2

Damped and Driven Harmonic Oscillation

2.1 A damped harmonic oscillation is written

$$x(t) = x_0\, e^{-\gamma t}\, \cos(\omega_r\, t) + \left(\frac{v_0 + \gamma\, x_0}{\omega_r}\right) e^{-\nu t} \sin(\omega_r\, t), \tag{1}$$

where x_0 is the initial displacement, v_0 the initial velocity, γ the damping rate, $\omega_r = (\omega_0^2 - \gamma^2)^{1/2}$, and ω_0 the undamped oscillation frequency. The zeros of the oscillation cycle correspond to $x = 0$. So, setting the previous expression to zero, we obtain

$$\tan(\omega_r\, t_m) = -\frac{x_0\, \omega_r}{v_0 + \nu\, x_0}, \tag{2}$$

where the t_m is the time of an individual zero. The previous equation has one solution for $\omega_r\, t$ in the range 0 to π. Moreover, $\tan(\omega_r\, t_m + \pi) = \tan(\omega_r\, t_m)$. It follows that successive maxima occur in a regular sequence whose period is $T = \pi/\omega_r$. This is half the time period between successive maxima. (See Exercise 2.2.)

2.2 A damped harmonic oscillation is written

$$x(t) = x_0\, e^{-\gamma t}\, \cos(\omega_r\, t) + \left(\frac{v_0 + \nu\, x_0}{\omega_r}\right) e^{-\gamma t} \sin(\omega_r\, t), \tag{3}$$

where x_0 is the initial displacement, v_0 the initial velocity, γ the damping rate, $\omega_r = (\omega_0^2 - \gamma^2)^{1/2}$, and ω_0 the undamped oscillation frequency. It follows that

$$\dot{x} = v_0\, e^{-\gamma t}\, \cos(\omega_r\, t) - \left(\frac{x_0\, \omega_0^2 + \gamma\, v_0}{\omega_r}\right) e^{-\gamma t} \sin(\omega_r\, t). \tag{4}$$

The maxima and minima of the oscillation cycle correspond to $\dot{x} = 0$. So, setting the previous expression to zero, we obtain

$$\tan(\omega_r\, t_m) = \frac{v_0\, \omega_r}{x_0\, \omega_0^2 + \gamma\, v_0}, \tag{5}$$

where the t_m is the time of an individual maximum or minimum. The previous equation has two solutions for $\omega_r\, t$ in the range 0 to 2π. One of these corresponds to a maximum, and the other to a minimum. It follows that successive maxima occur in a regular sequence whose period is $T = 2\pi/\omega_r$. Thus, the ratio of successive maxima is

$$\frac{x(t_m + T)}{x(t_m)} = \mathrm{e}^{-\gamma T}, \tag{6}$$

which is a constant. This result follows from Equation (3) because $\sin[\omega_r\,(t_m + T)] = \sin(\omega_r\,t)$ and $\cos[\omega_r\,(t_m + T)] = \cos(\omega_r\,t)$.

2.3 The amplitude of a damped harmonic oscillation decays as

$$a(t) = x_0\,\mathrm{e}^{-\gamma t}, \tag{7}$$

where γ is the damping constant. The amplitude decreases to $1/\mathrm{e}$ of its initial value when $t = 1/\gamma$. The angular frequency of the oscillation is $\omega_r = (\omega_0^2 - \gamma^2)^{1/2}$, where ω_0 is the undamped oscillation frequency. The ratio of the period of the oscillation to that when there is no damping is

$$\frac{T}{T_0} = \frac{\omega_0}{\omega} = \left(1 - \frac{\gamma^2}{\omega_0^2}\right)^{-1/2}. \tag{8}$$

However, we are told that $t = 1/\gamma = n\,T \simeq n\,T_0$ (because $T \simeq T_0$ when the damping is weak, so that $n \gg 1$). Hence,

$$\frac{T}{T_0} = \left(1 - \frac{1}{\omega_0^2\,T_0^2\,n^2}\right)^{-1/2} = \left(1 - \frac{1}{4\pi^2\,n^2}\right)^{-1/2} \simeq 1 + \frac{1}{8\pi^2\,n^2}, \tag{9}$$

because $T_0 = 2\pi/\omega_0$.

2.4 Given that $f_0 = 256\ \mathrm{s}^{-1}$, it follows that

$$\omega_0 = 2\pi\,f_0 = 1608.5\ \mathrm{rad.\,s^{-1}}. \tag{10}$$

Assuming that the mean energy varies as

$$\langle E\rangle = E_0\ \exp(-\nu\,t), \tag{11}$$

the time required for the mean energy to decay to half of its initial value is

$$t_{1/2} = \frac{\ln 2}{\nu}. \tag{12}$$

However, $t_{1/2} = 1\,\mathrm{s}$, so

$$\nu = \ln 2 = 0.6931\,\mathrm{s}^{-1}. \tag{13}$$

Hence, the effective quality factor is

$$Q_f = \frac{\omega_0}{\nu} = \frac{1608.5}{0.6931} \simeq 2321. \tag{14}$$

2.5 The displacement of the electron is

$$x = A\,\sin(\omega\,t), \tag{15}$$

where $\omega = 2\pi\,f$. Hence, the acceleration is

$$a = \ddot{x} = -A\,\omega^2\,\sin(\omega\,t). \tag{16}$$

By analogy with the analysis of Exercise 1.11, the average of a^2 over an oscillation cycle is

$$\langle a^2\rangle = \frac{1}{2}\,A^2\,\omega^4. \tag{17}$$

Thus, the mean energy loss rate of the oscillator due to radiation is

$$-\frac{d\langle E\rangle}{dt} = \frac{K\,e^2\,\langle a^2\rangle}{c^3} = \frac{K\,e^2\,A^2\,\omega^4}{2\,c^3}. \tag{18}$$

Hence, the energy loss per cycle is

$$\Delta E = -T\,\frac{d\langle E\rangle}{dt} = \frac{\pi\,K\,e^2\,A^2\,\omega^3}{c^3}, \tag{19}$$

where $T = 2\pi/\omega$ is the oscillation period. From Exercise 1.11, the mean energy of the oscillator is

$$\langle E\rangle = \langle U\rangle + \langle K\rangle = \frac{1}{2}\,m_e\,\omega^2\,A^2, \tag{20}$$

where m_e is the electron mass. The quality factor of the oscillator is thus

$$Q_f = 2\pi\,\frac{\langle E\rangle}{\Delta E} = \frac{m_e\,c^3}{K\,e^2\,\omega}. \tag{21}$$

As we saw in Exercise 2.4, the half-life of a damped oscillatory system is

$$t_{1/2} = \frac{\ln 2}{\nu}. \tag{22}$$

However,

$$Q_f = \frac{\omega}{\nu}, \tag{23}$$

so

$$t_{1/2} = \frac{\ln 2\,Q_f}{\omega}. \tag{24}$$

The number of oscillations in this time period is

$$N = \frac{\omega\,t_{1/2}}{2\pi}. \tag{25}$$

The wavelength of green light is $\lambda = 5.7\times 10^{-7}\,\text{m}$. Hence, $f = c/\lambda =$

$3\times 10^8/5.7\times 10^{-7} = 5.3\times 10^{14}\,\mathrm{s}^{-1}$, and $\omega = 2\pi f = 3.3\times 10^{15}\,\mathrm{rad.\,s}^{-1}$. It follows that

$$Q_f = \frac{m_e\,c^3}{K\,e^2\,\omega} = \frac{9.1\times 10^{-31}\,(3.0\times 10^8)^3}{6\times 10^9\,(1.6\times 10^{-19})^2\,3.3\times 10^{15}} \simeq 5\times 10^7. \tag{26}$$

Likewise,

$$t_{1/2} = \frac{\ln 2\,Q_f}{\omega} = \frac{\ln 2\,(5\times 10^7)}{3.3\times 10^{15}} \simeq 1\times 10^{-8}\,\mathrm{s}, \tag{27}$$

and

$$N = \frac{3.3\times 10^{15}\,(1\times 10^{-8})}{2\pi} \simeq 5\times 10^6. \tag{28}$$

2.6 The quality factor is defined

$$Q_f = \frac{2\pi\,E}{-\Delta E}, \tag{29}$$

where $-\Delta E$ is the energy lost in an oscillation cycle. We can write

$$-\Delta E = -\frac{dE}{dt}\,T, \tag{30}$$

where $-dE/dt$ is the mean energy loss rate, and T the period. Hence, we obtain

$$\frac{dE}{dt} = -\frac{2\pi}{Q_f\,T}\,E, \tag{31}$$

which can be solved to give

$$E(t) = E_0\,\exp\left(-\frac{2\pi}{Q_f}\,\frac{t}{T}\right). \tag{32}$$

Thus, when $t = Q_f\,T$,

$$E = E_0\,\mathrm{e}^{-2\pi} \simeq 1.9\times 10^{-3}. \tag{33}$$

Because $E \propto a^2$, the amplitude decays by a factor $\mathrm{e}^{-\pi} \simeq 0.04$ in the same time interval.

2.7 The solution to the damped harmonic oscillator equation can be written

$$x(t) = x_0\,\mathrm{e}^{-\nu\,t/2}\,\cos(\omega_1\,t) + \left(\frac{v_0 + \nu\,x_0/2}{\omega_1}\right)\mathrm{e}^{-\nu\,t/2}\,\sin(\omega_1\,t), \tag{34}$$

where $\omega_1 = (\omega_0^2 - \nu^2/4)^{1/2}$. Here, it is assumed that $\nu \leq 2\,\omega_0$. Furthermore, $x_0 = x(0)$ and $v_0 = \dot{x}(0)$. Taking the limit $\nu \to 2\,\omega_0$, which is equivalent to taking the limit $\omega_1 \to 0$, we find that

$$x(t) \to x_0\,\mathrm{e}^{-\nu\,t/2} + \left(\frac{v_0 + \nu\,x_0/2}{\omega_1}\right)\mathrm{e}^{-\nu\,t/2}\,\omega_1\,t, \tag{35}$$

because $\cos\theta \simeq 1$ and $\sin\theta \simeq \theta$ when $|\theta| \ll 1$. Hence,

$$x(t) \to (x_0 + [v_0 + (\nu/2)\,x_0]\,t)\,\mathrm{e}^{-\nu\,t/2}. \tag{36}$$

2.8 Let $I_1(t)$, $I_2(t)$, and $I_3(t)$ be the currents flowing in the left, middle, and right legs of the circuit, respectively. If $I(t) = I_0\,\cos(\omega\,t)$ is the current fed into the circuit then Kirchhoff's first circuital law requires that

$$I(t) = I_1(t) + I_2(t) + I_3(t). \tag{37}$$

Becuase the three legs of the circuit are connected in parallel, the potential drops across them are the same. The potential drop across the left leg is

$$L\,\frac{dI_1}{dt}, \tag{38}$$

whereas the potential drop across the middle leg is

$$R\,I_2, \tag{39}$$

and the potential drop across the right leg is

$$\int_0^t I_3(t')\,dt'\bigg/C. \tag{40}$$

Hence, the common potential drop across all three legs is

$$V(t) = L\,\frac{dI_1}{dt} = R\,I_2 = \int_0^t I_3(t')\,dt'\bigg/C. \tag{41}$$

It follows that

$$I_2 = \frac{L}{R}\,\frac{dI_1}{dt}, \tag{42}$$

and

$$L\,C\,\frac{d^2I_1}{dt^2} = I_3 = I - I_1 - I_2 = I - I_1 - \frac{L}{R}\,\frac{dI_1}{dt}. \tag{43}$$

Thus,

$$\frac{d^2I_1}{dt^2} + \nu\,\frac{dI_1}{dt} + \omega_0^{\,2}\,I_1 = \omega_0^{\,2}\,I_0\,\cos(\omega\,t), \tag{44}$$

where $\omega_0 = 1/\sqrt{L\,C}$ and $\nu = 1/R\,C$. This is the driven damped harmonic oscillator equation. The resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{L\,C}}, \tag{45}$$

whereas the quality factor takes the form

$$Q_f = \frac{\omega_0}{\nu} = \frac{R}{\sqrt{L/C}}. \tag{46}$$

At the resonant frequency ($\omega = \omega_0$), the first and third terms on the left-hand side of Equation (44) cancel (since $d^2/dt^2 \equiv -\omega^2$), and we are left with

$$\nu\,\frac{dI_1}{dt} = \omega_0^{\,2}\,I_0\,\cos(\omega_0\,t). \tag{47}$$

However, the potential drop across the circuit is

$$V(t) = L\,\frac{dI_1}{dt} = \frac{L\,\omega_0^2}{\nu}\,I_0\,\cos(\omega_0\,t) = R\,I_0\,\cos(\omega_0\,t). \tag{48}$$

Thus, the mean power absorbed by the circuit is

$$P = \langle I(t)\,V(t)\rangle = \langle R\,I_0^2\,\cos^2(\omega\,t)\rangle = \frac{1}{2}\,R\,I_0^2. \tag{49}$$

2.9 Minus the rate of work of the damping force, $f = -m\,\nu\,\dot{x}$, is

$$P = -f\,\dot{x} = m\,\nu\,\dot{x}^2. \tag{50}$$

Given that $x = x_0\,\cos(\omega\,t - \varphi)$, the average rate of work is

$$\langle P\rangle = \langle m\,\nu\,\omega^2\,x_0^2\,\sin^2(\omega\,t - \varphi)\rangle = \frac{1}{2}\,m\,\nu\,\omega^2\,x_0^2. \tag{51}$$

However, according to Equation (2.48) (in the book),

$$x_0 = \frac{\omega_0^2\,X_0}{[(\omega_0^2 - \omega^2)^2 + \nu^2\,\omega^2]^{1/2}}. \tag{52}$$

Hence,

$$\langle P\rangle = \frac{1}{2}\,m\,\nu\,X_0^2\left[\frac{\omega^2\,\omega_0^4}{(\omega_0^2 - \omega^2)^2 + \nu^2\,\omega^2}\right]. \tag{53}$$

In the vicinity of the resonant frequency, $(\omega_0^2 - \omega^2)^2 + \nu^2\,\omega^2 \simeq 4\,\omega_0^2\,(\omega_0 - \omega)^2 + \nu^2\,\omega_0^2$, and $\omega^2 \simeq \omega_0^2$, so

$$\langle P\rangle \simeq \frac{1}{2}\,m\,\nu\,X_0^2\left[\frac{\omega_0^4}{4\,(\omega_0 - \omega)^2 + \nu^2}\right]. \tag{54}$$

However, $\omega_0^2 = k/m$ and $Q_f = \omega_0/\nu$, so

$$\langle P\rangle \simeq \frac{1}{2}\,\omega_0\,k\,X_0^2\,Q_f\left[\frac{\nu^2}{4\,(\omega_0 - \omega)^2 + \nu^2}\right], \tag{55}$$

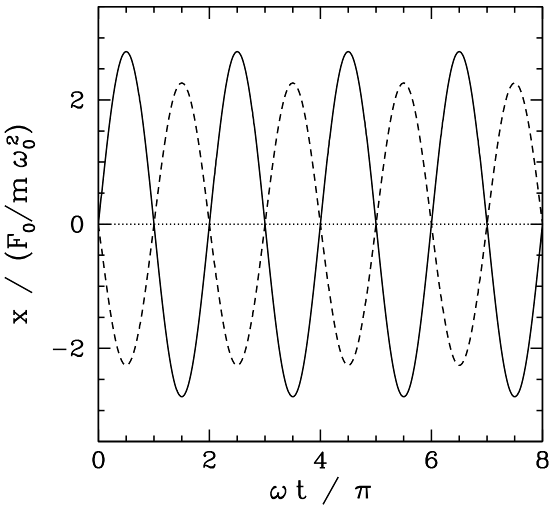
which is the same as Equation (2.59) (in the book).

2.10 Consider the driven undamped harmonic oscillator equation

$$m\,\ddot{x} + k\,x = F_0\,\sin(\omega\,t). \tag{56}$$

Let us search for a "time-asymptotic" solution of the form

$$x(t) = a\,\sin(\omega\,t - \phi). \tag{57}$$



Substitution into Equation (56) yields

$$-m\,\omega^2\,a\,\sin(\omega\,t-\phi)+k\,a\,\sin(\omega\,t-\phi)=F_0\,\sin(\omega\,t). \tag{58}$$

This equation is satisfied when $\phi=0$, and

$$a=\frac{F_0}{m\,(\omega_0^2-\omega^2)}, \tag{59}$$

where $\omega_0=\sqrt{k/m}$. Thus, the "time-asymptotic" solution is

$$x(t)=\frac{F_0\,\sin(\omega\,t)}{m\,(\omega_0^2-\omega^2)}. \tag{60}$$

The preceding figure shows the "time-asymptotic" solution calculated for $\omega=0.8\,\omega_0$ (solid curve) and $\omega=1.2\,\omega_0$ (dashed curve).

The "transient" solution (i.e., the solution that would be transient were a small amount of damping added to the system) satisfies Equation (56) with the right-hand side set to zero: that is,

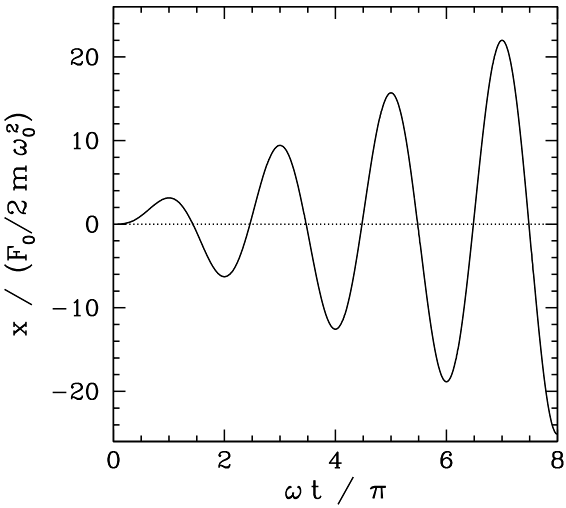
$$m\,\ddot{x}+k\,x=0. \tag{61}$$

Hence,

$$x(t)=A\,\cos(\omega_0\,t)+B\,\sin(\omega_0\,t), \tag{62}$$

where A and B are constants. The most general solution is the sum of the "time-asymptotic" and the "transient" solutions. In other words,

$$x(t)=\frac{F_0\,\sin(\omega\,t)}{m\,(\omega_0^2-\omega^2)}+A\,\cos(\omega_0\,t)+B\,\sin(\omega_0\,t). \tag{63}$$



Thus,

$$x(0) = A, \tag{64}$$

and

$$\dot{x}(0) = \frac{\omega\, F_0}{m\,(\omega_0^2 - \omega^2)} + \omega_0\, B. \tag{65}$$

Since $x(0) = \dot{x}(0) = 0$, we have

$$A = 0, \tag{66}$$

and

$$B = -\frac{F_0\,(\omega/\omega_0)}{m\,(\omega_0^2 - \omega^2)}. \tag{67}$$

Hence,

$$x(t) = \frac{F_0}{m\,(\omega_0^2 - \omega^2)}\left[\sin(\omega\, t) - \frac{\omega}{\omega_0}\,\sin(\omega_0\, t)\right]. \tag{68}$$

Taking the limit $\omega \rightarrow \omega_0$, we have $\omega_0^2 - \omega^2 \simeq 2\,\omega_0\,(\omega_0 - \omega)$, and

$$\sin(\omega\, t) \simeq \sin(\omega_0\, t) + (\omega\, t - \omega_0\, t)\,\cos(\omega_0\, t) + \cdots. \tag{69}$$

Thus,

$$x(t) \simeq \frac{F_0}{2\, m\, \omega_0\,(\omega_0 - \omega)}\left[\sin(\omega_0\, t) + (\omega - \omega_0)\, t\, \cos(\omega_0\, t) - \left(1 + \frac{\omega - \omega_0}{\omega_0}\right)\sin(\omega_0\, t)\right], \tag{70}$$

which yields

$$x(t) \simeq \frac{F_0}{2\,m\,\omega_0^2}\left[\sin(\omega_0\,t) - \omega_0\,t\,\cos(\omega_0\,t)\right]. \tag{71}$$

This solution is plotted in the preceding figure.