Chapter 2

Solutions to Exercise Problems

1. Classify the following differential equations according to their categories such as ODE or PDE, order, dimension, linear or nonlinear, time independent or dependent, and type of coefficient, etc. Also identify the dependent and independent variables.

   (a)
   $$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 7tx = 0$$

   Answer:
   2^{nd} order ODE, linear, time dependent, and with non-constant coefficients. It is a 0D equation because x is dependent variable and there is no independent dimensional variable. t is independent time variable and it is a non-constant coefficient of x.

   (b)
   $$\frac{\partial u}{\partial t} - 5\frac{\partial^2 u}{\partial y^2} = (1 + 5t)\sin(y)$$

   Answer:
   2^{nd} order PDE, linear, time dependent, and with constant coefficient. It is a 1D equation. u is dependent variable, y is independent dimensional variable and t is independent time variable.

   (c)
   $$\frac{d^3w}{dx^3} + w\frac{dw}{dx} + 9w = 0$$

   Answer:
   3^{rd} order ODE, nonlinear, time independent, and with constant coefficient. It is a 1D equation. w is dependent variable, x is independent dimensional variable. It contains a product term of w and $\frac{\partial w}{\partial x}$, hence nonlinear.

   (d)
   $$\frac{d^2x}{dt^2} - 2\frac{1}{t^2}x = 0$$

   Answer:
   2^{nd} order ODE, linear, time dependent, and with non-constant coefficient. It is a 0D equation. x is dependent variable with no independent dimensional variable and t is independent time variable. $1/t^2$ is the coefficient of x, hence non-constant.

(e)

$$\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)^2 + 8v = x + t^2 + 8xt$$

Answer:
1^{st} order PDE, nonlinear, time dependent, and with constant coefficient. It is a 1D equation. v is dependent variable, x is independent dimensional variable and t is independent time variable. It has the square term of $\frac{\partial v}{\partial x}$, hence nonlinear.

(f)

$$\frac{\partial w}{\partial t} - \frac{\partial}{\partial x}\left[2w\frac{\partial^2 w}{\partial x^2}\right] - 8w = 0$$

Answer:
3^{rd} order PDE, nonlinear, time dependent, and with constant coefficient. It is a 1D equation. w is dependent variable, x is independent dimensional variable and t is independent time variable. It contains the product term of w and $\frac{\partial w^2}{\partial x^2}$, hence nonlinear.

2. Classify the following differential equations according to their categories such as ODE or PDE, order, dimension, linearity, time dependency, and type of coefficient. Also identify the dependent and independent variables.

(a)

$$\frac{d^3\theta}{dt^3} + 5\theta + \sin(\theta) = 0$$

Answer:
3^{rd} order ODE, linear, time dependent, and with constant coefficient. It is a 0D equation. θ is dependent variable with no independent dimensional variable and t is independent time variable.

(b)

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + 15\frac{\partial u}{\partial y} = 1 + x + y$$

Answer:
2^{nd} order PDE, linear, time independent, and with constant coefficient. It is a 2D equation. u is dependent variable, x and y are the independent dimensional variables.

(c)

$$\rho(x,y,z)\frac{\partial^2 u}{\partial t^2} + \alpha(x,y,z)\frac{\partial u}{\partial t} - T(x,y,z)\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right\} = g(x,y,z)$$

Answer:
2^{nd} order PDE, linear, time dependent, and with non-constant coefficient. It is a 3D equation. u is dependent variable, x, y and z are the independent dimensional variables, and t is independent time variable. ρ, α and T are non-constant coefficients.

(d)

$$\frac{d^2 v}{dt^2} - 2\frac{dv}{dt} + t^2 v = 0$$

Answer:
2^{nd} order ODE, linear, time dependent, and with non-constant coefficient. It is a 0D equation. v is dependent variable with no independent dimensional variable and t is independent time variable. t^2 is the coefficient of v, hence non-constant.

(e)

$$\rho(x)c(x)\left(\frac{\partial y}{\partial t}\right)^3 - \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial y}{\partial x}\right) = 0$$

Answer:
2^{nd} order PDE, nonlinear, time dependent, and with non-constant coefficient. It is a 1D equation. y is dependent variable and x is independent dimensional variable and t is independent time variable.

(f)

$$\frac{d^2x}{dt^2} + 2\frac{d^4x}{dt^4} + 2015x^2 = \sin(\pi t)$$

Answer:
4^{th} order ODE, nonlinear, time dependent, and with constant coefficient. It is a 0D equation. x is dependent variable with no independent dimensional variable and t is independent time variable.

3. Determine whether each of the followings below is a solution of the corresponding differential equation in Exercise 1:

(a)

$$x(t) = te^t$$

Answer:
Substituting $x(t) = te^t$ into 1(a), we have

$$(t+2)e^t + 2(1+t)e^t + 7t^2e^t = 7t^2e^t + 3te^t + 4e^t \neq 0$$

thus, it is not a solution.

(b)

$$u(y,t) = t\sin(y)$$

Answer:
Substituting $u(y,t) = t\sin(y)$ into 1(b), we have

$$\sin(y) + 5t\sin(y) = (1+5t)\sin(y)$$

thus, it is a solution.

(c)

$$w(x) = x^3 + 5x$$

Answer:
Substituting $w(x) = x^3 + 5x$ into 1(c), we have

$$6 + (x^3+5x)(3x^2+5) + 9(x^3+5x) \neq 0$$

thus, it is not a solution.

(d)

$$x(t) = \frac{1}{t}$$

Answer:
Substituting $x(t) = \frac{1}{t}$ into 1(d), we have

$$\frac{2}{t^3} - \frac{2}{t^2}\frac{1}{t} = 0$$

thus, it is a solution.

(e)

$$v(x,t) = tx$$

Answer:
Substituting $v(x,t) = tx$ into 1(e), we have

$$x + t^2 + 8tx = x + t^2 + 8xt$$

thus, it is a solution.

(f)

$$w(x,t) = x^2t$$

Answer:
Substituting $u(x,t) = x^2t$ into 1(f), we have

$$x^2 - 8xt^2 - 8x^2t \neq 0$$

thus, it is not a solution.

4. Identify the size and the type of the given matrices and denote whether the matrix is a square, column, diagonal, row, identity, or symmetric.

(a)

$$\begin{bmatrix} 23 & 12 & 0 \\ 12 & 40 & 25 \\ 0 & 25 & 9 \end{bmatrix}$$

Answer:
It is a 3×3 square matrix; it is also a symmetric one.

(b)

$$\begin{Bmatrix} t \\ t^2 \\ t^3 \\ t^4 \end{Bmatrix}$$

Answer:
It is a 4×1 column vector.

(c)

$$\begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix}$$

Answer:
It is a 1×4 row vector.

(d)

$$\begin{bmatrix} 6 & 3 \\ 5 & 7 \\ 4 & 1 \end{bmatrix}$$

Answer:
It is a 3×2 matrix.

(e)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:
It is a 4×4 identity matrix.

(f)

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix}$$

Answer:
It is a 5×5 square matrix; it is also a symmetric and diagonal matrix.

5. Given matrices

$$[A] = \begin{bmatrix} 7 & 3 & 2 \\ 9 & 0 & -8 \\ 5 & -7 & 4 \end{bmatrix}, [B] = \begin{bmatrix} 5 & 4 & -3 \\ 9 & 4 & 6 \\ 2 & 1 & -6 \end{bmatrix}, \text{and } [C] = \begin{Bmatrix} 3 \\ -7 \\ 9 \end{Bmatrix}$$

perform the following operations:

(a) [A]+[B]=?

Answer: [A]+[B]= $\begin{bmatrix} 12 & 7 & -1 \\ 18 & 4 & -2 \\ 7 & -6 & -2 \end{bmatrix}$.

(b) 5[A]=?

Answer: 5[A]= $\begin{bmatrix} 35 & 15 & 10 \\ 45 & 0 & -40 \\ 25 & -35 & 20 \end{bmatrix}$.

(c) [A][B]=?

Answer: [A][B]= $\begin{bmatrix} 66 & 42 & -15 \\ 29 & 28 & 21 \\ -30 & -4 & -81 \end{bmatrix}$.

(d) [B][A]=?

Answer: [B][A]= $\begin{bmatrix} 56 & 36 & -34 \\ 129 & -15 & 10 \\ -7 & 48 & -28 \end{bmatrix}$.

(e) [A]{C}=?

Answer: [A]{C}= $\begin{Bmatrix} 18 \\ -45 \\ 100 \end{Bmatrix}$.

(f) $[B]^2$=?

Answer: $[B]^2$= $\begin{bmatrix} 55 & 33 & 27 \\ 93 & 58 & -39 \\ 7 & 6 & 36 \end{bmatrix}$.

(g) Show that [I][A]=[A][I]=[A]

Answer: [I][A]= $\begin{bmatrix} 7 & 3 & 2 \\ 9 & 0 & -8 \\ 5 & -7 & 4 \end{bmatrix}$ =[A][I].

6. Given the matrices

$$[A] = \begin{bmatrix} 1 & 6 & 9 \\ 7 & 3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \text{ and } [B] = \begin{bmatrix} 0 & 8 & -3 \\ -5 & 9 & 3 \\ 2 & 5 & -9 \end{bmatrix}$$

perform the following operations:

(a) $[A]^T$=? and $[B]^T$=?

Answer: $[A]^T$= $\begin{bmatrix} 1 & 7 & 5 \\ 6 & 3 & -1 \\ 9 & 2 & 4 \end{bmatrix}$, $[B]^T$= $\begin{bmatrix} 0 & -5 & 2 \\ 8 & 9 & 5 \\ -3 & 3 & -9 \end{bmatrix}$.

(b) Verify that $([A]+[B])^T=[A]^T+[B]^T$

Answer:$([A]+[B])^T$= $\begin{bmatrix} 1 & 2 & 7 \\ 14 & 12 & 4 \\ 6 & 5 & -5 \end{bmatrix}$ $=[A]^T+[B]^T$.

(c) Verify that $([A][B])^T=[B]^T[A]^T$

Answer: $([A][B])^T$= $\begin{bmatrix} -12 & 107 & -66 \\ -11 & 93 & -30 \\ 13 & 51 & -54 \end{bmatrix}^T = \begin{bmatrix} -12 & -11 & 13 \\ 107 & 93 & 51 \\ -66 & -30 & -54 \end{bmatrix}$ $=[B]^T[A]^T$.

7. Given the following matrices

$$[A] = \begin{bmatrix} 2 & 7 & -5 \\ 8 & 9 & 7 \\ 13 & -5 & 6 \end{bmatrix} \text{ and } [B] = \begin{bmatrix} 3 & 8 & -2 \\ 5 & 13 & 0 \\ 14 & -7 & 6 \end{bmatrix}$$

calculate

(a) determinant of [A] and [B]
Answer: determinant of [A] =1264, determinant of [B] =428.

(b) determinant of $[A]^T$
Answer: determinant of $[A]^T$ = 1264.

(c) determinant of 7[A]
Answer: determinant of 7 [A] = $7^3\times$ determinant of [A]=433552.

8. Given the following matrix

$$[A] = \begin{bmatrix} 0 & 7 & 0 \\ 4 & 3 & 5 \\ 9 & -4 & -7 \end{bmatrix}$$

calculate determinant of [A] and of $[A]^T$
Answer: determinant of [A]= 511; determinant of $[A]^T$ = 511.

9. Solve the following matrix equation by using Gauss elimination method and MATLAB

$$\begin{bmatrix} 2187500 & -937500 & 0 \\ -937500 & 2187500 & -1250000 \\ 0 & -1250000 & 1250000 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 500 \end{Bmatrix}$$

Answer:
By Gauss elimination method, one can express the matrix equation as follows

$$\begin{aligned} 2.1875u_2 - 0.9375u_3 &= 0 \\ -0.9375u_2 + 2.1875u_3 - 1.25u_4 &= 0 \\ -1.25u_3 + 1.25u_4 &= 5\times 10^{-4} \end{aligned}$$

Summing the last two equations, one obtains

$$-0.9375u_2 + 0.9375u_3 = 5\times 10^{-4}$$

Adding this equation to the first one above, one has

$$1.25u_2 = 5\times 10^{-4}$$

which leads to $u_2 = 0.4\times 10^{-3}$. Substituting u_2 to the first equation, one gets $u_3 = 0.93\times 10^{-3}$, and substituting u_3 to the third equation, we gets $u_4 = 1.33\times 10^{-3}$.

By MATLAB $\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 2187500 & -937500 & 0 \\ -937500 & 2187500 & -1250000 \\ 0 & -1250000 & 1250000 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ 500 \end{Bmatrix}$

$$= 10^{-5}\times \begin{bmatrix} 0.080 & 0.080 & 0.080 \\ 0.080 & 0.187 & 0.187 \\ 0.080 & 0.187 & 0.267 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 500 \end{Bmatrix} = 10^{-3} \begin{Bmatrix} 0.40 \\ 0.93 \\ 1.33 \end{Bmatrix}.$$

10. Calculate the inverse of the following matrices:

(a)

$$[A] = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Answer: $[\mathrm{A}]^{-1} = \begin{bmatrix} 1/7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/6 \end{bmatrix} = \begin{bmatrix} 0.143 & 0 & 0 & 0 \\ 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0.33 & 0 \\ 0 & 0 & 0 & 0.167 \end{bmatrix}$.

(b)

$$[B] = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 7 & 2 \\ -1 & 3 & 4 \end{bmatrix}$$

Answer: $[\mathrm{B}]^{-1} = \begin{bmatrix} 0.917 & -0.042 & -0.208 \\ -0.583 & 0.208 & 0.042 \\ 0.667 & -0.167 & 0.167 \end{bmatrix}$.

(c)

$$[C] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

Answer: $[\mathrm{C}]^{-1} = \frac{1}{k_{11}k_{22}-k_{12}k_{21}} \begin{bmatrix} k_{22} & -k_{12} \\ -k_{21} & k_{11} \end{bmatrix}$.