From an elementary perspective also we know that in order for the particle to move on the sphere a centra force has to act which is a function of the radial coordinate only. But this too is constant on a sphere so that,

$$m\ddot{\vec{r}} = -\lambda^2 \ \vec{r}$$

where $\lambda^2 R$ is the centripital force.

Q.18

$$g(x) = c_0 + c_1 x + c_2 x^2$$

$$f(x) = sin(x)$$

$$F[c] = \int_0^\pi dx \ (sin(x) - c_0 - c_1 x - c_2 x^2)^2$$

$$= -4c_0 + 8c_2 + \frac{\pi}{2} + (c_0^2 - 2c_1)\pi + (c_0 c_1 - 2c_2)\pi^2 + \frac{1}{3}(c_1^2 + 2c_0 c_2)\pi^3 + \frac{1}{2}c_1 c_2 \pi^4 + \frac{1}{5}(c_2^2 \pi^5)$$

Setting $\frac{\partial}{\partial c_i} F[c] = 0$ we get,

$$c_0 = \frac{12}{\pi^3}(-10 + \pi^2), c_1 = -\frac{60}{\pi^4}(-12 + \pi^2), c_2 = \frac{60}{\pi^5}(-12 + \pi^2)$$

II. CHAPTER-2

Q.1 $L(q_s \dot{q}_s) - L(q, \dot{q}) = \frac{dF(q_s, \dot{q}_s)}{dt}$. This means,

$$\frac{d}{ds} L(q_s \dot{q}_s) = \frac{dF_s(q_s, \dot{q}_s)}{dt}$$

where $F_s \equiv \frac{d}{ds} F$. But,

$$\frac{d}{ds} L(q_s \dot{q}_s) = \frac{dq_s}{ds} \frac{\partial L}{\partial q_s} + \frac{d\dot{q}_s}{ds} \frac{\partial L}{\partial \dot{q}_s} = \frac{dq_s}{ds} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} + \frac{d\dot{q}_s}{ds} \frac{\partial L}{\partial \dot{q}_s} = \frac{d}{dt} \left(\frac{dq_s}{ds} \frac{\partial L}{\partial \dot{q}_s} \right)$$

Therefore,

$$\frac{d}{dt} \left(\frac{dq_s}{ds} \frac{\partial L}{\partial \dot{q}_s} - F_s(q_s, \dot{q}_s) \right) = 0$$

This means the Noether constant is,

$$Q = \frac{dq_s}{ds} \frac{\partial L}{\partial \dot{q}_s} - F_s(q_s, \dot{q}_s)$$

Q.2 Make a transformation to polar coordinates. $x = rcos(\theta)$, $y = rsin(\theta)$. Then a rotation $x^{'} = xcos(\alpha) + ysin(\alpha)$ and $y^{'} = -xsin(\alpha) + ycos(\alpha)$ would be a translation in the new coordinates since $r^{'} = r$ but $\theta^{'} = \theta - \alpha$.

Q.3 $L(sq, s\dot{q}) = L(q, \dot{q})$. This is possible if $L(q, \dot{q}) = f(\frac{\dot{q}}{q})$. The Noether constant is,

$$Q = \left(\frac{dq_s}{ds} \frac{\partial L}{\partial \dot{q}_s} \right)_{s=1} = f^{'}(\frac{\dot{q}}{q})$$

Q.4 Consider $L(q_1, q_2, \dot{q}_1, \dot{q}_2)$. Here, $L(sq_1, sq_2, s\dot{q}_1, s\dot{q}_2) = L(q_1, q_2, \dot{q}_1, \dot{q}_2)$. This is possible only if,

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = f(\frac{q_2}{q_1}, \frac{\dot{q}_1}{q_1}, \frac{\dot{q}_2}{q_1})$$

Noether's constant is,

$$Q = \frac{dq_1, s}{ds} \frac{\partial L}{\partial \dot{q}_{1,s}} + \frac{dq_2, s}{ds} \frac{\partial L}{\partial \dot{q}_{2,s}}$$

$$Q = q_1 \frac{\partial L}{\partial \dot{q}_{1,s}} + q_2 \frac{\partial L}{\partial \dot{q}_{2,s}}$$

$$= f_2(\frac{q_2}{q_1}, \frac{\dot{q}_1}{q_1}, \frac{\dot{q}_2}{q_1}) + \frac{q_2}{q_1} f_3(\frac{q_2}{q_1}, \frac{\dot{q}_1}{q_1}, \frac{\dot{q}_2}{q_1})$$

where $f_2(x, y, z) = \frac{\partial}{\partial y} f(x, y, z)$ and $f_3(x, y, z) = \frac{\partial}{\partial z} f(x, y, z)$.

Q.5 The Lagrangian is, $L(\theta, \dot{\theta}, ..) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$. The total energy is,

$$E = \frac{1}{2} m(\dot{r}^2 + r^2\omega^2) + V(r, \theta)$$

Define,

$$\Omega = \frac{n}{2\pi} \int_{\theta}^{\theta + \frac{2\pi}{n}} d\alpha \, |\omega(\alpha)|$$

where $\omega(\alpha)$ is the angular velocity of the particle when its making an angle α with the x-axis. It is easy to see that Ω is conserved.

Q.6 Now $L(x^\mu(s, q), \dot{x}^\mu(s, q)) = L(x^\mu(s), \dot{x}^\mu(s))$.

$$x^\mu(s, q) = \Lambda^\mu_\rho(q) x^\rho(s)$$

Noether's constant is,

$$Q = \left(\frac{dx^\mu(s, q)}{dq} \frac{\partial L}{\partial \dot{x}^\mu(s, q)} \right)_{q=0} = \left(\frac{d\Lambda^\mu_\rho(q)}{dq} \right)_{q=0} x^\rho(s) m\dot{x}_\mu(s)$$

For boost in the x-direction,

$$Q = \left(\frac{d\Lambda^0_0(q)}{dq} \right)_{q=0} x^0(s) m\dot{x}_0(s) + \left(\frac{d\Lambda^0_1(q)}{dq} \right)_{q=0} x^1(s) m\dot{x}_0(s) + \left(\frac{d\Lambda^1_0(q)}{dq} \right)_{q=0} x^0(s) m\dot{x}_1(s) + \left(\frac{d\Lambda^1_1(q)}{dq} \right)_{q=0} x^1(s) m\dot{x}_1(s)$$

$$x^0(s, q) = \Lambda^0_0(q) x^0(s) + \Lambda^0_1(q) x^1(s)$$

$$x^1(s, q) = \Lambda^1_0(q) x^0(s) + \Lambda^1_1(q) x^1(s)$$

so that ($q = v/c$),

$$\Lambda_0^0(q) = \Lambda_1^1(q) = (1-q^2)^{-\frac{1}{2}}$$

$$\Lambda_0^1(q) = \Lambda_1^0(q) = -q\ (1-q^2)^{-\frac{1}{2}}$$

$$Q = -x^1(s)m\dot{x}_0(s) - x^0(s)m\dot{x}_1(s) = -x(s)mct^{'}(s) + ct(s)mx^{'}(s)$$

A boost is a rotation in space and time. A rotation in just space leads to a conservation of angular momentum $L_z = xp_y - yp_x = xmv_y - ymv_x$. The above quantity is similar. The most general situation would be

$$Q^{\mu\nu} = mx^{'\mu}(s)x^{\nu}(s) - mx^{'\nu}(s)x^{\mu}(s)$$

This antisymmetric tensor is conserved. The the space-space component is the usual angular momentum. The space time component is the quantity in the problem.

Q.7 $L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}\lambda^2 q^2$ where $q = (q_1, q_2, ..., q_d)$. where λ is a Lagrange multiplier. Since there is a rotation symmetry O(d), the corresponding component of the generalised angular momentum are conserved.

III. CHAPTER-3

Q.1 Consider the relations in the text,

$$E_x^{'} = E_x;\ E_y^{'} = \gamma(E_y - \frac{v}{c}B_z);\ E_z^{'} = \gamma(E_z + \frac{v}{c}B_y)$$

$$B_x^{'} = B_x;\ B_y^{'} = \gamma(B_y + \frac{v}{c}E_z);\ B_z^{'} = \gamma(B_z - \frac{v}{c}E_y)$$

$$\mathbf{E}^{'} \cdot \mathbf{B}^{'} = E_xB_x + \gamma^2(E_y - \frac{v}{c}B_z)(B_y + \frac{v}{c}E_z) + \gamma^2(E_z + \frac{v}{c}B_y)(B_z - \frac{v}{c}E_y) = \mathbf{E}\cdot\mathbf{B}$$

Q.2

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}} \mathbf{a}(\mathbf{k})e^{i(\mathbf{k}.\mathbf{r}-\omega_k)t} + \sum_{\mathbf{k}} \mathbf{a}^*(\mathbf{k})e^{-i(\mathbf{k}.\mathbf{r}-\omega_k)t}$$

$$\partial_t\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}}(-i\omega_k)\mathbf{a}(\mathbf{k})e^{i(\mathbf{k}.\mathbf{r}-\omega_k)t} + \sum_{\mathbf{k}}(i\omega_k)\mathbf{a}^*(\mathbf{k})e^{-i(\mathbf{k}.\mathbf{r}-\omega_k)t}$$

$$\mathbf{P} = V\ \sum_{\mathbf{k}}(-i\omega_k)(\mathbf{a}(\mathbf{k})\times\mathbf{a}(-\mathbf{k}))e^{-2i\omega_k t} + V\ \sum_{\mathbf{k}}(-i\omega_k)\mathbf{a}(\mathbf{k})\times\mathbf{a}^*(\mathbf{k})$$

$$+V\ \sum_{\mathbf{k}}(i\omega_k)(\mathbf{a}^*(\mathbf{k})\times\mathbf{a}^*(-\mathbf{k}))e^{2i\omega_k t} + V\ \sum_{\mathbf{k}}(i\omega_k)\mathbf{a}^*(\mathbf{k})\times\mathbf{a}(\mathbf{k})$$