\ t-\pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

Notice that this function can be written in the following form

$f(t) = (t-\pi)H(t-\pi) + \big[0-(t-\pi)\big]H(t-2\pi) = (t-\pi)H(t-\pi)$

$-(t-2\pi)H(t-2\pi) - H(t-2\pi).$

Using the translation property in Theorem 1.2.10 we have

$F(s) = \dfrac{e^{-s\pi}}{s^2} - \dfrac{e^{-2s\pi}}{s^2} - \dfrac{e^{-2s\pi}}{s}.$

(e) $f(t) = H(t-1) + 2H(t-3) - H(t-4).$

Using the translation property in Theorem 1.2.10 we have

$F(s) = \dfrac{e^{-s}}{s} + 2\dfrac{e^{-3s}}{s} - \dfrac{e^{-4s\pi}}{s}.$

(f) $f(t) = (t-3)H(t-3) - (t-2)H(t-2).$

Using the translation property in Theorem 1.2.10 we have

$F(s) = \dfrac{e^{-3s}}{s} - 3\dfrac{e^{-2s\pi}}{s}.$

(g) $f(t) = t - (t-1)H(t-1).$

Using the translation property in Theorem 1.2.10 we have

$F(s) = \dfrac{1}{s^2} - \dfrac{e^{-s}}{s^2}.$

3. Let $f = \mathcal{L}^{-1}\{F\}.$

(a) $F(s) = \dfrac{3!}{(s-2)^4}$.

Using the shifting property we have

$f(t) = e^{2t}t^3$.

(b) $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$.

First write $F(s)$ as

$$F(s) = -\frac{1}{3}e^{-2s}\frac{1}{s+2} + \frac{1}{3}e^{-2s}\frac{1}{s-1}.$$

Using the translation property in Theorem 1.2.10 we have

$$f(t) = \frac{1}{3}\Big[e^{-2(t-2)}H(t-2) + e^{t-2}H(t-2)\Big].$$

(c) $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$. First write $F(s)$ as

$$F(s) = e^{-2s}\,\frac{2(s-1)}{(s-1)^2+1}.$$

By the translation property of Laplace transform we have

$$\frac{2(s-1)}{(s-1)^2+1} = 2\mathcal{L}^{-1}\Big(e^t \sin t\Big).$$

Now, using the translation property in Theorem 1.2.10 we have

$$f(t) = \Big(2e^{t-2}\sin(t-2)\Big)\,H(t-2).$$

(d) $F(s) = \dfrac{2e^{-2s}}{s^2-4}$. First write $F(s)$ as

$$F(s) = \frac{1}{2}e^{-2s}\,\frac{1}{s-2} - \frac{1}{2}e^{-2s}\,\frac{1}{s+2}.$$

By the translation property of Laplace transform we have

$$\frac{1}{s-2} = \mathcal{L}^{-1}\Big(e^{2t}\Big) \text{ and } \frac{1}{s+2} = \mathcal{L}^{-1}\Big(e^{-2t}\Big).$$

Now, using the translation property in Theorem 1.2.10 we have

$$f(t) = \frac{1}{2}e^{2(t-2)}\,H(t-2) - \frac{1}{2}e^{-2(t-2)}\,H(t-2).$$

4. Let $F(s)$ be Laplace transforms of the given functions.

(a) $\delta(t-\pi)$.

By Formula (2.1.14) we have

$F(s) = e^{-\pi s}$.

(c) $2\delta(t-3) - \delta(t-1)$.

By Formula (2.1.14) we have

$F(s) = e^{-3s} - e^{-s}$.

(d) $\sin\, t\, u_{2\pi}(t) - \delta\big(t - \frac{\pi}{2}\big)$.

Since $\sin\, t$ is 2π-periodic we have

$\sin\, t\, u_{2\pi}(t) - \delta\big(t - \frac{\pi}{2}\big) = \sin\,(t - 2\pi)\, u_{2\pi}(t) - \delta\big(t - \frac{\pi}{2}\big)$.

Therefore, by the translation property in Theorem 2.1.10 and Formula (2.1.14) we have

$$F(s) = e^{-2\pi s}\, \frac{1}{s^2+1}\, H(t-2\pi) - \exp\Big(\big(-\tfrac{\pi}{2}s\big).$$

(e) $\cos\, 2t\, u_\pi(t) + \delta(t-\pi)$.

From $\cos\, 2t = \cos\, 2(t-\pi)$ we have newline
$\cos\, 2t\, u_\pi(t) + \delta(t-\pi) = \cos\, 2t = \cos\, 2(t-\pi)\, u_\pi(t) + \delta(t-\pi)$.

Therefore, by the translation property in Theorem 2.1.10 and Formula (2.1.14) we have

$$F(s) = e^{-\pi s}\, \frac{s}{s^2+4}\, H(t-\pi) + e^{-\pi s}.$$

2.1.3 Initial–Value Problems and the Laplace Transform.

1. Let $Y(s) = \mathcal{L}\{y(t)\}$.

(a) $y' + y = e^{-2t},\ y(0) = 2$.

Taking Laplace transform from both sides of the differential equation and use Corollary 2.1.2 we have

$$sY(s) - 2 + Y(s) = \frac{1}{s+2}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = \frac{1}{(s+2)(s+1)} + \frac{2}{s+1} = \frac{3}{s+1} - \frac{1}{s+2}.$$

Taking thew inverse Laplace transform we have

$$y(t) = 3e^{-t} - e^{-2t}.$$

(b) $y' + 7y = H(t-2),\ y(0) = 3.$

Taking Laplace transform from both sides of the differential equation and use Corollary 2.1.2 we have

$$sY(s) - 3 + 7Y(s) = e^{-2s}\frac{1}{s}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = e^{-2s}\frac{1}{s(s+7)} + \frac{3}{s+7}$$
$$= \frac{1}{7}e^{-2s}\,\frac{1}{s} - \frac{1}{7}e^{-2s}\,\frac{1}{s+7} + \frac{3}{s+7}.$$

Taking thew inverse Laplace transform we have

$$y(t) = \frac{1}{7}\,H(t-2) - \frac{1}{7}\,e^{-7(t-2)}\,H(t-2) + 3e^{-7t}.$$

(c) $y' + 7y = H(t-2)e^{-2(t-2)},\ y(0) = 1.$

Taking Laplace transform from both sides of the differential equation and use Corollary 2.1.2 we have

$$sY(s) - 1 + 7Y(s) = e^{-2s}\frac{1}{s+2}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = e^{-2s}\frac{1}{(s+2)(s+7)} + \frac{1}{s+7}$$
$$= \frac{1}{5}e^{-2s}\,\frac{1}{s+2} - \frac{1}{5}e^{-2s}\,\frac{1}{s+7} + \frac{3}{s+7}.$$

Taking the inverse Laplace transform we have

$$y(t) = \frac{1}{5}\,e^{-2(t-2)}\,H(t-2) - \frac{1}{5}\,e^{-7(t-2)}\,H(t-2) + 3e^{-7t}.$$

2. Let $Y(s) = \mathcal{L}\{y(t)\}$.

(a) $y'' + y' + 7y = \sin 3t,\ y(0) = 2,\ y'(0) = 0.$

Taking Laplace transform from both sides of the differential equation and use Corollary 2.1.2 we have

$$s^2Y(s) - 2s + sY(s) - 2 + 7Y(s) = \frac{3}{s^2+9}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = \frac{2s+2}{s^2+s+7} + \frac{3}{(s^2+s+7)(s^2+9)}.$$

Decomposing into partial fractions we have

$$Y(s) = \frac{-3(s+2)}{13(s^2+9)} + \frac{3(s+3)}{13(s^2+s+7)}$$
$$= -\frac{3}{13}\frac{s}{s^2+9} - \frac{6}{13}\frac{1}{s^2+9}$$
$$+\frac{16}{13}\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{27}{4}} + \frac{27}{13}\frac{1}{(s+\frac{1}{2})^2+\frac{27}{4}}.$$

Taking inverse Laplace transform we have

$$y(t) = -\frac{3}{13}\cos 3t - \frac{2}{13}\sin 3t + \frac{16}{13}e^{-\frac{1}{2}t}\cos\left(\tfrac{3\sqrt{3}t}{2}\right)$$
$$+\frac{18}{13\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\tfrac{3\sqrt{3}t}{2}\right).$$

(b) $y'' + 3y = H(t-4)\cos 5(t-4),\ y(0) = 0,\ y'(0) = -2.$

Taking Laplace transform from both sides of the differential equation and use Corollary 2.1.2 we have

$$s^2Y(s) + 2 + 3Y(s) = e^{-4s}\frac{5}{s^2+25}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = -\frac{2}{s^2+3} + e^{-4s}\frac{3}{(s^2+3)(s^2+25)}.$$

Decomposing into partial fractions we have

$$Y(s) = e^{-4s}\frac{s}{s^2+3} - e^{-4s}\frac{s}{s^2+25} - \frac{2}{s^2+3}.$$

Taking inverse Laplace transform we have

$$y(t) = \frac{1}{22}\Big[\cos\big(\sqrt{3}(t-4)\big) - \cos 5(t-4)\Big]H(t-4) - \frac{2}{\sqrt{3}}\sin\big(\sqrt{3}t\big).$$

(c) $y'' + 9y = H(t-5)\sin 3(t-5),\ y(0) = 2,\ y'(0) = 0.$

Taking Laplace transform from both sides of the differential equation and use Corollary 2.1.2 we have

$$s^2Y(s) - 2s + 9Y(s) = e^{-5s}\frac{3}{s^2+9}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = \frac{2s}{s^2+9} + e^{-5s}\frac{3}{(s^2+9)^2}.$$

Taking inverse Laplace transform, together with the initial conditions, we have

$y(t) = 2\cos 3t + H(t-5)f(t-5)$, where

$f(t) = \mathcal{L}^{-1}\{\dfrac{1}{(s^2+9)^2}\}.$

To find $\mathcal{L}^{-1}\{\dfrac{1}{(s^2+9)^2}\}$ we use the formula

$\mathcal{L}\{tf(t)\} = \dfrac{d}{ds}\big(\mathcal{L}\{f\}\big)(s).$

$\mathcal{L}\{t\cos 3t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+9}\right) = \dfrac{s^2-9}{(s^2+9)^2}.$

$$\mathcal{L}^{-1}\{\frac{1}{(s^2+9)^2}\} = \frac{1}{9}\mathcal{L}^{-1}\{\frac{1}{s^2+9}\} - \frac{1}{9}\mathcal{L}^{-1}\{\frac{s^2-9}{(s^2+9)^2}\}$$
$$= \frac{1}{9^2}\sin 3t - \frac{1}{9}t\cos 3t.$$

Therefore

$$y(t) = 2\cos 3t + \left[\frac{1}{81}\sin 3(t-5) - \frac{1}{9}(t-5)\cos 3(t-5)\right]H(t-5).$$

(d) $y^{iv} - y = H(t-1) - H(t-2)$, $y(0) = y'(0) = y''(0) = y'''(0) = 0$

Taking inverse Laplace transform, together with the initial conditions, we have

$(s^4-1)Y(s) = \dfrac{(e^{-s}-e^{-2s})}{s}$ and so

$$Y(s) = e^{-s}\frac{1}{s(s^4-1)} - e^{2-s}\frac{1}{s(s^4-1)}$$
$$= e^{-s}\frac{1}{s(s-1)(s+1)(s^2+1)} - e^{-2s}\frac{1}{s(s-1)(s+1)(s^2+1)}.$$

By partial fraction decomposition we have

$$e^{-s}\frac{1}{s(s-1)(s+1)(s^2+1)} - e^{-2s}\frac{1}{s(s-1)(s+1)(s^2+1)}$$
$$= -e^{-s}\frac{1}{s} + \frac{1}{4}\,e^{-s}\,\frac{1}{s-1} + \frac{1}{4}\,e^{-s}\frac{1}{s+1} + \frac{1}{2}\,e^{-s}\,\frac{1}{s^2+1}$$
$$= e^{-2s}\,\frac{1}{s} - \frac{1}{4}\,e^{-2s}\,\frac{1}{s-1} - \frac{1}{4}\,e^{-2s}\,\frac{1}{s+1} - e^{-2s} - \frac{1}{2}\,e^{-2s}\,\frac{1}{s^2+1}$$

Taking inverse Laplace transform gives

$$f(t) = \big[1 + \frac{1}{4}\,e^{t-1} + \frac{1}{4}\,e^{-(t-1)} + \frac{1}{2}\,\sin{(t-1)}\big]\,H(t-1)$$
$$-\big[1 + \frac{1}{4}\,e^{t-2} + \frac{1}{4}\,e^{-(t-2)} + \frac{1}{2}\,\sin{(t-2)}\big]\,H(t-2).$$

(e) $y'' + 3y = w(t)$, $y(0) = 0$, $\; y'(0) = -2$, $w(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & t \ge 1. \end{cases}$

Taking Laplace transform of both sides of the equation and using the given initial conditions we have

$$s^2Y(s) + 2 + 3Y(s) = \mathcal{L}\{w\}.$$

From $w(t) = t - (t-1)H(t-1)$ it follows

$$\mathcal{L}\{w\} = \frac{1}{s^2} - e^{-s}\,\frac{1}{s^2}.$$

Therefore

$$(s^2+3)Y(s) = -2 + \frac{1}{s^2} - e^{-s}\,\frac{1}{s^2}.$$

Solving for $Y(s)$ we obtain

$$Y(s) = -2\frac{1}{s^2+3} - e^{-s}\,\frac{1}{s^2(s^2+3)}.$$

By decomposition we have

$$Y(s) = -2\frac{1}{s^2+3} + \frac{1}{3}\,e^{-s}\,\frac{1}{s^2} - \frac{1}{3}\,e^{-s}\,\frac{1}{s^2+3}.$$

Taking inverse Laplace transform we have

$$y(t) = -\frac{2}{\sqrt{3}}\,\sin\,\big(\sqrt{3}\,t\big) + \frac{1}{3}\,(t-1)H(t-1) - \frac{1}{\sqrt{3}}\,\sin\,\big(\sqrt{3}\,(t-1)\big).$$

3. Let $F = \mathcal{L}\{f\}$. $f(t+T) = f(t)$ for every t since f is T-periodic. Then

$$F(s) = \int_0^\infty e^{-st}\,f(t)\,dt = \int_0^T]\,e^{-st}\,f(t)\,dt + \int_T^\infty e^{-st}\,f(t)\,dt$$
$$= \int_0^T e^{-st}\,f(t)\,dt + \int_T^\infty e^{-st}\,f(t+T)\,dt$$
$$= \int_0^T e^{-st}\,f(t)\,dt + \int_0^\infty e^{-s(u-T)}\,f(u)\,du$$
$$= \int_0^T e^{-st}\,f(t)\,dt + e^{sT}\int_0^\infty e^{-su}\,f(u)\,du$$
$$= \int_0^T e^{-st}\,f(t)\,dt + e^{sT}\,F(s).$$

Therefore, $\big(1-e^{sT}\big)F(s) = \int_0^T e^{-st}\,f(t)\,dt$

from where the result follows. newline

4. Let $W(s) = \mathcal{L}\{w\}$, where $w(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \end{cases}$ is 2-periodic. By the previous exercise we have

$$W(s) = \frac{1}{1-e^{2s}}\int_0^2 e^{-st}\,w(t)\,dt$$
$$= \frac{1}{1-e^{2s}}\int_0^1 e^{-st}\,dt - \frac{1}{1-e^{2s}}\int_1^2 e^{-st}\,dt$$
$$= \frac{2}{2}\,\tanh\,\frac{s}{2}.$$

5. Let $Y(s) = \mathcal{L}\{y(t)\}$.

(a) $y'' - 2y' + y = 3\delta_2(t),\ y(0) = 0,\ y'(0) = 1.$

Taking Laplace transform from both sides of the differential equation and use Corollary 2.1.2 we have

$$s^2Y(s) - 2 - 2sY(s) + Y(s) = 3e^{-2s}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = \frac{2}{s^2 - 2s + 1} + 3e^{-2s}\,\frac{1}{s^2 - 2s + 1} = \frac{1}{(s-1)^2} + 3e^{-2s}\,\frac{1}{(s-1)^2}.$$

Taking inverse Laplace transform we have

$$y(t) = te^t + 33(t-2)e^{t-2}H(t-2).$$

(b) $y'' + 2y' + 6y = 3\delta_2(t) - 4\delta_5(t),\ y(0) = 0,\ y'(0) = 1.$

Taking Laplace transform from both sides of the differential equation and use the given initial conditions we have

$$s^2Y(s) - 1 + 2sY(s) + 6Y(s) = 3e^{-2s} - 4e^{-5s}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = \frac{1}{s^2 + 2s + 6} + 3e^{-2s}\,\frac{1}{s^2 + 2s + 6} - 4e^{-5s}\,\frac{1}{s^2 + 2s + 6}$$
$$= \frac{1}{(s+1)^2 + 5} + 3e^{-2s}\,\frac{1}{(s+1)^2 + 5} - 4e^{-5s}\,\frac{1}{(s+1)^2 + 5}.$$

Taking inverse Laplace transform we have

$$y(t) = \frac{1}{\sqrt{5}}\,e^{-t}\sin\left(\sqrt{5}\,t\right) + \frac{3}{\sqrt{5}}\,e^{-(t-2)}\sin\left(\sqrt{5}\,(t-2)\right)H(t-2) - \frac{4}{\sqrt{5}}\,e^{-(t-5)}\sin\left(\sqrt{5}\,(t-5)\right)H(t-5).$$

(c) $y'' + 2y' + y = \delta_\pi,\ y(0) = 1,\ y'(0) = 0.$

Taking Laplace transform from both sides of the differential equation and use the given initial conditions we have

$$s^2Y(s) - s + 2sY(s) - 2 + Y(s) = e^{-\pi s}.$$

Solving the above equation for $Y(s)$ we obtain

$$Y(s) = e^{-\pi s}\,\frac{1}{(s+1)^2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}.$$

Taking inverse Laplace transform we have

$$y(t) = (t-\pi)e^{-(t-\pi)}H(t-\pi) + e^{-t} + te^{-t}.$$

(d) $y'' + 2y' + 3y = \sin t + \delta_\pi,\ y(0) = 0,\ y'(0) = 1.$

Taking Laplace transform, together with the initial conditions, we have

$$s^2Y(s) - 1 + 2sY(s) - 2 + 3Y(s) = \frac{1}{s^2+1} + e^{-\pi s}.$$

Solving for $Y(s)$ we obtain

$$Y(s) = \frac{3}{(s^2+1)(s^2+2s+3)} + e^{-\pi s}\,\frac{1}{s^2+2s+3}.$$

By partial fraction decomposition we have

$$\begin{aligned}
Y(s) &= \frac{3}{s^2+2s+3} + \frac{3}{4}\,\frac{1}{s^2+1} - \frac{3}{4}\,\frac{s}{s^2+1} \\
&+\frac{3}{4}\,e^{-\pi s}\,\frac{1}{s^2+2s+3} \\
&= 3\frac{1}{(s+1)^2+2} + \frac{3}{4}\,\frac{1}{s^2+1} - \frac{3}{4}\,\frac{s}{s^2+1} \\
&+\frac{3}{4}\,e^{-\pi s}\,\frac{1}{(s+1)^2+2}.
\end{aligned}$$

Taking inverse Laplace transform we obtain

$$\begin{aligned}
y(t) &= \frac{3}{\sqrt{2}}\,e^{-t}\,\sin\left(\sqrt{2}\,t\right) + \frac{3}{4}\,\sin t - \frac{3}{4}\,\cos t \\
&+\frac{3}{4\sqrt{2}}\,e^{-(t-\pi)}\,\sin\left(\sqrt{2}\,(t-\pi)\right)H(t-\pi).
\end{aligned}$$

(e) $y'' + 4y = \delta_\pi(t) - \delta_{2\pi},\ y(0) = 0,\ y'(0) = 0.$

Taking Laplace transform, together with the initial conditions, we have

$$s^2Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Solving for $Y(s)$ we obtain

$$Y(s) = e^{-\pi s}\,\frac{1}{s^2+4} - e^{-2\pi s}\,\frac{1}{s^2+4}.$$

Taking inverse Laplace transform we obtain

$$y(t) = \frac{1}{2}\sin 2(t-\pi)\,H(t-\pi) - \frac{1}{2}\sin 2(t-2\pi)\,H(t-2\pi)$$
$$= \frac{1}{2}\sin 2t\,H(t-\pi) - \frac{1}{2}\sin 2t\,H(t-2\pi).$$

(f) $y'' + y = \delta_\pi(t)\cos t,\ y(0) = 0,\ y'(0) = 1.$

Taking Laplace transform, together with the initial conditions, we have

$$s^2Y(s) - 1 + Y(s) = \int_0^\infty e^{-st}\cos t\,\delta_\pi(t)\,dt.$$

By formula (2.1.15) we have

$$\int_0^\infty e^{-st}\cos t\,\delta_\pi(t)\,dt = e^{-\pi s}\cos\pi = -e^{-\pi s}.$$

Therefore

$$s^2Y(s) - 1 + Y(s) = -e^{-\pi s}.$$

Solving for $Y(s)$ we obtain

$$Y(s) = \frac{1}{s^2+1} - e^{-\pi s}\frac{1}{s^2+1}.$$

Taking inverse Laplace transform we obtain

$$y(t) = \sin t - \sin(t-\pi)\,H(t-\pi).$$

2.1.4 The Convolution Theorem.

1. Commutative Property.

$$(f*g)(x) = \int_0^x f(x-y)g(y)\,dy.$$

Introduce the new variable t by $t = x - y$. Then $y = x - t$ and $dy = -dt$.

$$(f*g)(x) = \int_0^x f(x-y)g(y)\,dy = \int_x^0 f(t)g(x-t)\,(-dt)$$
$$= \int_0^x g(x-t)f(t)\,dt = (g*f)(x).$$

Distributive Property.

$$\big(f*(g+h)\big)(x)=\int_0^x f(x-y)\big(g(y)+h(y)\big)\,dy$$

$$=\int_0^x f(x-y)g(y)\,dy+\int_0^x f(x-y)h(y)\,dy$$

$$=(f*g)(x)+(f*h)(x).$$

Associative Property.

$$\big(f*(g*h)\big)(x)=\int_0^x f(y)(g*h)(x-y)\,dy$$

$$=\int_0^x f(y)\bigg(\int_0^{x-y} g(t)h(x-y-t)\,dt\bigg)\,dy.$$

In the first integral introduce a new variable z by $z=y+t$. Then

$$\big(f*(g*h)\big)(x)=\int_0^x \bigg(\int_y^x f(y)g(z-y)h(x-z)\,dz\bigg)\,dy.$$

Interchanging the order of integration we obtain

$$\big(f*(g*h)\big)(x)=\int_0^x \bigg(\int_0^z f(y)g(z-y)h(x-z)\,dy\bigg)\,dz$$

$$=\int_0^x \bigg(\int_0^z f(y)g(z-y)\,dy\bigg)h(x-z)\,dz=\int_0^x (f*g)(z)h(x-z)\,dz$$

$$=\big((f*g)*h\big)(x).$$

2. (a) $f(t)=1,\ \ g(t)=t^2.$

$$(f*g)(t)=\int_0^t f(t-y)g(y)\,dy=\int_0^t (t-y)^2\,dy=\int_0^t (t^2-2ty+y^2\,dy$$

$$t^3-t^3+\frac{1}{3}t^3=\frac{1}{3}t^3.$$

(b) $f(t) = e^{-3t},\ \ g(t) = 2.$

$$(f * g)(t) = \int_0^t f(t-y)g(y)\,dy = \int_0^t 2e^{-3(t-y)}\,dy$$
$$= 2e^{-3t}\int_0^t e^{3y}\,dy = \frac{2}{3}\,e^{-3t}\,e^{3y}\Big|_{y=0}^{t} = \frac{2}{3}\,e^{-3t}\left(e^{3t}-1\right)$$
$$= \frac{2}{3}\left(1 - e^{-tx}\right).$$

(c) $f(t) = t,\ \ g(t) = e^{-t}.$

$$(f * g)(t) = \int_0^t f(y)g(t-y)\,dy = \int_0^t ye^{-(t-y)}\,dy$$
$$= e^{-t}\int_0^t ye^{y}\,dy = e^{-t}\,e^{y}\big(-1+y\big)\Big|_{y=0}^{t}$$
$$= e^{-t}\,e^{t}\big(-1+t\big) - e^{-t}\,\big(-1+0\big)$$
$$= t - 1 + e^{-t}.$$

(d) $f(t) = \cos\,t,\ \ g(t) = \sin\,t.$

$$(f * g)(t) = \int_0^t f(y)g(t-y)\,dy = \int_0^t \cos\,y\,\sin\,(t-y)\,dy$$
$$= \frac{1}{4}\big(\cos\,(t-2y) + 2y\sin\,t\big)\Big|_{y=0}^{y=t} = \frac{1}{4}\big(\cos\,t + 2t\sin\,t - \cos\,t\big)$$
$$= \frac{1}{2}t\sin\,t.$$

(e) $f(t) = t,\ \ g(t) = H(t) - H(t-2).$

First write $g(t)$ in the following worm

$$g(t) = \begin{cases} 1, & 0 \le t < 2 \\ 0, & t > 2. \end{cases}$$

$$(f * g)(t) = \int_0^t g(y) f(t-y)\, dy = \int_0^t (t-y) g(y)\, dy$$

$$= \begin{cases} \displaystyle\int_0^t (t-y)\, dy, & t < 2 \\ 0, & t > 2 \end{cases} = \begin{cases} \dfrac{1}{2} t^2, & t < 2 \\ 0, & t > 2. \end{cases}$$

$$= \frac{1}{2} t^2 - \frac{1}{2} t^2 H(t-2).$$

3. Let $F = \mathcal{L}\{f\}$.

(a) $f(t) = \displaystyle\int_0^t (t-x)^2 \cos 2x\, dx.$

Notice that $f(t)$ can be written as

$f = g_1 * g_2$, where $g_1(t) = t^2$ and $g_2(t) = \cos 2t$.

Since

$$G_1(s) = \mathcal{L}\{t^2\} = \frac{2}{s^3} \text{ and } G_2(s) = \mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

from the Convolution Theorem for Laplace transform we have

$$F(s) = G_1(s)\, G_2(s) = \frac{2}{s^2(s^2+4)}.$$

(b) $f(t) = \displaystyle\int_0^t e^{-(t-x)} \sin x\, dx.$

Notice that $f(t)$ can be written as

$f = g_1 * g_2$, where $g_1(t) = e^{-t}$ and $g_2(t) = \sin t$.

Since

$$G_1(s) = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1} \text{ and } G_2(s) = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

from the Convolution Theorem for Laplace transform we have

$$F(s) = G_1(s)\, G_2(s) = \frac{1}{(s+1)(s^2+1)}.$$

(c) $f(t) = \int\limits_0^t (t-x)e^x \, dx.$

Notice that $f(t)$ can be written as

$f = g_1 * g_2$, where $g_1(t) = t$ and $g_2(t) = e^t$.

Since

$$G_1(s) = \mathcal{L}\{t\} = \frac{1}{s^2} \text{ and } G_2(s) = \mathcal{L}\{e^t\} = \frac{1}{s-1}$$

from the Convolution Theorem for Laplace transform we have

$$F(s) = G_1(s)\, G_2(s) = \frac{1}{s^2(s-1)}.$$

(d) $f(t) = \int\limits_0^t \sin(t-x) \cos \; dx.$

Notice that $f(t)$ can be written as

$f = g_1 * g_2$, where $g_1(t) = \sin t$ and $g_2(t) = \cos t$.

Since

$$G_1(s) = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \text{ and } G_2(s) = \mathcal{L}\{\cos\} = \frac{s}{s^2+1}$$

from the Convolution Theorem for Laplace transform we have

$$F(s) = G_1(s)\, G_2(s) = \frac{s}{(s^2=1)^2}.$$

(e) $f(t) = \int\limits_0^t (t-x)^2 e^x \, dx.$

Notice that $f(t)$ can be written as

$f = g_1 * g_2$, where $g_1(t) = t^2$ and $g_2(t) = e^t$.

Since

$G_1(s) = \mathcal{L}\{t^2\} = \dfrac{2}{s^3}$ and $G_2(s) = \mathcal{L}\{e^t\} = \dfrac{1}{s-1}$

from the Convolution Theorem for Laplace transform we have

$$F(s) = G_1(s)\, G_2(s) = \frac{2}{s^3(s-1)}.$$

(f) $f(t) = \displaystyle\int_0^t e^{-(t-x)} \sin^2 x\, dx.$

Notice that $f(t)$ can be written as

$f = g_1 * g_2$, where $g_1(t) = e^{-t}$ and $g_2(t) = \sin^2 t$.

Since

$$G_1(s) = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1} \text{ and } G_2(s) = \mathcal{L}\{\sin^2 t\}$$
$$= \frac{1}{2}\mathcal{L}\{1 - \cos 2t\} = \frac{1}{2s} - \frac{s}{2(s^2+4)}$$

from the Convolution Theorem for Laplace transform we have

$$F(s) = G_1(s)\, G_2(s) = \frac{1}{2s(s+1)} - \frac{1}{2s(s^2+4)}$$
$$= \frac{2}{s(s+1)(s^2+4)}.$$

4. Let $f = \mathcal{L}^{-1}\{F\}$.

(a) $F(s) = \dfrac{1}{s^2(s+1)}$.

Write $F(s)$ in the following form

$$F(s) = \frac{1}{s^2}\,\frac{1}{s+1}.$$

Then from

$$g_1(t) = \mathcal{L}^{-1}\{G_1(s)\} = \mathcal{L}^{-1}\Big\{\frac{1}{s^2}\Big\} = t,$$
$$g_2(t) = \mathcal{L}^{-1}\{G_2(s)\} = \mathcal{L}^{-1}\Big\{\frac{1}{s+1}\Big\} = e^{-t},$$

and the Convolution Theorem for Laplace transform we have

$$f(t) = \big(g_1 * g_2\big)(t) = \int\limits_0^t g_1(x) g_2(t-x)\,dx$$

$$= \int\limits_0^t x\,e^{t-x}\,dx = te^{-x}\Big|_{x=0}^{x=t} - \big[-xe^{-x} - e^{-x}\big]\Big|_{x=0}^{x=t}$$

$$-1 + e^{-t} + t.$$

(b) $F(s) = \dfrac{s^2}{(s^2+1)^2}$.

Write $F(s)$ in the following form

$$F(s) = \frac{s}{s^2+1}\,\frac{s}{s^2+1}.$$

Then from

$$g(t) = \mathcal{L}^{-1}\{G_(s)\} = \mathcal{L}^{-1}\Big\{\frac{s}{s^2+1}\Big\} = \sin\,t$$

and the Convolution Theorem for Laplace transform we have

$$f(t) = \big(g * g\big)(t) = \int\limits_0^t g(x) g(t-x)\,dx$$

$$= \int\limits_0^t \cos\,x\,\cos\,(t-x)\,dx = \frac{1}{2}\int\limits_0^t \big[\cos\,t + \cos\,(t-2x)\big]\,dx$$

$$= \frac{1}{2}\big[x\cos\,t - \frac{1}{2}\,\sin\,(t-2x)\big]_{x=0}^{x=t}$$

$$= \frac{1}{2}\big(t\cos\,t + \sin\,t\big).$$

(c) $F(s) = \dfrac{1}{(s^2+4)(s+1)}$.

Write $F(s)$ in the following form

$$F(s) = \frac{1}{s^2+4}\,\frac{1}{s+1}.$$

Then from

$$g_1(t) = \mathcal{L}^{-1}\{G_1(s)\} = \mathcal{L}^{-1}\Big\{\frac{1}{s^2+4}\Big\} = \frac{1}{2}\,\sin\,2t,$$

$$g_2(t) = \mathcal{L}^{-1}\{G_2(s)\} = \mathcal{L}^{-1}\Big\{\frac{1}{s+1}\Big\} = e^{-t},$$

and the Convolution Theorem for Laplace transform we have

$$f(t) = \big(g_1 * g_2\big)(t) = \int\limits_0^t g_1(x) g_2(t-x)\, dx = \frac{1}{2}\int\limits_0^t e^{-(t-x)} \sin 2x\, dx$$
$$= \frac{1}{2}\, e^{-t} \int\limits_0^t e^x \sin 2x\, dx = \frac{1}{10}\, e^{-t}\, e^x \big(\sin 2x - 2\cos 2x\big)\Big|_{x=0}^{t}$$
$$= \frac{1}{5}\, e^{-t} + \frac{1}{10}\,\big(-2\cos 2t + \sin 2t\big).$$

(d) $F(s) = \dfrac{1}{s^3(s^2+16)}$.

Write $F(s)$ in the following form

$$F(s) = \frac{1}{s^2+16}\,\frac{1}{s^3}.$$

Then from

$$g_1(t) = \mathcal{L}^{-1}\{G_1(s)\} = \mathcal{L}^{-1}\Big\{\frac{1}{s^2+16}\Big\} = \frac{1}{4}\sin 4t,$$
$$g_2(t) = \mathcal{L}^{-1}\{G_2(s)\} = \mathcal{L}^{-1}\Big\{\frac{1}{s^3}\Big\} = \frac{1}{2}t^2,$$

and the Convolution Theorem for Laplace transform we have

$$f(t) = \big(g_1 * g_2\big)(t) = \int\limits_0^t g_1(x) g_2(t-x)\, dx = \frac{1}{8}\int\limits_0^t x^2 \sin 4(t-x)\, dx$$
$$= \frac{1}{8}\,\frac{1}{32}\,\Big[(8x^2-1)\cos 4(t-x) + 4x\sin 4(t-x)\Big]\Big|_{x=0}^{x=t}$$
$$= \frac{1}{256}\,\big[8t^2 - 1 + \cos 4t\big].$$

(e) $F(s) = \dfrac{s}{(s^2+4)(s+1)}$.

Write $F(s)$ in the following form

$$F(s) = \frac{s}{s^2+4}\,\frac{1}{s+1}.$$

Then from

$$g_1(t) = \mathcal{L}^{-1}\{G_1(s)\} = \mathcal{L}^{-1}\Big\{\frac{s}{s^2+4}\Big\} =, \cos\, 2t,$$

$$g_2(t) = \mathcal{L}^{-1}\{G_2(s)\} = \mathcal{L}^{-1}\Big\{\frac{1}{s+1}\Big\} = e^{-t},$$

and the Convolution Theorem for Laplace transform we have

$$f(t) = \big(g_1 * g_2\big)(t) = \int\limits_0^t g_1(x)g_2(t-x)\,dx = \int\limits_0^t \cos\, 2x\, e^{-(t-x)}\,dx$$

$$= e^{-t}\int\limits_0^t \cos\, 2x\, e^x\,dx = \frac{1}{5}\,e^{-t}\left[e^x\big(\cos\, 2x + 2\sin\, 2x\big)\right]\Big|_{x=0}^{x=t}$$

$$= \frac{1}{5}\,\big(e^{-t} + \cos\, 2t + 2\sin\, 2t\big).$$

5. Let $Y = \mathcal{L}\{y\}$.

(a) $y - 4t = -3\displaystyle\int\limits_0^t y(x)\sin{(t-x)}\,dx.$

Notice that the above equation can be written in the following form

$$y - 4t = -3\big(y(t) * \sin\, t\big).$$

Taking Laplace transform of both sides of the above equation and using the Convolution Theorem we have

$$Y(s) - \frac{8}{s^2} = -3Y(s)\,\frac{1}{s^2+1}.$$

Solving for $Y(s)$ we have

$$Y(s) = 8\frac{s^2+1}{s^2(s^2+4)} = 6\,\frac{1}{s^2+4} + 2\,\frac{1}{s^2}.$$

Taking inverse Laplace we obtain

$$y(t) = 3\,\sin\, 2t + 2\,t.$$

(b) $y = \dfrac{t^2}{2} - \displaystyle\int\limits_0^t (t-x)\,y(x)\,dx.$

Notice that the above equation can be written in the following form

$$y = \frac{t^2}{2} - \big(y(t) * t\big).$$

Taking Laplace transform of both sides of the above equation and using the Convolution Theorem we have

$$Y(s) = \frac{1}{s^3} - Y(s)\,\frac{1}{s^2}.$$

Solving for $Y(s)$ we have

$$Y(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Taking inverse Laplace we obtain

$$y(t) = 1 - \cos t.$$

(c) $y - e^{-t} = -2\int_0^t y(x)\cos(t-x)\,dx.$

Notice that the above equation can be written in the following form

$$y - e^{-t} = -2\big(y(t) * \cos t\big).$$

Taking Laplace transform of both sides of the above equation and using the Convolution Theorem we have

$$Y(s) - \frac{1}{s+1} = -2Y(s)\,\frac{s}{s^2+1}.$$

Solving for $Y(s)$ we have

$$Y(s) = \frac{s^2+1}{(s+1)^3} = \frac{1}{s+1} - 2\,\frac{1}{(s+1)^2} + 2\,\frac{1}{(s+1)^3}.$$

Taking inverse Laplace we obtain

$$y(t) = e^{-t} - 2te^{-t} + t^2e^{-t} = (t-1)^2\,e^{-t}.$$

(d) $y(t) = t^3 + \int_0^t y(t-x)\sin x\,dx.$

Notice that the above equation can be written in the following form

$$y = t^3 + \big(y(t) * \sin t\big).$$

Taking Laplace transform of both sides of the above equation and using the Convolution Theorem we have

$$Y(s) = \frac{6}{s^4} + Y(s)\,\frac{1}{s^2+1}.$$

Solving for $Y(s)$ we have

$$Y(s) = 6\frac{s^2+1}{s^6} = 6\frac{1}{s^4} + \frac{6}{s^6}.$$

Taking inverse Laplace we obtain

$$y(t) = t^3 + \frac{6}{5!}t^5 = t^3 + \frac{1}{20}t^5.$$

(e) $y(t) = 1 + 2\int\limits_0^t y(x)e^{-2(t-x)}\,dx.$

Notice that the above equation can be written in the following form

$$y = 1 + 2\big(y(t) * e^{-2t}\big).$$

Taking Laplace transform of both sides of the above equation and using the Convolution Theorem we have

$$Y(s) = \frac{1}{s} + 2Y(s)\,\frac{1}{s+2}.$$

Solving for $Y(s)$ we have

$$Y(s) = \frac{s+2}{s^2} = \frac{1}{s} + \frac{2}{s^2}.$$

Taking inverse Laplace we obtain

$$y(t) = 1 + 2t.$$

6. Let $Y(s) = \mathcal{L}\{y(t)\}$.

(a) $y'(t) - t = \int\limits_0^t y(x)\cos(t-x)\,dx,\;\; y(0) = 4.$

Notice that the above equation can be written in the following form

$$y'(t) - t = \big(y(t) * \cos\, t\big),\;\; y(0) = 4.$$

Taking Laplace transform of both sides of the above equation and using the Convolution Theorem and the given initial condition we have

$$sY(s) - 4 - \frac{1}{s^2} + Y(s)\,\frac{s}{s^2+1}.$$

Solving for $Y(s)$ we have

$$Y(s) = \frac{(4s^2+1)(s^2+1)}{s^5} = \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}.$$

Taking inverse Laplace we obtain

$$y(t) = 4 + \frac{5}{2}t^2 + \frac{1}{24}t^4.$$

(b) $\displaystyle\int_0^t \frac{y'(\tau)}{\sqrt{t-\tau}}\,d\tau = 1 - 2\sqrt{t},\ \ y(0) = 0.$

Notice that the above equation can be written in the following form

$$\left(y'(t) * \frac{1}{\sqrt{t}}\right) = 1 - 2\sqrt{t}.$$

Taking Laplace transform of both sides of the above equation and using the Convolution Theorem and the given initial condition we have

$$\left(\mathcal{L}\{y'\}\right)\left(\mathcal{L}\{\frac{1}{\sqrt{t}}\}\right) = \frac{1}{s} - 2\frac{\Gamma(3/2)}{s^{3/2}},$$

$$sY(s)\,\frac{\Gamma(1/2)}{s^{1/2}} = \frac{1}{s} - 2\frac{\Gamma(3/2)}{s^{3/2}}.$$

Solving for $Y(s)$ and using the facts

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \ \text{ and } \ \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

we have

$$Y(s) = \frac{1}{\sqrt{\pi}}\,\frac{1}{s^{3/2}} - \frac{1}{s^2}.$$

Taking inverse Laplace transform we have

$$y(t) = \frac{2}{\pi}\,\sqrt{\pi} - t.$$

2.2 Fourier Transforms.

2.2.1 Definition of Fourier Transform.

1. Let $F = \mathcal{F}\{f\}$.

(a) $f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 < x < 1 \\ 0, & x > 1. \end{cases}$

Notice that $f(x) = g(x) + 2H(x-1)$ where

$$g(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 < x < 1 \\ 0, & x > 0. \end{cases}$$

and $H(x)$ is the Heaviside function.

$$\big(\mathcal{F}\{g\}\big)(\omega) = \int_{-\infty}^{\infty} g(x)\, e^{-i\omega x}\, dx = \int_{-1}^{1} -e^{-i\omega x}\, dx$$

$$\frac{1}{i\omega}\, e^{-i\omega x}\Big|_{x=-1}^{x=1} = \frac{1}{i\omega}\Big[e^{-i\omega} - e^{i\omega}\Big] = -\frac{2\sin\,\omega}{\omega}.$$

Now from $2H(x) = 1 + sgn(x)$ and the result in Example 2.2.6 we have

$$\big(\mathcal{F}\{H\}\big)(\omega) = 2\pi\delta(\omega) + \frac{2}{i\omega}$$

Therefore

$$F(\omega) = -\frac{2\sin\,\omega}{\omega} + e^{-i\omega}\left(2\pi\delta(\omega) + \frac{2}{i\omega}\right)$$

$$= -\frac{4\sin\,\omega}{\omega} + 2\pi e^{-i\omega}\delta(\omega) - \frac{2i\cos\,\omega}{\omega}.$$

(b) $f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 3 \\ 0, & x > 3. \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int_{0}^{3} x\, e^{-i\omega x}\, dx$$

$$= e^{-i\omega x}\,\frac{1 + i\omega x}{\omega^2}\Big|_{x=0}^{x=3} = e^{-3i\omega}\,\frac{1 - e^{3i\omega} + 3i\omega}{\omega^2}.$$

(c) $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0. \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int_{0}^{\infty} e^{-x}\, e^{-i\omega x}\, dx$$

$$= -\frac{1}{1+i\omega}\, e^{(-1-i\omega)x}\Big|_{x=0}^{\infty} = \frac{1}{1+i\omega}.$$

(d) $f(x) = e^{-(bx)^2}$ where $b = \sqrt{\dfrac{a}{2}}$.

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int_{-\infty}^{\infty} e^{-(bx)^2}\, e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-(bx)^2 - i\omega x}\, dx = \int_{-\infty}^{\infty} \exp\Big(-\big(bx + \frac{i\omega}{2b}\big)^2 - \frac{\omega^2}{4b^2}\Big)\, dx$$

$$= \exp\Big(-\frac{\omega^2}{4b^2}\Big) \int_{-\infty}^{\infty} \exp\Big(-\big(bx + \frac{i\omega}{2b}\big)^2\Big)\, dx.$$

If we change the variable x in the last integral by $t = bx + \dfrac{i\omega}{2b}$ we have

$$F(\omega) = \exp\Big(-\frac{\omega^2}{4b^2}\Big)\frac{1}{b}\int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{\sqrt{\pi}}{b}\exp\Big(-\frac{\omega^2}{4b^2}\Big) = \sqrt{\frac{2\pi}{a}}\exp\Big(-\frac{\omega^2}{2a}\Big).$$

Since $\exp\big(-\dfrac{ax^2}{2}\big)$ is continuous everywhere, from Theorem 2.2.1 it follows

$$\exp\Big(-\frac{ax^2}{2}\Big) = \frac{1}{2\pi}\sqrt{\frac{2}{a}}\int_{-\infty}^{\infty} \exp\Big(-\frac{\omega^2}{2a}\Big)\, e^{i\omega x}\, d\omega$$

$= \dfrac{1}{2\pi}\sqrt{\dfrac{2}{a}}\displaystyle\int_{-\infty}^{\infty} \exp\Big(-\frac{\omega^2}{2a}\Big)\Big[\cos(\omega x) + i\sin(\omega x)\Big]\, d\omega$. Comparing the real and imaginary parts, from the last equation we have

$$\int_{-\infty}^{\infty} \exp\Big(-\frac{\omega^2}{4b^2}\Big)\cos(\omega x)\, d\omega = 2\pi\sqrt{\frac{2}{a}}\exp\Big(-\frac{ax^2}{2}\Big).$$

2. $f(x) = \begin{cases} e^{-x}\, x^{a-1}, & x > 0 \\ 0, & x \le 0. \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int_{0}^{\infty} e^{-x}\, x^{a-1}\, e^{-i\omega x}\, dx$$
$$= \int_{0}^{\infty} e^{-(1+i\omega)x}\, x^{a-1}\, dx$$

If we introduce variable t by $(1+i\omega)x = t$ we have

$$F(\omega) = \frac{1}{(1+i\omega)^a} \int_{0}^{\infty} e^{-t}\, t^{a-1}\, dt = \frac{\Gamma(a)}{(1+i\omega)^a}.$$

3. $f(x) = \begin{cases} e^{2x}, & x < 0 \\ e^{-2x}, & x > 0. \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int_{-\infty}^{0} e^{2x}\, e^{-i\omega x}\, dx + \int_{0}^{\infty} e^{-2x}\, e^{-i\omega x}\, dx$$
$$= \int_{-\infty}^{0} e^{(2-i\omega)x}\, dx + \int_{0}^{\infty} e^{-(2+i\omega)x}\, dx$$
$$= \frac{1}{2-i\omega}\, e^{(2-i\omega)x}\Big|_{-\infty}^{x=0} - \frac{1}{2+i\omega}\, e^{-(2+i\omega)x}\Big|_{x=0}^{\infty}$$
$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega} = \frac{4}{(2-i\omega)(2+i\omega)}.$$

4. $f(x) = \begin{cases} \cos ax, & |x| < 1 \\ 0, & |x| > 1. \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int_{-1}^{1} \cos ax\, e^{-i\omega x}\, dx$$
$$= \frac{1}{2} \int_{-1}^{1} \left(e^{iax} + e^{-iax}\right) e^{-i\omega x}\, dx$$
$$= \frac{1}{2} \int_{-1}^{1} \left(e^{i(a-\omega)x} + e^{-i(a+\omega)x}\right) dx$$

$$= \frac{1}{2}\Big(\frac{1}{i(a-\omega)}\, e^{i(a-\omega)x} - \frac{1}{i(a+\omega)}\, e^{-i(a\omega)x}\Big)\Bigg|_{x=-1}^{x=1}$$

$$= \frac{\sin(\omega-a)}{\omega-a} + \frac{\sin(\omega+a)}{\omega+a}.$$

5. $f(x) = \begin{cases} 0, & x < 0 \\ g(x), & x > 0. \end{cases}$

$$\big(\mathcal{F}\{f\}\big)(-i\omega) = \int\limits_{-\infty}^{\infty} f(x)\, e^{-i(-i\omega)x}\, dx = \int\limits_{0}^{\infty} g(x) e^{\omega x}\, dx$$

$$= \big(\mathcal{L}\{g\}\big)(\omega).$$

6. $f(x) = \begin{cases} \cos x, & |x| < \pi \\ 0, & |x| > \pi. \end{cases}$

$$F(\omega) = \int\limits_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int\limits_{-\pi}^{\pi} \cos x\, e^{-i\omega x}\, dx$$

$$= \frac{1}{2}\int\limits_{-\pi}^{\pi} \Big(e^{ix} + e^{-ix}\Big)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{2}\int\limits_{-\pi}^{\pi} \Big(e^{i(1-\omega)x} + e^{-i(1+\omega)x}\Big)\, dx$$

$$= \frac{1}{2}\Big(\frac{1}{i(1-\omega)}\, e^{i(1-\omega)x} - \frac{1}{i(1+\omega)}\, e^{-i(1-\omega)x}\Big)\Bigg|_{x=-\pi}^{x=\pi}$$

$$= \frac{\sin(\omega-1)\pi}{\omega-1} + \frac{\sin(\omega+1)\pi}{\omega+1} = 2\frac{\omega\,\sin(\omega\pi)}{1-\omega^2}.$$

For $|x| < \pi$ the function $\cos x$ is continuous, and so by Theorem 2.2.1 it follows

$$\cos x = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\omega\,\sin(\omega\pi)}{1-\omega^2}\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\omega\,\sin(\omega\pi)}{1-\omega^2}\,\Big[\cos(\omega x) + i\sin(\omega x)\Big]\, d\omega.$$

Comparing the real and imaginary parts, from the last equation we have

$$\int_{-\infty}^{\infty} \frac{\omega \, \cos(\omega x)\, \sin(\omega\pi)}{1-\omega^2}\, d\omega = \pi \cos x$$

For $|x| > \pi$ we have

$$\int_{-\infty}^{\infty} \frac{\omega \, \cos(\omega x)\, \sin(\omega\pi)}{1-\omega^2}\, d\omega = 0.$$

7. (a) $f(x) = e^{-ax}, \;\; a > 0.$

$$\big(\mathcal{F}_c\{f\}\big)(\omega) = \int_0^{\infty} f(x)\, \cos(\omega x)\, dx = \frac{1}{2}\int_0^{\infty} e^{-ax}\left[e^{i\omega x} + e^{-i\omega x}\right] dx$$

$$= \frac{1}{2}\int_0^{\infty} \left[e^{(-a+i\omega)x} + e^{(-a-i\omega)x}\right] dx = \frac{1}{2}\left[\frac{e^{(-a+i\omega)x}}{-a+i\omega} + \frac{e^{(-a-i\omega)x}}{-a-i\omega}\right]_{x=0}^{\infty}$$

$$= \frac{a}{a^2+\omega^2}.$$

(b) $f(x) = e^{-ax}, \;\; a > 0.$

$$\big(\mathcal{F}_s\{f\}\big)(\omega) = \int_0^{\infty} f(x)\, \sin(\omega x)\, dx = \frac{1}{2i}\int_0^{\infty} e^{-ax}\left[e^{i\omega x} - e^{-i\omega x}\right] dx$$

$$= \frac{1}{2i}\int_0^{\infty} \left[e^{(-a+i\omega)x} - e^{(-a-i\omega)x}\right] dx = \frac{1}{2i}\left[\frac{e^{(-a+i\omega)x}}{-a+i\omega} - \frac{e^{(-a-i\omega)x}}{-a-i\omega}\right]_{x=0}^{\infty}$$

$$= \frac{\omega}{a^2+\omega^2}.$$

(c) $f(x) = xe^{-ax}, \;\; a > 0.$

$$\big(\mathcal{F}_s\{f\}\big)(\omega) = \int_0^{\infty} f(x)\, \sin(\omega x)\, dx = \frac{1}{2i}\int_0^{\infty} xe^{-ax}\left[e^{i\omega x} - e^{-i\omega x}\right] dx$$

$$= \frac{1}{2i}\int_0^{\infty} x\left[e^{(-a+i\omega)x} - e^{(-a-i\omega)x}\right] dx$$

$$\frac{1}{2i}\, e^{-(a+i\omega)x} \left[e^{2i\omega x} \frac{-1 - ax + i\omega x}{(a - i\omega)^2} + \frac{1 + ax + i\omega x}{(a + i\omega)^2} \right]_{x=0}^{\infty}$$
$$= \frac{1}{2i} \left[\frac{1}{(a - i\omega)^2} - \frac{1}{(a + i\omega)^2} \right] = 2 \frac{a\omega}{(a^2 + \omega^2)^2}.$$

(d) $f(x) = (1 + x)e^{-ax}, \;\; a > 0.$

$$\big(\mathcal{F}_c\{f\}\big)(\omega) = \int_0^\infty f(x)\, \cos(\omega x)\, dx$$
$$= \frac{1}{2} \int_0^\infty (1 + x)e^{-ax} \left[e^{i\omega x} + e^{-i\omega x} \right]$$
$$= \frac{1}{2} \int_0^\infty (1 + x) \left[e^{(-a+i\omega)x} + e^{(-a-i\omega)x} \right] dx$$
$$= \frac{1}{2}\, e^{-(a+i\omega)x} \left[\frac{1 + a + i\omega}{(a + i\omega)^2} - \frac{x}{a + i\omega} \right]_{x=0}^{\infty}$$
$$+ \frac{1}{2}\, e^{-(a+i\omega)x}\, e^{2i\omega x}\, \frac{-1 - (1 + x)a + (1 + x)i\omega}{(a - i\omega)^2} \Bigg]_{x=0}^{\infty}$$
$$= \frac{2}{(a^2 + \omega^2)^2}.$$

8. $H(x) \equiv u_0(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$

It is easy to see that

$$H(x) = \frac{1}{2} + \frac{1}{2}\, sgn(x),$$

where the signum function $sgn(x)$ is defined by

$$sgn(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Using Example 2.2.6 and result (2.2.12) we have

$$\big(\mathcal{F}\{H\}\big)(\omega) = \frac{1}{2}\, \big(\mathcal{F}\{1\}\big)(\omega) + \frac{1}{2}\, \big(\mathcal{F}\{sgn\}\big)(\omega)$$
$$= \pi\delta(\omega) + \frac{i}{\omega}.$$

9. (a) $\mathcal{F}\{\sin(\omega_0 x)H(x)\}(\omega) = \int\limits_{-\infty}^{\infty} \sin(\omega_0 x)H(x)\, e^{-i\omega x}\, dx$

$$= \frac{1}{2i}\int\limits_{-\infty}^{\infty} e^{-(\omega-\omega_0)ix}H(x)\, dx - \frac{1}{2i}\int\limits_{-\infty}^{\infty} e^{-(\omega_0+\omega)ix}H(x)\, dx$$

$$= \frac{1}{2i}\big(\mathcal{F}\{H\}\big)(\omega-\omega_0) - \frac{1}{2i}\big(\mathcal{F}\{H\}\big)(\omega+\omega_0).$$

Applying the result of Exercise 8 of this section we obtain

$$\big(\mathcal{F}\{\sin(\omega_0 x)H(x)\}\big)(\omega)$$

$$= \frac{1}{2i}\left(\pi\delta(\omega-\omega_0) + \frac{i}{\omega-\omega_0}\right) - \frac{1}{2i}\left(\pi\delta(\omega+\omega_0) + \frac{i}{\omega+\omega_0}\right)$$

$$= \frac{\pi}{2i}\left(\pi\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right) + \frac{\omega_0}{\omega^2-\omega_0^2}.$$

(b) $\mathcal{F}\{\cos(\omega_0 x)H(x)\}(\omega) = \int\limits_{-\infty}^{\infty} \cos(\omega_0 x)H(x)\, e^{-i\omega x}\, dx$

$$= \frac{1}{2}\int\limits_{-\infty}^{\infty} e^{-(\omega-\omega_0)ix}H(x)\, dx + \frac{1}{2}\int\limits_{-\infty}^{\infty} e^{-(\omega_0+\omega)ix}H(x)\, dx$$

$$= \frac{1}{2}\big(\mathcal{F}\{H\}\big)(\omega-\omega_0) + \frac{1}{2i}\big(\mathcal{F}\{H\}\big)(\omega+\omega_0).$$

Applying the result of Exercise 8 of this section we obtain

$$\big(\mathcal{F}\{\cos(\omega_0 x)H(x)\}\big)(\omega)$$

$$= \frac{1}{2}\left(\pi\delta(\omega-\omega_0) + \frac{i}{\omega-\omega_0}\right) + \frac{1}{2}\left(\pi\delta(\omega+\omega_0) + \frac{i}{\omega+\omega_0}\right)$$

$$= \frac{\pi}{2}\left(\pi\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right) + i\,\frac{\omega}{\omega^2-\omega_0^2}.$$

2.2.2 Properties of Fourier Transforms.

1. Let $f(x)$ be an integrable function on $\mathbb{R}$ $\left(\int\limits_{-\infty}^{\infty} \big|f(x)\big|\, dx = M < \infty\right)$

   and let $F(\omega) = \big(\mathcal{F}\{f\}\big)(\omega)$.

   Bounded:

   For every ω we have

$$\left|F(\omega)\right| = \left|\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx\right| \leq \int_{-\infty}^{\infty} \left|f(x)\, e^{-i\omega x}\right| dx = \int_{-\infty}^{\infty} \left|f(x)\right| dx.$$

Continuity:

Fix ω_0. Then

$$\left|F(\omega) - F(\omega_0)\right| = \left|\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} - f(x)\, e^{-i\omega_0 x}\, dx\right|$$

$$\leq \int_{-\infty}^{\infty} \left|f(x)\right| \left|e^{-i\omega x} - e^{-i\omega_0 x}\right| dx.$$

Since $e^{-i\omega x}$ is a continuous function we have that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|e^{-i\omega x} - e^{-i\omega_0 x}\right| < \epsilon \text{ for every } \omega \text{ such that } |\omega - \omega_0| < \delta.$$

For $|\omega - \omega_0| < \delta$ we have

$$\left|F(\omega) - F(\omega_0)\right| \leq \int_{-\infty}^{\infty} \left|f(x)\right| \left|e^{-i\omega x} - e^{-i\omega_0 x}\right| dx \leq \epsilon \int_{-\infty}^{\infty} \left|f(x)\right| dx =$$

$M\epsilon$.

2. Let $F(\omega) = \big(\mathcal{F}\{\frac{1}{1+x^2}\}\big)(\omega) = \pi\, e^{-|\omega|}$.

(a) $g(x) = \dfrac{1}{1+a^2x^2}$.

By the dilation property of Fourier transform we have

$$\big(\mathcal{F}\{g(x)\}\big)(\omega) = \frac{1}{|a|}\, F\big(\frac{\omega}{a}\big) = \frac{1}{|a|}\, \pi\, e^{-\dfrac{|\omega|}{a}}.$$

(b) $g(x) = \dfrac{\cos ax}{1+x^2}$.

Write $f(x)$ in the following form

$$g(x) = \frac{1}{2}\, \frac{e^{iax}}{1+x^2} + \frac{1}{2}\, \frac{e^{-iax}}{1+x^2}.$$

From the modulation property for Fourier transform we have

$$\big(\mathcal{F}\{g(x)\}\big)(\omega) = \frac{1}{2}\,F(\omega - a) + \frac{1}{2}\,F(\omega + a)$$
$$= \frac{\pi}{2}\,e^{-|\omega-a|} + \frac{\pi}{2}\,e^{-|\omega+a|}$$

3. Let $F(\omega) = \big(\mathcal{F}\{e^{-ax}\,H(x)\}\big)(\omega) = \dfrac{1}{a+i\omega}$.

Then

$$\big(\mathcal{F}\{e^{-ax}\,\sin\,bx\,H(x)\}\big)(\omega)$$
$$= \frac{1}{2i}\,\big(\mathcal{F}\{e^{ibx}\,e^{-ax}\,H(x)\}\big)(\omega) - \frac{1}{2i}\,\big(\mathcal{F}\{e^{-ibx}\,e^{-ax}\,H(x)\}\big)(\omega).$$

From the modulation property for Fourier transform we have

$$\big(\mathcal{F}\{e^{-ax}\,\sin\,bx\,H(x)\}\big)(\omega) = \frac{1}{2i}\,F(\omega - b) - \frac{1}{2i}\,F(\omega + b)$$
$$= \frac{1}{2i}\,\frac{1}{a+i(\omega-b)} - \frac{1}{2i}\,\frac{1}{a+i(\omega+b)} = \frac{b}{(a+i\omega)^2+b^2}.$$

4. From the previous exercise and Parseval's identity we have

$$\int_{-\infty}^{\infty} e^{-2ax}\,sin^2\,(2bx)H^2(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b^2}{\big|(a+i\omega)^2+b^2\big|^2}\,d\omega.$$

Since

$$|(a+i\omega)^2+b^2\big|^2 = (a^2+b^2-\omega^2)^2 + 4a^2\omega^2 = (\omega^2+a^2-b^2)^2 + 4a^2b^2$$

and

$$\int_{-\infty}^{\infty} e^{-2ax}\,sin^2\,(2bx)H^2(x)\,dx = \int_{0}^{\infty} e^{-2ax}\,sin^2\,(2bx)\,dx$$
$$= \frac{e^{-2ax}}{a^2+b^2}\,\Big(-a^2-b^2+a^2\cos{(2bx)} - ab\sin{(2bx)}\Big)\Big|_{x=0}^{\infty}$$
$$= \frac{b^2}{4a(a^2+b^2)}$$

we have

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2+a^2-b^2)^2+4a^2b^2} = \frac{\pi}{2a(a^2+b^2)}.$$

5. Let Let $F(\omega) = \big(\mathcal{F}\{\dfrac{1}{1+x^2}\}\big)(\omega) = \pi e^{-|\omega|}$.

   Parseval's identity implies

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi^2 e^{-2|\omega|}\,d\omega$$
$$= \frac{\pi}{2}\int_{-\infty}^{0} e^{2\omega}\,d\omega + \frac{\pi}{2}\int_{0}^{\infty} e^{-2\omega}\,d\omega = \frac{\pi}{2}.$$

6. Let $f(x) = \Big(\mathcal{F}^{-1}\big\{F(\omega)\big\}\Big)(x)$.

   (a) $F(\omega) = \dfrac{1}{\omega^2 - 2ib\omega - a^2 - b^2}$.

   Notice that $F(\omega)$ can be written as

$$F(\omega) = \frac{1}{2a}\,\frac{1}{\omega-(a+bi)} - \frac{1}{2a}\,\frac{1}{\omega-(-a+bi)}$$

   Using the modulation property and the Table of Fourier transforms we have

$$\Big(\mathcal{F}^{-1}\Big\{\frac{1}{\omega-(a+bi)}\Big\}\Big)(x) = \frac{i}{2}\,e^{(a+bi)ix}\,sgn(x),$$
$$\Big(\mathcal{F}^{-1}\Big\{\frac{1}{\omega-(-a+bi)}\Big\}\Big)(x) = \frac{i}{2}\,e^{(-a+bi)ix}\,sgn(x).$$

   Therefore,

$$f(x) = \Big(\mathcal{F}^{-1}\big\{F(\omega)\big\}\Big)(x)$$
$$= \frac{1}{2a}\,\Big(\mathcal{F}^{-1}\Big\{\frac{1}{\omega-(a+bi)}\Big\}\Big)(x) - \frac{1}{2a}\,\Big(\mathcal{F}^{-1}\Big\{\frac{1}{\omega-(-a+bi)}\Big\}\Big)(x)$$
$$\frac{1}{2a}\,\frac{i}{2}\,e^{axi-bx}\,sgn(x) - \frac{1}{2a}\,\frac{i}{2}\,e^{-axi-bx}\,sgn(x)$$
$$= -\frac{1}{2a}\,e^{-bx}\,\sin(ax)\,sgn(x).$$

   (b) $F(\omega) = \dfrac{i\omega}{(1+i\omega)(1-i\omega)}$.

   Notice that $F(\omega)$ can be written as

$$F(\omega) = \frac{1}{2}\,\frac{1}{1+i\omega} - \frac{1}{2}\,\frac{1}{1-i\omega}$$

Using the Table of Fourier transforms we have

$$\begin{aligned}
f(x) &= \Big(\mathcal{F}^{-1}\big\{F(\omega)\big\}\Big)(x)\\
&= \frac{1}{2}\left(\mathcal{F}^{-1}\Big\{\frac{1}{1+i\omega}\Big\}\right)(x) - \frac{1}{2}\left(\mathcal{F}^{-1}\Big\{\frac{1}{1-i\omega}\Big\}\right)(x)\\
&= \frac{1}{2}\,e^{-x}\,H(x) - \frac{1}{2}\,e^{x}\,H(-x).
\end{aligned}$$

(c) $F(\omega) = \dfrac{i\omega}{(1+i\omega)(1+2i\omega)^2}$.

Notice that $F(\omega)$ can be written as

$$F(\omega) = \frac{1}{1+i\omega} - 2\frac{1}{1+2i\omega} + 2\frac{1}{(1+2i\omega)^2}.$$

Using the Table of Fourier transforms we have

$$\begin{aligned}
f(x) &= \Big(\mathcal{F}^{-1}\big\{F(\omega)\big\}\Big)(x)\\
&= \left(\mathcal{F}^{-1}\Big\{\frac{1}{1+i\omega}\Big\}\right) - 2\left(\mathcal{F}^{-1}\Big\{\frac{1}{1+2i\omega}\Big\}\right) + 2\left(\mathcal{F}^{-1}\Big\{\frac{1}{(1+2i\omega)^2}\Big\}\right)\\
&= \left(\mathcal{F}^{-1}\Big\{\frac{1}{1+i\omega}\Big\}\right) - \left(\mathcal{F}^{-1}\Big\{\frac{1}{\frac{1}{2}+i\omega}\Big\}\right) - \frac{1}{2}\left(\mathcal{F}^{-1}\Big\{\frac{1}{(\frac{1}{2}+i\omega)^2}\Big\}\right)\\
&= e^{-x}\,H(x) - e^{x/2}\,H(x) - \frac{1}{2}\,x\,e^{-x/2}\,H(x).
\end{aligned}$$

7. Let $f(x) = \Big(\mathcal{F}^{-1}\big\{F(\omega)\big\}\Big)(x)$.

(a) $F(\omega) = \dfrac{\omega}{\omega^2+a^2}$.

From the Fourier Transform Inversion Theorem it follows

$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(\omega)\,e^{i\omega x}\,d\omega = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\omega}{\omega^2+a^2}\,e^{i\omega x}\,d\omega.$$

First let $x > 0$. To evaluate the above improper integral consider the upper semicircle $\{z = Re^{i\theta},\ 0 \le \theta \le \pi\} \cup \{x:\ -R \le x \le R\}$, where R is sufficiently large such that the point $z = ai$ is inside the semicircle.

From the Cauchy Residue Theorem we have

$$\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega = \int_{-\infty}^{\infty} \frac{\omega}{\omega^2 + a^2}\, e^{i\omega x}\, d\omega$$

$$= 2\pi Res\left(\frac{z}{z^2+a^2}\, e^{izx},\, z = ai\right) = 2\pi \frac{ai}{2ai}\, \frac{e^{-ax}}{2} = \pi i\, e^{-ax}.$$

Therefore

$$f(x) = i\,\frac{\pi}{2}\, e^{-ax}.$$

If $x < 0$ working similarly but using the lower semicircle and the pole $-ai$ we obtain

$$f(x) = -i\,\frac{\pi}{2}\, e^{-ax}.$$

(b) $F(\omega) = \dfrac{3}{(2-i\omega)(1+i\omega)}$.

From the Fourier Transform Inversion Formula it follows

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{3}{(2-i\omega)(1+i\omega)}\, e^{i\omega x}\, d\omega.$$

If $x > 0$, then consider the contour integral

$$\int_{C_R \cup [-R,R]} F(z)\, e^{izx}\, dz,$$

where C_R is the upper semicircle and R is large enough such that the only simple pole of the function $F(z)$ in the upper half-plane is the point $z = i$. Since

$$Res\Big(\frac{3\, e^{izx}}{(2-iz)(1+iz)},\ \ z = i\Big) = \lim_{z\to i} \frac{3(z-i)}{(2-iz)(1+iz)e^{izx}} = -i\, e^{-x}.$$

Therefore, by Cauchy Residue Theorem we have

$$f(x) = \frac{1}{2\pi}\, 2\pi i\big(-i\, e^{-x}\big) = e^{-x}.$$

If $x < 0$, then consider the lower semicircle in which case the only simple pole of the function $F(z)$ in the lower half-plane is the point $z = -2i$.

From

$$Res\Big(\frac{3\, e^{izx}}{(2-iz)(1+iz)},\ \ z = -2i\Big) = \lim_{z\to -2i} \frac{3(z+2i)\, e^{izx}}{(2-iz)(1+iz)} = i\, e^{2x}.$$

and Cauchy Residue Theorem we have

$$f(x) = \frac{1}{2\pi}\, 2\pi i \big(-i\, e^{2x}\big) = e^{2x}.$$

(c) $F(\omega) = \dfrac{\omega^2}{(\omega^2+a^2)^2}, \;\; a > 0.$

From the Fourier Transform Inversion Theorem it follows

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\omega^2}{(\omega^2+a^2)^2}\, d\omega.$$

If $x > 0$, then consider the contour integral

$$\int\limits_{C_R \cup [-R,R]} F(z)\, e^{izx}\, dz,$$

where C_R is the upper semicircle and R is large enough such that the only pole of second order of the function $F(z)$ in the upper half-plane is the point $z = ai$. Since

$$Res\Big(\frac{z^2}{(z^2+a^2)^2}\, e^{izx}, \;\; z = ai\Big) = \lim_{z\to ai} \frac{d}{dz}\Big(\frac{(z-ai)^2 z^2}{(z^2+a^2)^2}\, e^{izx}\Big)$$
$$= -i\, \frac{e^{-ax}}{4a}\, \big(1 - ax\big),$$

by Cauchy Residue Theorem we have

$$f(x) = \frac{1}{2\pi}\, 2\pi i\, (-i)\, \frac{e^{-ax}}{4a}\, \big(1 - ax\big) = \frac{e^{-ax}}{4a}\, \big(1 - ax\big).$$

If $x < 0$, then consider the lower semicircle in which case the only double order pole of the function $F(z)$ in the lower half-plane is the point $z = -ai$.

From

$$Res\Big(\frac{z^2}{(z^2+a^2)^2}\, e^{izx}, \;\; z = -ai\Big) = \lim_{z\to -ai} \frac{d}{dz}\Big(\frac{(z+ai)^2 z^2}{(z^2+a^2)^2}\, e^{izx}\Big)$$
$$= -i\, \frac{e^{ax}}{4a}\, \big(1 + ax\big),$$

by Cauchy Residue Theorem we have

$$f(x) = \frac{1}{2\pi}\, 2\pi i\, (-i)\, \frac{e^{ax}}{4a}\, \big(1 + ax\big) = \frac{e^{+ax}}{4a}\, \big(1 + ax\big).$$

8. $F(\omega)=\dfrac{e^{-i\omega}}{\omega^2+a^2},\;\; a>0.$

From the Fourier Inversion Theorem we have

$$f(x)=\Big(\mathcal{F}^{-1}\big\{F(\omega)\big\}\Big)(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}F(\omega)\,e^{i\omega x}\,d\omega$$
$$=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-i\omega}}{\omega^2+a^2}\,e^{i\omega x}\,d\omega=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{i\omega(x-1)}}{\omega^2+a^2}\,d\omega.$$

If $x>1$, then consider the contour integral

$$\int\limits_{C_R\cup[-R,R]}F(z)\,e^{izx}\,dz,$$

where C_R is the upper semicircle and R is large enough such that the only simple pole of the function $F(z)$ in the upper half-plane is the point $z=ai$. Since

$$Res\Big(\frac{e^{i(x-1)z}}{z^2+a^2},\; z=ai\Big)=\lim_{z\to ai}(z-ai)\frac{e^{i(x-1)z}}{z^2+a^2}=-i\frac{1}{2a}\,e^{-a(x-1)}.$$

Therefore, by Cauchy Residue Theorem we have

$$f(x)=\frac{1}{2\pi}\,2\pi(-i)\frac{1}{2a}\,e^{-a(x-1)}=\frac{1}{2a}\,e^{-a(x-1)}.$$

If $x<1$, then working with the lower semicircle and the simple pole at the point $-ai$ we obtain

$$f(x)=\frac{1}{2a}\,e^{a(x-1)}.$$

9. Let $Y=\mathcal{F}\{y\}$.

(a) $y'(x)+y=\dfrac{1}{2}\,e^{-|x|}.$

Taking Fourier transform of both sides of the given equation and using Theorem 2.2.3. (differentiation) and the Table of Fourier transforms we obtain

$$i\omega Y(\omega)+Y(\omega)=\frac{1}{\omega^2+1}.$$

Solving for $Y(\omega$ we have

$$Y(\omega) = \frac{1}{(1+\omega i)(\omega^2+1)}.$$

By partial fractional decomposition we have

$$Y(\omega) = \frac{1}{2}\,\frac{1}{(\omega-i)^2} - \frac{i}{4}\,\frac{1}{\omega-i} + \frac{i}{4}\,\frac{1}{\omega+i}$$
$$= -\frac{1}{2}\,\frac{1}{(1+i\omega)^2} + \frac{1}{4}\,\frac{1}{1+i\omega} + \frac{1}{4}\,\frac{1}{1-i\omega}$$

Taking inverse Fourier transform and using the Table for Fourier transforms we obtain

$$y(x) = \frac{1}{2}\,xe^{-x}\,H(x) + \frac{1}{4}\,e^{-x}\,H(x) + \frac{1}{4}\,e^{x}\,H(-x).$$

Alternatively, we can find the inverse Fourier transform $y(x)$ of $Y(\omega$ by the Fourier Inverse Theorem:

$$y(x) = \Big(\mathcal{F}^{-1}\big\{Y(\omega)\big\}\Big)(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} Y(\omega)\,e^{i\omega x}\,d\omega$$
$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{1}{(1+\omega i)(\omega^2+1)}\,e^{i\omega x}\,d\omega$$

If $x > 0$, then consider the contour integral

$$\int\limits_{C_R\cup[-R,R]} F(z)\,e^{izx}\,dz,$$

where C_R is the upper semicircle and R is large enough such that the only pole of order 2 of the function $F(z)$ in the upper half-plane is the point $z = i$. Since

$$Res\Big(\frac{e^{izx}}{(z^2+1)(1+iz)},\ z=i\Big) = \lim_{z\to i}\,\frac{d}{dz}\Big((z-i)^2\frac{e^{izx}}{(z^2+1)(1+iz)}\Big)$$
$$= -i\frac{1}{4}\,e^{-x}\big(2x+1\big)$$

by Cauchy Residue Theorem we have

$$y(x) = \frac{1}{2\pi}\,2\pi i(-i)\frac{1}{4}\,e^{-x}\big(2x+1\big) = \frac{1}{4}\,e^{-x}\big(2x+1\big).$$

If $x < 0$, then working with the lower semicircle and the simple pole at the point $-i$ we obtain

$y(x) = \frac{1}{4}\, e^x.$

(b) $y'(x) + y = f(x), \;\; F = \mathcal{F}\{f\}.$

Taking Fourier transform of both sides of the given equation and using Theorem 2.2.3. (differentiation) and the Table of Fourier transforms we obtain

$i\omega Y(\omega) + Y(\omega) = F(\omega).$

Solving for $Y(\omega)$ we obtain

$Y(\omega) = \frac{1}{\omega i + 1}\, F(\omega).$

Using the convolution property for Fourier transform we have

$y(x) = g(x) * f(x),$

where

$g(x) = \big(\mathcal{F}^{-1}\{\frac{1}{\omega i + 1}\}\big)(x) = e^{-x} H(x).$

Therefore,

$$y(x) = g(x) * f(x) = \int\limits_{-\infty}^{\infty} g(y) f(x-y)\, dy = \int\limits_{0}^{\infty} e^{-y} f(x-y)\, dy.$$

(c) $y''(x) + 4y'(x) + 4y = e^{-|x|}.$

Taking Fourier transform of both sides of the given equation and using Theorem 2.2.3. (differentiation) and the Table of Fourier transforms we obtain

$(i\omega)^2 Y(\omega) + 4i\omega Y(\omega) + 4Y(\omega) = 2\frac{1}{\omega^2 + 1}.$

Solving for $Y(\omega)$ we obtain

$Y(\omega) = \frac{2}{\omega^2 + 1}\, \frac{1}{(\omega i + 2)^2}.$

Using the convolution property for Fourier transform we have

$y(x) = g(x) * h(x),$

$$g(x) = \left(\mathcal{F}^{-1}\left\{\frac{2}{\omega^2+1}\right\}\right)(x) = e^{-|x|},$$
$$h(x) = \left(\mathcal{F}^{-1}\left\{\frac{1}{(\omega i+2)^2}\right\}\right)(x) = x\,e^{-2x}\,H(x).$$

Therefore

$$y(x) =, \left(e^{-|x|} * xe^{-2x}\,H(x)\right) = \int\limits_{-\infty}^{\infty} e^{-|x-y|}y\,e^{-2y}\,H(y)\,dy$$
$$= \int\limits_{0}^{\infty} e^{-|x-y|}y\,e^{-2y}\,dy$$

If $x < 0$, then $|x-y| = -x+y$ and so

$$y(x) = e^{x}\int\limits_{0}^{\infty} y\,e^{-3y}\,dy = \frac{1}{9}\,e^{x}.$$

If $x > 0$, then

$$y(x) = \int\limits_{0}^{x} e^{x-y}\,y\,e^{-2y}\,dy + \int\limits_{x}^{\infty} e^{-x+y}\,y\,e^{-2y}\,dy$$
$$= e^{x}\int\limits_{0}^{x} y\,e^{-3y}\,dy + e^{-x}\int\limits_{x}^{\infty} y\,e^{-y}\,dy$$
$$\frac{e^{x}}{9}\left(1-e^{-3x}-3xe^{-3x}\right) + e^{-2x}\,(1+x)$$
$$= \frac{e^{x}}{9} + \frac{8e^{-2x}}{9} - x\frac{e^{-2x}}{3} + xe^{-2x}.$$

(d) $y''(x) + 3y'(x) + 2y = e^{-x}\,H(x)$.

Taking Fourier transform of both sides of the given equation we obtain

$$(i\omega)^2 Y(\omega) + 3i\omega Y(\omega) + 2Y(\omega) = \frac{1}{\omega i+1}.$$

Solving for $Y(\omega)$ we obtain

$$Y(\omega) = \frac{1}{\omega i+1}\,\frac{1}{-\omega^2+3i\omega+2} = \frac{1}{(\omega i+1)^2(\omega i+2)}.$$

By partial fractions decomposition we have

$$Y(\omega) = \frac{1}{(\omega i + 1)^2} - \frac{1}{\omega i + 1} + \frac{2}{\omega i + 2}.$$

Taking inverse Fourier transform we obtain

$$y(x) = xe{-}xH(x) - e{-}xH(x) + 2e^{-2x}H(x).$$

10. Let $F_c(\omega) = \big(\mathcal{F}\{f\}\big)(\omega)$.

Since f is integrable we have

$$\lim_{x\to\infty} f(x) = 0$$

Using the integration by parts formula we have

$$\big(\mathcal{F}_s\{f'\}\big)(\omega) = \int\limits_0^\infty f'(x)\, \sin(\omega x)\, dx$$

$$= f(x)\, \sin(\omega x)\Big|_{x=0}^{\infty} - \omega \int\limits_0^\infty f(x)\, \cos(\omega x)\, dx = \omega F_c(\omega).$$

11. If f is a function which is integrable and square integrable on $(0, \infty)$

with Fourier cosine transform F_c and Fourier sine transform F_s, then

$$\int\limits_0^\infty \Big|F_c(\omega)\Big|^2 d\omega = \frac{\pi}{2} \int\limits_0^\infty \Big|f(x)\Big|^2 dx,$$

$$\int\limits_0^\infty \Big|F_s(\omega)\Big|^2 d\omega = \frac{\pi}{2} \int\limits_0^\infty \Big|f(x)\Big|^2 dx.$$

We will prove Parseval's formula for the Fourier sine transform.

Let

$$f_o(x) = \begin{cases} -f(-x), & x < 0 \\ f(x), & x > 0 \end{cases}$$

be the odd extension of $f(x)$. Then

$$
\begin{aligned}
\big(\mathcal{F}\{f_o\}\big)(\omega) &= \int_{-\infty}^{\infty} f_o(x)e^{-i\omega x}\,dx \\
&= \int_{-\infty}^{0} f_o(x)e^{-i\omega x}\,dx + \int_{0}^{\infty} f_o(x)e^{-i\omega x}\,dx \\
&= -\int_{-\infty}^{0} f_o(-x)e^{-i\omega x}\,dx + \int_{0}^{\infty} f_o(x)e^{-i\omega x}\,dx \\
&= -\int_{-\infty}^{0} f_o(-x)e^{-i\omega x}\,dx + \int_{0}^{\infty} f_o(x)e^{-i\omega x}\,dx \\
&= -\int_{\infty}^{0} f_o(u)e^{i\omega u}\,(-du) + \int_{0}^{\infty} f_o(x)e^{-i\omega x}\,dx \\
&= -\int_{0}^{\infty} f(u)e^{i\omega u}\,du + \int_{0}^{\infty} f(x)e^{-i\omega x}\,dx \\
&= \int_{0}^{\infty} f(x)\Big(e^{-i\omega x} - e^{i\omega x}\Big)\,dx \\
&= -2i\int_{0}^{\infty} f(x)\sin(\omega x)\,dx = -2i\,\big(\mathcal{F}_s\{f\}\big)(\omega).
\end{aligned}
$$

Therefore,

$$
\big(\mathcal{F}_s\{f\}\big)(\omega) = \frac{i}{2}\big(\mathcal{F}\{f_o\}\big)(\omega).
$$

$$
\int_{0}^{\infty} \Big|\mathcal{F}_s\{f\}(\omega)\Big|^2\,d\omega = \frac{1}{4}\int_{0}^{\infty} \Big|\big(\mathcal{F}\{f_o\}\big)(\omega)\Big|^2\,d\omega
$$

$$
= \frac{1}{8}\int_{-\infty}^{\infty} \Big|\big(\mathcal{F}\{f_o\}\big)(\omega)\Big|^2\,d\omega
$$

by Parseval's formula for Fourier transform

$$
= \frac{1}{8}\,2\pi\int_{-\infty}^{\infty} \Big|f_o(x)\Big|^2\,dx = \frac{\pi}{2}\int_{\infty}^{\infty} \Big|f_o(x)\Big|^2\,dx.
$$