Chapter 2

Basic Probability Theory

2.1 Solutions to Even-Numbered Exercises

Solution 2.2.

(a)

$$\begin{aligned}
\mathrm{pr}(\bar{A} \cap \bar{B}) &= \mathrm{pr}[\overline{A \cup B}] = 1 - \mathrm{pr}(A \cup B) \\
&= 1 - \mathrm{pr}(A) - \mathrm{pr}(B) + \mathrm{pr}(A \cap B) = 1 - \mathrm{pr}(A) - \mathrm{pr}(B) + \mathrm{pr}(A)\mathrm{pr}(B) \\
&= [1 - \mathrm{pr}(A)] - \mathrm{pr}(B)[1 - \mathrm{pr}(A)] \\
&= [1 - \mathrm{pr}(A)][1 - \mathrm{pr}(B)] = \mathrm{pr}(\bar{A})\mathrm{pr}(\bar{B}),
\end{aligned}$$

which completes the proof.

(b)

$$\mathrm{pr}(A \cap B) = \mathrm{pr}(A)\mathrm{pr}(B) = \mathrm{pr}(A)[1 - \mathrm{pr}(\bar{B})] = \mathrm{pr}(A) - \mathrm{pr}(A)\mathrm{pr}(\bar{B}).$$

Hence,

$$\mathrm{pr}(A)\mathrm{pr}(\bar{B}) = \mathrm{pr}(A) - \mathrm{pr}(A \cap B) = \mathrm{pr}(A \cap \bar{B}),$$

since $\mathrm{pr}(A) = \mathrm{pr}(A \cap B) + \mathrm{pr}(A \cap \bar{B})$.

The second result follows in a completely analogous manner.

Solution 2.4.

(a)

$$\mathrm{pr}(\text{lot is purchased}) = \frac{C_0^5 C_{10}^{95}}{C_{10}^{100}} = 0.5838.$$

(b) Let k denote the *smallest* number of defective kidney dialysis machines that can be in the lot of 100 machines so that the probability is no more than 0.20 that the hospital will purchase the entire lot of 100 machines. Then, we need to find smallest positive integer k such that

$$\frac{\mathrm{C}_0^k \mathrm{C}_{10}^{100-k}}{\mathrm{C}_{10}^{100}} = \frac{\mathrm{C}_{10}^{100-k}}{\mathrm{C}_{10}^{100}} \leq 0.20.$$

By computer, or by trial-and-error, we obtain $k = 15$.

Solution 2.6.

(a) pr(car door breaks during the 1,000-th trial)=pr[(car door does not break during any of the first 999 trials)$\cap$(car door breaks during the 1,000-th trial)]$= (0.9995)^{999}(0.0005) = 0.0003$.

(b) pr(car door breaks before the 1,001-th trial starts)=1-pr(car door does not break during the first 1,000 trials)$= 1 - (0.9995)^{1,000} = 1 - 0.6065 = 0.3935$.

(c) The assumption that the probability of the car door breaking does not change from trial-to-trial is probably an unrealistic one. As the number of trials increases, the probability of breakage would be expected to slowly increase, negating the assumption of mutually independent trials.

Solution 2.8. First,

$$\begin{aligned} \mathrm{pr(A)} &= \mathrm{pr(A|C)pr(C)} + \mathrm{pr(A|\bar{C})pr(\bar{C})} \\ &= (0.90)(0.01) + (0.06)(0.99) = 0.0684. \end{aligned}$$

And,

$$\begin{aligned} \mathrm{pr(B)} &= \mathrm{pr(B|C)pr(C)} + \mathrm{pr(B|\bar{C})pr(\bar{C})} \\ &= (0.95)(0.01) + (0.08)(0.99) = 0.0887. \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{pr(A \cap B)} &= \mathrm{pr(A \cap B|C)pr(C)} + \mathrm{pr(A \cap B|\bar{C})pr(\bar{C})} \\ &= \mathrm{pr(A|C)pr(B|C)pr(C)} + \mathrm{pr(A|\bar{C})pr(B|\bar{C})pr(\bar{C})} \\ &= (0.90)(0.95)(0.01) + (0.06)(0.08)(0.99) = 0.0134. \end{aligned}$$

Finally,

$$\mathrm{pr(A \cap B)} = 0.0134 \neq \mathrm{pr(A)pr(B)} = (0.0684)(0.0887) = 0.0061,$$

so that events A and B are unconditionally dependent.

This simple numerical example illustrates the general principle that conditional independence between two events does not imply unconditional independence between these same two events.

Solution 2.10. Now,

$$\begin{aligned}\mathrm{pr}(\mathrm{A}) &= 1 - \mathrm{pr}(\text{no heads among the } n \text{ tosses}) - \mathrm{pr}(\text{no tails among the } n \text{ tosses}) \\ &= 1 - \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n = 1 - \left(\frac{1}{2}\right)^{(n-1)}.\end{aligned}$$

And,

$$\begin{aligned}\mathrm{pr}(\mathrm{B}) &= \mathrm{pr}(\text{no tails among the } n \text{ tosses}) + \mathrm{pr}(\text{exactly one tail among the } n \text{ tosses}) \\ &= \left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n = (n+1)\left(\frac{1}{2}\right)^n.\end{aligned}$$

Also,

$$\mathrm{pr}(\mathrm{A} \cap \mathrm{B}) = \mathrm{pr}(\text{exactly one tail among the } n \text{ tosses}) = n\left(\frac{1}{2}\right)^n.$$

Finally, for A and B to be independent events, we require $\mathrm{pr}(\mathrm{A} \cap \mathrm{B}) = \mathrm{pr}(\mathrm{A})\mathrm{pr}(\mathrm{B})$, or

$$n\left(\frac{1}{2}\right)^n = \left[1 - \left(\frac{1}{2}\right)^{(n-1)}\right]\left[(n+1)\left(\frac{1}{2}\right)^n\right],$$

or equivalently, $\left(\frac{n}{n+1}\right) = 1 - \left(\frac{1}{2}\right)^{(n-1)}$, giving $n = 3$.

Solution 2.12.

(a) pr(all four are the same race) = pr(all four are C) + pr(all four are H) + pr(all four are A) + pr(all four are N) = $(0.45)^4 + (0.25)^4 + (0.20)^4 + (0.10)^4 = 0.0466$.

(b) pr[exactly 2 (and only 2) are the same race] = pr(2 C's and any two other races) + pr(2 H's and any two other races) + pr(2 A's and any two other races) + pr(2 N's and any two other races) = $6(0.45)^2(0.55)^2 + 6(0.25)^2(0.75)^2 + 6(0.20)^2(0.80)^2 + 6(0.10)^2(0.90)^2 = 0.7806$.

(c) pr(at least 2 are not Caucasian) $=\sum_{j=2}^{4} \mathrm{C}_j^4 (0.55)^j (0.45)^{4-j} = 0.7585$.

(d) Let $\mathrm{E}_1 \equiv$ "exactly 2 of 4 are C", and let $\mathrm{E}_2 \equiv$ "all 4 are each either C or H". So,

$$\begin{aligned}\mathrm{pr}(\mathrm{E}_1|\mathrm{E}_2) &= \frac{\mathrm{pr}(\mathrm{E}_1 \cap \mathrm{E}_2)}{\mathrm{pr}(\mathrm{E}_2)} = \frac{\mathrm{pr}[\text{2C's and 2H's}]}{\mathrm{pr}(\mathrm{E}_2)} \\ &= \frac{6(0.45)^2(0.25)^2}{(0.45+0.25)^4} = 0.3161.\end{aligned}$$

Solution 2.14.

(a) $\mathrm{pr}(\mathrm{C}|\mathrm{H} \cap \mathrm{D}) = 100/150 = 2/3$; or,

$$\mathrm{pr}(\mathrm{C}|\mathrm{H} \cap \mathrm{D}) = \frac{\mathrm{pr}(\mathrm{C} \cap \mathrm{H} \cap \mathrm{D})}{\mathrm{pr}(\mathrm{H} \cap \mathrm{D})} = \frac{100/300}{150/300} = \frac{2}{3}.$$

(b)

$$\begin{aligned}\mathrm{pr}(\mathrm{C} \cup \mathrm{D}|\bar{\mathrm{H}}) &= \mathrm{pr}(\mathrm{C}|\bar{\mathrm{H}}) + \mathrm{pr}(\mathrm{D}|\bar{\mathrm{H}}) - \mathrm{pr}(\mathrm{C} \cap \mathrm{D}|\bar{\mathrm{H}}) \\ &= \frac{50}{100} + \frac{60}{100} - \frac{40}{100} = 0.70;\end{aligned}$$

or,

$$\mathrm{pr}(\mathrm{C} \cup \mathrm{D}|\bar{\mathrm{H}}) = 1 - \mathrm{pr}(\bar{\mathrm{C}} \cap \bar{\mathrm{D}}|\bar{\mathrm{H}}) = 1 - \frac{30}{100} = 0.70.$$

(c) $\mathrm{pr}(\mathrm{H}|\bar{\mathrm{C}}) = 90/(90+50) = 9/14$; or,

$$\mathrm{pr}(\mathrm{H}|\bar{\mathrm{C}}) = \frac{\mathrm{pr}(\mathrm{H} \cap \bar{\mathrm{C}})}{\mathrm{pr}(\bar{\mathrm{C}})} = \frac{90/300}{140/300} = 9/14.$$

(d)

$$\mathrm{pr}(\overline{\mathrm{C} \cap \mathrm{H}}|\mathrm{D}) = 1 - \mathrm{pr}(\mathrm{C} \cap \mathrm{H}|\mathrm{D}) = 1 - \frac{\mathrm{pr}(\mathrm{C} \cap \mathrm{H} \cap \mathrm{D})}{\mathrm{pr}(\mathrm{D})} = 1 - \frac{100/300}{210/300} = \frac{11}{21}.$$

(e)

$$\begin{aligned}\mathrm{pr}(\mathrm{C} \cup \mathrm{D} \cup \mathrm{H}) &= \mathrm{pr}(\mathrm{C}) + \mathrm{pr}(\mathrm{D}) + \mathrm{pr}(\mathrm{H}) - \\ &\quad \mathrm{pr}(\mathrm{C} \cap \mathrm{D}) - \mathrm{pr}(\mathrm{C} \cap \mathrm{H}) - \mathrm{pr}(\mathrm{D} \cap \mathrm{H}) + \mathrm{pr}(\mathrm{C} \cap \mathrm{D} \cap \mathrm{H}) \\ &= \frac{160}{300} + \frac{210}{300} + \frac{200}{300} - \frac{140}{300} - \frac{110}{300} - \frac{150}{300} + \frac{100}{300} = 0.90;\end{aligned}$$

or,

$$\mathrm{pr}(\mathrm{C} \cup \mathrm{D} \cup \mathrm{H}) = 1 - \mathrm{pr}(\overline{\mathrm{C} \cup \mathrm{D} \cup \mathrm{H}}) = 1 - \mathrm{pr}(\bar{\mathrm{C}} \cap \bar{\mathrm{D}} \cap \bar{\mathrm{H}}) = 1 - \frac{30}{300} = 0.90.$$

(f)

$$\begin{aligned} \mathrm{pr}[\mathrm{C} \cup (\mathrm{H} \cap \mathrm{D})] &= \mathrm{pr}(\mathrm{C}) + \mathrm{pr}(\mathrm{H} \cap \mathrm{D}) - \mathrm{pr}(\mathrm{C} \cap \mathrm{H} \cap \mathrm{D}) \\ &= \frac{160}{300} + \frac{150}{300} - \frac{100}{300} = 0.70 \end{aligned}$$

Solution 2.16.

(a) $\pi^n + (1-\pi)^n$.

(b) $\pi^r(1-\pi)^{n-r}, 0 \leq r \leq n$.

(c) $\sum_{j=0}^{r} \mathrm{C}_j^n \pi^j (1-\pi)^{n-j}, 0 \leq r \leq n$.

(d) $(\pi^s)[\mathrm{C}_{r-s}^{n-s}\pi^{r-s}(1-\pi)^{n-r}] = \mathrm{C}_{r-s}^{n-s}\pi^r(1-\pi)^{n-r}, 0 \leq s \leq r \leq n$.

(e) $(\pi^s)[\sum_{j=r-s}^{n-s} \mathrm{C}_j^{n-s}\pi^j(1-\pi)^{(n-s)-j}] = \sum_{j=r-s}^{n-s} \mathrm{C}_j^{n-s}\pi^{j+s}(1-\pi)^{(n-s)-j}$, $0 \leq s \leq r \leq n$.

Solution 2.18.

(a) First,

$$\begin{aligned} \mathrm{pr}(\mathrm{A} \cap \mathrm{B}|\mathrm{C}) &= \mathrm{pr}(\mathrm{A}|\mathrm{C})\mathrm{pr}(\mathrm{B}|\mathrm{C}) \Leftrightarrow \frac{\mathrm{pr}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{C})} = \mathrm{pr}(\mathrm{A}|\mathrm{C}) \left[\frac{\mathrm{pr}(\mathrm{B} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{C})}\right] \\ &\Leftrightarrow \frac{\mathrm{pr}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{B} \cap \mathrm{C})} = \mathrm{pr}(\mathrm{A}|\mathrm{C}) \Leftrightarrow \mathrm{pr}(\mathrm{A}|\mathrm{B} \cap \mathrm{C}) = \mathrm{pr}(\mathrm{A}|\mathrm{C}). \end{aligned}$$

And,

$$\begin{aligned} \mathrm{pr}(\mathrm{A}|\mathrm{B} \cap \mathrm{C}) &= \mathrm{pr}(\mathrm{A}|\mathrm{C}) \Leftrightarrow \frac{\mathrm{pr}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{B} \cap \mathrm{C})} = \frac{\mathrm{pr}(\mathrm{A} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{C})} \\ &\Leftrightarrow \frac{\mathrm{pr}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{A} \cap \mathrm{C})} = \frac{\mathrm{pr}(\mathrm{B} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{C})} \Leftrightarrow \mathrm{pr}(\mathrm{B}|\mathrm{A} \cap \mathrm{C}) = \mathrm{pr}(\mathrm{B}|\mathrm{C}), \end{aligned}$$

which completes the proof that the three equalities are equivalent.

(b) For $i = 1, 2, \ldots, 6$, let E_i be the event that "the number i is rolled"; clearly, $\mathrm{pr}(\mathrm{E}_i) = 1/6$ and the events $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_6$ are pairwise mutually exclusive. Then,

$$\mathrm{pr}(\mathrm{A}|\mathrm{B} \cap \mathrm{C}) = \frac{\mathrm{pr}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{B} \cap \mathrm{C})} = \frac{\mathrm{pr}(\mathrm{E}_6)}{\mathrm{pr}(\mathrm{E}_5 \cup \mathrm{E}_6)} = \frac{1/6}{2/6} = \frac{1}{2},$$

and

$$\mathrm{pr}(\mathrm{A}|\mathrm{C}) = \frac{\mathrm{pr}(\mathrm{A} \cap \mathrm{C})}{\mathrm{pr}(\mathrm{C})} = \frac{\mathrm{pr}(\mathrm{E}_6)}{\mathrm{pr}(\mathrm{E}_5 \cup \mathrm{E}_6)} = \frac{1/6}{2/6} = \frac{1}{2},$$

so that events A and B are conditionally independent given that event C has occurred.

However,

$$\mathrm{pr}(\mathrm{A}|\mathrm{B}\cap\bar{\mathrm{C}}) = \frac{\mathrm{pr}(\mathrm{A}\cap\mathrm{B}\cap\bar{\mathrm{C}})}{\mathrm{pr}(\mathrm{B}\cap\bar{\mathrm{C}})} = \frac{\mathrm{pr}(\mathrm{E}_4)}{\mathrm{pr}(\mathrm{E}_4)} = 1,$$

and

$$\mathrm{pr}(\mathrm{A}|\bar{\mathrm{C}}) = \frac{\mathrm{pr}(\mathrm{A}\cap\bar{\mathrm{C}})}{\mathrm{pr}(\bar{\mathrm{C}})} = \frac{\mathrm{pr}(\mathrm{E}_2\cup\mathrm{E}_4)}{\mathrm{pr}(\mathrm{E}_1\cup\mathrm{E}_2\cup\mathrm{E}_3\cup\mathrm{E}_4)} = \frac{2/6}{4/6} = \frac{1}{2},$$

so that events A and B are conditionally dependent given that event C has *not* occurred.

Solution 2.20. Let A be the event that Joe gets at least one hit during each of the 13 games in which he had 3 official at bats, let B be the event that Joe gets at least one hit during each of the 31 games in which he had 4 official at bats, and let C be the event that Joe gets at least one hit during each of the 12 games in which he had 5 official at bats.

Now, under the stated assumptions, the probability that Joe does *not* get a hit in a game where he has 3 official at bats is equal to $(1-0.408)^3 = (0.592)^3 = 0.2075$, so that

$$\mathrm{pr}(\mathrm{A}) = (1-0.2075)^{13} = 0.0486.$$

Using this same strategy to compute pr(B) and pr(C), we have

$$\begin{aligned} \pi &= \mathrm{pr}\left(\mathrm{A}\cap\mathrm{B}\cap\mathrm{C}\right) = \mathrm{pr}(\mathrm{A})\mathrm{pr}(\mathrm{B})\mathrm{pr}(\mathrm{C}) \\ &= (0.0486)\left[1-(0.592)^4\right]^{31}\left[1-(0.592)^5\right]^{12} \\ &= (0.0486)(0.0172)(0.4042) = 0.0003. \end{aligned}$$

This approximate calculation provides strong evidence for why a hitting streak of 56 games has occurred only once during the entire history of major league baseball.

Solution 2.22. Let H be the event that heads is observed on the coin that is randomly selected, and let A be the event that the other side of this coin is also heads. Further, let C_1 be the event that the coin selected has heads on both sides, and let C_2 be the event that the coin selected has heads on one side and tails on the other.

So, we wish to compute the numerical value of $\mathrm{pr}(\mathrm{A}|\mathrm{H}) = \mathrm{pr}(\mathrm{A}\cap\mathrm{H})/\mathrm{pr}(\mathrm{H})$. Now,

$$\begin{aligned}
\mathrm{pr}(\mathrm{A}\cap\mathrm{H}) &= \mathrm{pr}(\mathrm{A}\cap\mathrm{H}|\mathrm{C}_1)\mathrm{pr}(\mathrm{C}_1) + \mathrm{pr}(\mathrm{A}\cap\mathrm{H}|\mathrm{C}_2)\mathrm{pr}(\mathrm{C}_2) \\
&= (1)\left(\frac{1}{2}\right) + (0)\left(\frac{1}{2}\right) = \frac{1}{2}.
\end{aligned}$$

And,

$$\begin{aligned}
\mathrm{pr}(\mathrm{H}) &= \mathrm{pr}(\mathrm{H}|\mathrm{C}_1)\mathrm{pr}(\mathrm{C}_1) + \mathrm{pr}(\mathrm{H}|\mathrm{C}_2)\mathrm{pr}(\mathrm{C}_2) \\
&= (1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}.
\end{aligned}$$

Finally,

$$\mathrm{pr}(\mathrm{A}|\mathrm{H}) = \frac{\mathrm{pr}(\mathrm{A}\cap\mathrm{H})}{\mathrm{pr}(\mathrm{H})} = \frac{(1/2)}{(3/4)} = \frac{2}{3}.$$

Solution 2.24.

(a) For $i = 1, 2, 3$, let A_i be the event that this randomly chosen adult resident plays course #i. Then, if C is the event that this randomly chosen adult plays none of these three courses, we have

$$\begin{aligned}
\mathrm{pr}(\mathrm{C}) &= \mathrm{pr}\left(\overline{\mathrm{A}_1}\cap\overline{\mathrm{A}_2}\cap\overline{\mathrm{A}_3}\right) = \mathrm{pr}\left(\overline{\cup_{i=1}^{3}\mathrm{A}_i}\right) \\
&= 1 - \mathrm{pr}\left(\cup_{i=1}^{3}\mathrm{A}_i\right) \\
&= 1 - \mathrm{pr}(\mathrm{A}_1) - \mathrm{pr}(\mathrm{A}_2) - \mathrm{pr}(\mathrm{A}_3) + \mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2) \\
&+ \mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_3) + \mathrm{pr}(\mathrm{A}_2\cap\mathrm{A}_3) - \mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2\cap\mathrm{A}_3) \\
&= 1 - 0.18 - 0.15 - 0.12 + 0.09 + 0.06 + 0.05 - 0.02 \\
&= 0.73.
\end{aligned}$$

(b) Now, let B be the event that this randomly chosen adult resident plays exactly one of these three courses, and let D be the event that this randomly chosen resident plays at least two of these three courses. Then,

$$\mathrm{pr}(\mathrm{B}) = 1 - \mathrm{pr}(\mathrm{C}) - \mathrm{pr}(\mathrm{D}) = 1 - 0.73 - \mathrm{pr}(\mathrm{D}).$$

Now,

$$\begin{aligned}
\mathrm{pr}(\mathrm{D}) &= \mathrm{pr}\left[(\mathrm{A}_1\cap\mathrm{A}_2)\cup(\mathrm{A}_1\cap\mathrm{A}_3)\cup(\mathrm{A}_2\cap\mathrm{A}_3)\right] \\
&= \mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2) + \mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_3) + \mathrm{pr}(\mathrm{A}_2\cap\mathrm{A}_3) \\
&- 3\mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2\cap\mathrm{A}_3) + \mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2\cap\mathrm{A}_3) \\
&= 0.09 + 0.06 + 0.05 - 2(0.02) = 0.16.
\end{aligned}$$

Finally, $\mathrm{pr}(\mathrm{B}) = 1 - 0.73 - 0.16 = 0.11$.

(c) Let E be the event that this randomly chosen adult resident plays only #1 and #2. Then,

$$\begin{aligned}
\mathrm{pr}(\mathrm{E}|\bar{\mathrm{C}}) &= \frac{\mathrm{pr}(\mathrm{E}\cap\bar{\mathrm{C}})}{\mathrm{pr}(\bar{\mathrm{C}})} \\
&= \frac{\mathrm{pr}(\bar{\mathrm{C}}|\mathrm{E})\mathrm{pr}(\mathrm{E})}{\bar{\mathrm{C}}} \\
&= \frac{(1)\mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2\cap\overline{\mathrm{A}_3})}{(1-0.73)}.
\end{aligned}$$

Now, since

$$\mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2) = \mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2\cap\mathrm{A}_3) + \mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2\cap\overline{\mathrm{A}_3}),$$

it follows that $\mathrm{pr}(\mathrm{A}_1\cap\mathrm{A}_2\cap\overline{\mathrm{A}_3}) = 0.09 - 0.02 = 0.07$.

Finally, $\mathrm{pr}(\mathrm{E}|\bar{\mathrm{C}}) = (1)(0.07)/0.27 = 0.26$.

Solution 2.26. The probability that no two of these k dice show the same number (i.e., that all k numbers showing are different) is equal to

$$\alpha_k = \left(\frac{6}{6}\right)\left(\frac{5}{6}\right)\left(\frac{4}{6}\right)\cdots\left(\frac{6-k+1}{6}\right), 2 \le k \le 6.$$

And, the probability that no two of these k dice show the same number *and* that one of these k dice shows the number 6 is equal to

$$\beta_k = k\left[\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{4}{6}\right)\cdots\left(\frac{6-k+1}{6}\right)\right], 2 \le k \le 6.$$

Thus,

$$\theta_k = \frac{\beta_k}{\alpha_k} = \frac{k}{6}, 2 \le k \le 6.$$

Solution 2.28.

(a) Since each of the k balls can end up in any one of the n urns, the total number of possible configurations of k balls and n urns is n^k, and each of these possible configurations has probability n^{-k} of occurring. And, among these n^k equally likely configurations, there are $n(n-1)\cdots(n-k+1) = n!/(n-k)!$ configurations for which no urn contains more than one ball. Hence,

$$\theta(n,k) = \frac{n(n-1)\cdots(n-k+1)}{n^k} = \frac{n!}{(n-k)!n^k}, 1 \le k \le n.$$

(b) If we think of the 12 months as 12 urns and the 5 people as 5 balls, then

$$\begin{aligned}\gamma &= 1-\theta(12,5) = 1-\frac{(12)!}{(12-5)!(12)^5}\\ &= 1-0.382 = 0.618.\end{aligned}$$

Solution 2.30*. For $1 \leq i < j$, the event $\mathrm{A}(i,j)$ of interest can be written as
$\mathrm{A}(i,j) = \cap_{k=1}^{4}\mathrm{A}_k$, where A_1 is the event that the first $(i-1)$ tosses do not produce either the number 1 or the number 2, where A_2 is the event that the i-th toss produces the number 1, where A_3 is the event that the next $[(j-i)-1]$ tosses do not produce the number 2, and event A_4 is the event that the j-th toss produces the number 2. Since the events $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$, and A_4 are mutually independent, it follows that

$$\begin{aligned}\mathrm{pr}[\mathrm{A}(i,j)] &= \mathrm{pr}\left(\cap_{k=1}^{4}\mathrm{A}_k\right) = \prod_{k=1}^{4}\mathrm{pr}(\mathrm{A}_k)\\ &= \left[\left(\frac{4}{6}\right)^{(i-1)}\right]\left(\frac{1}{6}\right)\left[\left(\frac{5}{6}\right)^{(j-i-1)}\right]\left(\frac{1}{6}\right)\\ &= \frac{1}{20}\left(\frac{4}{5}\right)^{i}\left(\frac{5}{6}\right)^{j}.\end{aligned}$$

By symmetry, it follows that

$$\mathrm{pr}[\mathrm{A}(j,i)] = \frac{1}{20}\left(\frac{4}{5}\right)^{j}\left(\frac{5}{6}\right)^{i}.$$

Thus, the probability of either of the two scenarios $i < j$ and $j < i$ can be written succinctly as

$$\frac{1}{20}\left(\frac{4}{5}\right)^{\min\{i,j\}}\left(\frac{5}{6}\right)^{\max\{i,j\}}.$$

Solution 2.32*. For $i = 1, 2, \ldots, 6$, let E_i be the event that the number i appears on exactly two of the three dice when the experiment is conducted. Then,

$$\begin{aligned}\mathrm{pr}(\mathrm{A}) &= \mathrm{pr}\left(\cup_{i=1}^{6}\mathrm{E}_i\right) = \sum_{i=1}^{6}\mathrm{pr}(\mathrm{E}_i)\\ &= 6\left[3\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)\right] = \frac{5}{12} = 0.4167.\end{aligned}$$

Let C_n be the event that event A occurs at least twice during n repetitions of the experiment. Then,

$$\begin{aligned}\mathrm{pr}(\mathrm{C}_n) &= 1 - \mathrm{pr}\left(\overline{\mathrm{C}_n}\right) \\ &= 1 - \mathrm{pr}(\text{event A occurs at most once during } n \text{ repetitions of the experiment}) \\ &= 1 - \left[(0.5833)^n + n(0.5833)^{n-1}(0.4167)\right].\end{aligned}$$

By trial-and-error, the smallest value of n, say n^*, such that

$$\mathrm{pr}(\mathrm{C}_n) = 1 - \left[(0.5833)^n + n(0.5833)^{n-1}(0.4167)\right] \geq 0.90$$

is equal to $n^* = 8$.

Solution 2.34*. First, note that $\theta_2 = \alpha$ given that the first repetition results in outcome A, and that $\theta_2 = (1-\beta)$ given that the first repetition results in outcome B.

Now,

$$\theta_3 = \alpha\theta_2 + (1-\beta)(1-\theta_2) = k_0 + k_1\theta_2,$$

where $k_0 = (1-\beta)$ and $k_1 = (\alpha + \beta - 1)$.

Next,

$$\begin{aligned}\theta_4 &= \alpha\theta_3 + (1-\beta)(1-\theta_3) = k_0 + k_1\theta_3 \\ &= k_0 + k_1(k_0 + k_1\theta_2) \\ &= k_0(1+k_1) + k_1^2\theta_2.\end{aligned}$$

Using a similar strategy, we have

$$\begin{aligned}\theta_5 &= \alpha\theta_4 + (1-\beta)(1-\theta_4) = k_0 + k_1\theta_4 \\ &= k_0 + k_1\left[k_0(1+k_1) + k_1^2\theta_2\right] \\ &= k_0(1 + k_1 + k_1^2) + k_1^3\theta_2.\end{aligned}$$

So, in general, for $n = 3, 4, \ldots$, we have

$$\begin{aligned}
\theta_n &= k_0 \left(\sum_{j=0}^{n-3} k_1^j \right) + k_1^{n-2}\theta_2 \\
&= \frac{k_0 \left(1 - k_1^{n-2}\right)}{(1 - k_1)} + k_1^{n-2}\theta_2,
\end{aligned}$$

where $k_0 = (1 - \beta), k_1 = (\alpha + \beta - 1)$, and where θ_2 equals α if the first repetition of the experiment results in outcome A and equals $(1 - \beta)$ if the first repetition of the experiment results in outcome B.

Finally, since $0 < k_1 < 1$, we have

$$\begin{aligned}
\lim_{n\to\infty}\theta_n &= \frac{k_0}{(1 - k_1)} \\
&= \frac{(1 - \beta)}{(2 - \alpha - \beta)} = \frac{(1 - \beta)}{(1 - \alpha) + (1 - \beta)}.
\end{aligned}$$

Note that this limiting value is the same regardless of the outcome on the first repetition of the experiment.

Solution 2.36*.

(a) In words, the event that Player A is ruined with x dollars remaining is the union of two mutually exclusive events, namely, the event that Player A wins the next game and then is ruined with $(x + 1)$ dollars remaining and the event that Player A loses the next game and then is ruined with $(x - 1)$ dollars remaining. This leads to the desired difference equation

$$\theta_x = \pi\theta_{x+1} + (1 - \pi)\theta_{x-1}, x = 1, 2, \ldots, (a + b - 1);$$

clearly, $\theta_0 = 1$ since Player A has no money, and $\theta_{a+b} = 0$ since Player A has all of Player B's money.

(b) Using the difference equation given in part (a), we have

$$\begin{aligned}
\alpha+\beta\left(\frac{1-\pi}{\pi}\right)^{x} &= \pi\left[\alpha+\beta\left(\frac{1-\pi}{\pi}\right)^{x+1}\right]+(1-\pi)\left[\alpha+\beta\left(\frac{1-\pi}{\pi}\right)^{x-1}\right] \\
&= \alpha+\beta\left[\frac{(1-\pi)^{x+1}}{\pi^{x}}+\frac{(1-\pi)^{x}}{\pi^{x-1}}\right] \\
&= \alpha+\beta\left(\frac{1-\pi}{\pi}\right)^{x}[(1-\pi)+\pi] \\
&= \alpha+\beta\left(\frac{1-\pi}{\pi}\right)^{x}.
\end{aligned}$$

(c) Now,

$$\theta_0 = 1 = \alpha+\beta\left(\frac{1-\pi}{\pi}\right)^{0} = \alpha+\beta,$$

so that $\alpha = 1-\beta$.

And,

$$\theta_{a+b} = 0 = \alpha+\beta\left(\frac{1-\pi}{\pi}\right)^{a+b} = (1-\beta)+\beta\left(\frac{1-\pi}{\pi}\right)^{a+b}$$

so that

$$\beta = \left[1-\left(\frac{1-\pi}{\pi}\right)^{a+b}\right]^{-1},$$

and

$$\alpha = 1-\left[1-\left(\frac{1-\pi}{\pi}\right)^{a+b}\right]^{-1}.$$

Using these expressions for α and β, we obtain

$$\begin{aligned}
\theta_x &= 1-\left[1-\left(\frac{1-\pi}{\pi}\right)^{a+b}\right]^{-1}+\left[1-\left(\frac{1-\pi}{\pi}\right)^{a+b}\right]^{-1}\left(\frac{1-\pi}{\pi}\right)^{x} \\
&= \frac{\left(\frac{1-\pi}{\pi}\right)^{x}-\left(\frac{1-\pi}{\pi}\right)^{a+b}}{1-\left(\frac{1-\pi}{\pi}\right)^{a+b}}.
\end{aligned}$$

(d) Based on the expression for θ_x derived in part (c), it follows that the probability that Player A is ruined when Player A begins the competition with a dollars is

$$\theta_a = \frac{\left(\frac{1-\pi}{\pi}\right)^{a}-\left(\frac{1-\pi}{\pi}\right)^{a+b}}{1-\left(\frac{1-\pi}{\pi}\right)^{a+b}}.$$

And, by symmetry,

$$\begin{aligned}
\text{pr(Player B is ruined)} &= \frac{\left(\frac{\pi}{1-\pi}\right)^{b} - \left(\frac{\pi}{1-\pi}\right)^{a+b}}{1-\left(\frac{\pi}{1-\pi}\right)^{a+b}} \\
&= \frac{1-\left(\frac{1-\pi}{\pi}\right)^{a}}{1-\left(\frac{1-\pi}{\pi}\right)^{a+b}} = (1-\theta_a).
\end{aligned}$$

Since pr(Player A is ruined)+pr(Player B is ruined)=1, it is certain that either Player A or Player B will eventually lose all of his or her money.

When $\pi = 1/2$, one can use L'Hôpital's Rule to show that $\theta_a = b/(a+b)$ and $(1-\theta_a) = a/(a+b)$.

(e) If $\pi \leq 0.50$, so that *the house* has no worse than an even chance of winning each game, then $\lim_{b\to\infty}\theta_a = 1$, so that Player A will eventually lose all of his or her money if Player A continues to play. If $\pi > 0.50$, then $\lim_{b\to\infty}\theta_a = \left(\frac{1-\pi}{\pi}\right)^a$. As a word of caution, π is always less than 0.50 for any casino game.

Solution 2.38*.

(a) Let A_{xy} be the event that a person matches winning pair (x, y). Then,

$$\pi_{x0} = \mathrm{pr}(\mathrm{A}_{x0}) = \left[\frac{\mathrm{C}_x^5 \mathrm{C}_{5-x}^{51}}{\mathrm{C}_5^{56}}\right]\left(\frac{45}{46}\right), x = 3, 4, 5;$$

and,

$$\pi_{x1} = \mathrm{pr}(\mathrm{A}_{x1}) = \left[\frac{\mathrm{C}_x^5 \mathrm{C}_{5-x}^{51}}{\mathrm{C}_5^{56}}\right]\left(\frac{1}{46}\right), x = 0, 1, 2, 3, 4, 5.$$

Then, it follows directly that $\pi_{30} = 0.0033, \pi_{40} = 0.0001, \pi_{50} = 2.5610\text{x}10^{-7}$, $\pi_{01} = 0.0134, \pi_{11} = 0.0071, \pi_{21} = 0.0012, \pi_{31} = 0.0001, \pi_{41} = 1.4512\text{x}10^{-6}$, and $\pi_{51} = 5.6911\text{x}10^{-9}$.

(b) Let θ be the overall probability of winning if a person plays this Mega Millions lottery game one time. Then,

$$\begin{aligned}
\theta &= 1 - \mathrm{pr}(\mathrm{A}_{00}) - \mathrm{pr}(\mathrm{A}_{10}) - \mathrm{pr}(\mathrm{A}_{20}) \\
&= 1 - 0.9749 = 0.0251.
\end{aligned}$$

(c) We want to choose the smallest positive integer value of n, say n^*, that satisfies the inequality

$$1 - (1-0.0251)^n = 1 - (0.9749)^n \geq 0.90,$$

or equivalently that

$$n\ln(0.9749) \leq \ln(0.10).$$

It then follows easily that $n^* = 91$.

Solution 2.40*.

(a) There are two possible equally likely outcomes (namely, "evens" and "odds") for each game, and the total number of games played is equal to $\mathrm{C}_2^k = k(k-1)/2$. So, there are $2^{k(k-1)/2}$ total possible outcomes for all C_2^k games that are played. Of these $2^{k(k-1)/2}$ outcomes, there are $k!$ outcomes that produce the outcome of interest. So, since all outcomes are equally likely to occur, the desired probability is equal to

$$\theta_k = \frac{k!}{2^{k(k-1)/2}}.$$

Note that $\theta_2 = 1$, as expected, and that $\theta_6 = 6!/2^{15} = 0.0220$.

(b) Appealing to Stirling's approximation to $k!$ for large k, we have

$$\begin{aligned} \theta_k &\approx \frac{\sqrt{2\pi k}\left(\frac{k}{\mathrm{e}}\right)^k}{2^{k(k-1)/2}} \\ &\approx \frac{\sqrt{2\pi k}\left(\frac{k}{\mathrm{e}}\right)^k}{2^{k^2/2}} \\ &\approx \left(\frac{\sqrt{2\pi k}}{\mathrm{e}^k}\right)\left(\frac{k}{2^{k/2}}\right)^k, \end{aligned}$$

which converges to the value 0 as $k \to \infty$.

Solution 2.42*.

(a) Let A be the event that the gambler has a dollars to bet, let B be the event that the gambler accumulates b dollars, and let W be the event that the gambler wins the next play of the game. Then, we have

$$\begin{aligned} \theta_a &= \mathrm{pr(B|A)} = \mathrm{pr(B \cap W|A)} + \mathrm{pr(B \cap \overline{W}|A)} \\ &= \mathrm{pr(W|A)pr(B|W \cap A)} + \mathrm{pr(\overline{W}|A)pr(B|\overline{W} \cap A)} \\ &= \mathrm{pr(W)pr(B|W \cap A)} + \mathrm{pr(\overline{W})pr(B|\overline{W} \cap A)} \\ &= \pi\theta_{a+1} + (1-\pi)\theta_{a-1}, a = 1, 2, \ldots, (b-1). \end{aligned}$$

(b) Using direct substitution and simple algebra, it is straightforward to show that the stated solutions satisfy the difference equations given in part (a).

(c) For Scenario I, we have

$$\theta_{100} = \frac{100}{10,000} = 0.01.$$

For Scenario II, we have

$$\theta_{100} = \frac{\left(\frac{0.52}{0.48}\right)^{100} - 1}{\left(\frac{0.52}{0.48}\right)^{200} - 1} = 0.0003.$$

This result is clearly counterintuitive, since it is over $33(\approx 0.01/0.0003)$ times more likely that the gambler will accumulate b dollars under Scenario I than under Scenario II.