Problem Set 2.1

In Problems 1–4,

(*a*) Perform z_1 / z_2 and express the result in rectangular form,

(*b*) Verify that $\left|z_1 / z_2\right| = \left|z_1\right| / \left|z_2\right|$,

(*c*) Repeat Part (*a*) in MATLAB.

1. $\dfrac{-3-j}{2j}$

Solution

(*a*) $\dfrac{-3-j}{2j} \cdot \dfrac{-j}{-j} = \dfrac{-1+3j}{2} = \dfrac{-1}{2} + \dfrac{3}{2} j$

(*b*) $\left|\dfrac{-1}{2} + \dfrac{3}{2} j\right| = \sqrt{\left(\dfrac{-1}{2}\right)^2 + \left(\dfrac{3}{2}\right)^2} = \sqrt{\dfrac{10}{4}} = \dfrac{\sqrt{10}}{2}$, $\dfrac{|-3-j|}{|2j|} = \dfrac{\sqrt{10}}{2}$

(*c*)

```
>> z1 = -3-j; z2 = 2*j; z1/z2
ans =
  -0.5000 + 1.5000i
```

2. $\dfrac{2+j}{1-2j}$

Solution

(*a*) $\dfrac{2+j}{1-2j} = j$

(*b*) $|j| = 1$, $\dfrac{|2+j|}{|1-2j|} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1$

(*c*)

```
>> z1 = 2+j; z2 = 1-2*j; z1/z2
ans =
        0 + 1.0000i
```

3. $\dfrac{-3j}{2+3j}$

Solution

(*a*) $\dfrac{-3j}{2+3j} = \dfrac{-9}{13} - \dfrac{6}{13} j$

(*b*) $\left|\dfrac{-9}{13} - \dfrac{6}{13} j\right| = \sqrt{\dfrac{117}{169}} = \dfrac{3\sqrt{13}}{13}$, $\dfrac{|-3j|}{|2+3j|} = \dfrac{3}{\sqrt{13}} = \dfrac{3\sqrt{13}}{13}$

(*c*)

```
>> z1 = -3*j; z2 = 2+3*j; z1/z2
ans =
  -0.6923 - 0.4615i
```

4. $\dfrac{4}{-4+3j}$

Solution

$(a)\ \dfrac{4}{-4+3j}\cdot\dfrac{-4-3j}{-4-3j}=\dfrac{-16-12j}{25}=\dfrac{-16}{25}-\dfrac{12}{25}j$

$(b)\ \left|\dfrac{-16}{25}-\dfrac{12}{25}j\right|=\sqrt{\left(\dfrac{-16}{25}\right)^2+\left(\dfrac{-12}{25}\right)^2}=\sqrt{\dfrac{400}{625}}=\dfrac{4}{5}\quad,\qquad \dfrac{|4|}{|-4+3j|}=\dfrac{4}{5}$

(c)

```
>> z1 = 4; z2 = -4+3*j; z1/z2
ans =
  -0.6400 - 0.4800i
```

In Problems 5–8 express each complex number in its polar form.

5. $-\sqrt{3}-3j$

Solution

To calculate phase, we first find $\tan^{-1}\sqrt{3}=\frac{1}{3}\pi$. Since $-\sqrt{3}-3j$ is located in the 3rd quadrant, the phase is taken as either $\pi+\frac{1}{3}\pi$ in the positive sense (counterclockwise) or $\frac{1}{2}\pi+\frac{1}{6}\pi=\frac{2}{3}\pi$ in the negative (clockwise). In summary,

$$-\sqrt{3}-3j \overset{\text{3rd quadrant}}{=} 2\sqrt{3}\ e^{-(2\pi/3)j}.$$

6. $1-\frac{3}{2}j$

Solution

$$1-\frac{3}{2}j \overset{\text{4th quadrant}}{=} \frac{\sqrt{13}}{2}e^{-0.9828j}$$

7. $3+j\sqrt{3}$

Solution

$$3+j\sqrt{3} \overset{\text{1st quadrant}}{=} 2\sqrt{3}\ e^{j\pi/6}$$

8. $-1+\frac{1}{2}j$

Solution

$$-1+\frac{1}{2}j \overset{\text{2nd quadrant}}{=} \frac{\sqrt{5}}{2}e^{2.6779j}$$

In Problems 9 – 16 perform using polar form and express the result in rectangular form.

9. $\dfrac{3+2j}{-1+3j}$

Solution

$$\frac{3+2j}{-1+3j}=\frac{\sqrt{13}e^{0.5880j}}{\sqrt{10}e^{1.8925j}}=\frac{\sqrt{13}}{\sqrt{10}}e^{-1.3045j}=0.3-1.1j$$

10. $\dfrac{\sqrt{3}+3j}{3-j\sqrt{3}}$

Solution

$$\frac{\sqrt{3}+3j}{3-j\sqrt{3}}=\frac{2\sqrt{3}\ e^{(\pi/3)j}}{2\sqrt{3}\ e^{-(\pi/6)j}}=e^{(\pi/2)j}=j$$

11. $\dfrac{3-5j}{2j}$

Solution

$$\frac{3-5j}{2j}=\frac{\sqrt{34}e^{-1.0304j}}{2e^{(\pi/2)j}}=\frac{\sqrt{34}}{2}e^{-2.6012j}=-2.5-1.5j$$

12. $\dfrac{3j}{1-j}$

Solution

$$\frac{3j}{1-j}=\frac{3e^{(\pi/2)j}}{\sqrt{2}e^{-(\pi/4)j}}=\frac{3}{\sqrt{2}}e^{(3\pi/4)j}=-1.5+1.5j$$

13. $(4+3j)^3$

Solution

$$\left(4+3i\right)^3=\left[5e^{j(0.6435)}\right]^3=5^3e^{j(1.9305)}=-44+117j$$

14. $(0.9511+0.3090j)^{10}$

Solution

$$(0.9511+0.3090j)^{10}=\left[e^{j(\pi/10)}\right]^{10}=e^{j\pi}=-1$$

15. $\dfrac{(1+3j)^3}{(-1+2j)^2}$

Solution

$$\frac{(1+3j)^3}{(-1+2j)^2}=\frac{\left(\sqrt{10}\ e^{1.2490j}\right)^3}{\left(\sqrt{5}\ e^{2.0344j}\right)^2}=\frac{10\sqrt{10}\ e^{3.7470j}}{5e^{4.0688j}}=2\sqrt{10}e^{-0.3218j}=6-2j$$

16. $\dfrac{5j}{(1+4j)^3}$

Solution

$$\frac{5j}{(1+4j)^3}=\frac{5e^{(\pi/2)j}}{\left(\sqrt{17}e^{1.3258j}\right)^3}=\frac{5}{17\sqrt{17}}e^{-2.4066j}=-0.0529-0.0478j$$

In Problems 17–20, find all possible values for each expression.

17. $(-1)^{1/6}$

Solution

The goal is to find $w=\sqrt[6]{z}$ where $z=-1$. Noting that $z=-1$ is located on the negative real axis, one unit from the origin, we have $r=1$ and $\theta=\pi$, hence $z=-1=e^{j\pi}$. Then,

$$\sqrt[6]{-1}=\sqrt[6]{1}\left(\cos\frac{\pi+2k\pi}{6}+j\sin\frac{\pi+2k\pi}{6}\right)\ ,\qquad k=0,1,2,3,4,5$$

Therefore, the six roots are $\pm j,\ \frac{\sqrt{3}}{2}\pm\frac{1}{2}j,\ -\frac{\sqrt{3}}{2}\pm\frac{1}{2}j$, located on the unit circle, the vertices of a six-sided polygon.

18. $(-1+j)^{1/3}$

Solution

$$(-1+j)^{1/3}=\sqrt{2}^{1/3}\left[\cos\frac{\frac{3}{4}\pi+2k\pi}{3}+j\sin\frac{\frac{3}{4}\pi+2k\pi}{3}\right],\ k=0,1,2$$

The three values are obtained as $0.7937+0.7937j$, $-1.0842+0.2905j$, $0.2905-1.0842j$.

19. $(\sqrt{3}-3j)^{1/2}$

Solution

$$(\sqrt{3}-3j)^{1/2}=(2\sqrt{3})^{1/2}\left[\cos\frac{-\frac{1}{3}\pi+2k\pi}{2}+j\sin\frac{-\frac{1}{3}\pi+2k\pi}{2}\right],\quad k=0,1$$

The two values are $1.6119-0.9306j$, $-1.6119+0.9306j$.

20. $\sqrt{1+i\sqrt{3}}$

Solution

The goal is to find $w=\sqrt{z}$ where $z=1+i\sqrt{3}$. Since $z=1+i\sqrt{3}=2e^{j\pi/3}$, we have

$$\sqrt{1+i\sqrt{3}}=\sqrt{2}\left(\cos\frac{\frac{1}{3}\pi+2k\pi}{2}+j\sin\frac{\frac{1}{3}\pi+2k\pi}{2}\right)\ ,\qquad k=0,1$$

Therefore, the two roots are $\pm\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$, located on the circle of radius $\sqrt{2}$ and centered at the origin.

Problem Set 2.2

In Problems 1–10 solve the initial-value problem.

1. $\dot{x}+x=\sin t$, $x(0)=-1$

Solution

Since $g(t)=1$, we have $h(t)=t$ and

$$x(t)=e^{-t}\left[\int e^{t}\sin t\,dt+c\right]=e^{-t}\left[\tfrac{1}{2}e^{t}(\sin t-\cos t)+c\right]=\tfrac{1}{2}(\sin t-\cos t)+ce^{-t}$$

Using the initial condition $c=-\frac{1}{2}$ so that $x(t)=\frac{1}{2}\left[\sin t-\cos t-e^{-t}\right]$.

2. $\frac{1}{3}\dot{x}+x=0$, $x(0)=\frac{1}{3}$

Solution

Writing the ODE in standard form yields $g(t)=3$ so that $h(t)=3t$ and

$$x(t) = e^{-3t}\left[\int e^{3t} 0 \, dt + c\right] = ce^{-3t}$$

By the initial condition $c = \frac{1}{3}$ and the solution is obtained as $x(t) = \frac{1}{3}e^{-3t}$.

3. $2\dot{y} + ty = t$, $y(0) = 2$

Solution

Writing the ODE in standard form yields $g(t) = \frac{1}{2}t = f(t)$ so that $h(t) = \frac{1}{4}t^2$ and

$$y(t) = e^{-t^2/4}\left[\int e^{t^2/4}\tfrac{1}{2}t dt + c\right] = e^{-t^2/4}\left[e^{t^2/4} + c\right] = 1 + ce^{-t^2/4}$$

By the initial condition we have $c = 1$, hence $y(t) = 1 + e^{-t^2/4}$.

4. $\dot{u} = (1-u)\sin t$, $u\left(\frac{1}{2}\pi\right) = 2$

Solution

Writing the ODE in standard form yields $g(t) = \sin t = f(t)$ so that $h(t) = -\cos t$ and

$$u(t) = e^{\cos t}\left[\int e^{-\cos t}\sin t dt + c\right] = e^{\cos t}\left[e^{-\cos t} + c\right] = 1 + ce^{\cos t}$$

By the initial condition we have $c = 1$, hence $u(t) = 1 + e^{\cos t}$.

5. $(t-1)\dot{y} + ty = 2t$, $y(0) = 1$

Solution

Writing the ODE in standard form yields $g(t) = \dfrac{t}{t-1}$ and $f(t) = \dfrac{2t}{t-1}$ so that $h(t) = \int \dfrac{t}{t-1}dt = t + \ln(t-1)$ and

$$y(t) = e^{-t-\ln(t-1)}\left[\int e^{t+\ln(t-1)}\frac{2t}{t-1}dt + c\right] = e^{-t-\ln(t-1)}\left[2(t-1)e^t + c\right] = 2 + \frac{ce^{-t}}{t-1}$$

By the initial condition we have $c = 1$, hence $y(t) = 2 + \dfrac{e^{-t}}{t-1}$.

6. $\ddot{x} + 2\dot{x} + x = e^{-2t}$, $x(0) = 1$, $\dot{x}(0) = 1$

Solution

Characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$ so that $\lambda = -1, -1$ and $x_h \overset{\text{Case (2)}}{=} (c_1 + c_2 t)e^{-t}$. Based on $f(t) = e^{-2t}$ we pick $x_p = Ke^{-2t}$, no special case, and insert into the ODE to get $K = 1$ hence $x_p = e^{-2t}$. A general solution is $x(t) = (c_1 + c_2 t)e^{-t} + e^{-2t}$. Initial conditions yield $c_1 = 0, c_2 = 3$, and the solution is $x(t) = 3te^{-t} + e^{-2t}$.

7. $\ddot{x} + 4\dot{x} = 17\cos t$, $x(0) = -1$, $\dot{x}(0) = 0$

Solution

Characteristic values are $\lambda = 0, -4$ hence $x_h \overset{\text{Case (1)}}{=} c_1 + c_2 e^{-4t}$. Pick $x_p = A\cos t + B\sin t$ and insert into the original ODE to find $(4B - A)\cos t - (B + 4A)\sin t \equiv 17\cos t$. This implies $A = -1$, $B = 4$ so that a general solution is $x = c_1 + c_2 e^{-4t} - \cos t + 4\sin t$. By the initial conditions, $c_1 = -1$, $c_2 = 1$ and thus $x = -1 + e^{-4t} - \cos t + 4\sin t$.

8. $\ddot{u}+u=\sin 2t$, $u(0)=1$, $\dot{u}(0)=0$

Solution

Characteristic values are $\lambda=\pm j$ so that $u_h \overset{\text{Case (3)}}{=} c_1\cos t+c_2\sin t$. Based on the nature of $f(t)$ we pick $u_p = A\cos 2t+B\sin 2t$, no special case, and insert into the ODE to get $A=0$, $B=-\frac{1}{3}$ hence $u_p=-\frac{1}{3}\sin 2t$. A general solution is $u(t)=c_1\cos t+c_2\sin t-\frac{1}{3}\sin 2t$. By initial conditions, $c_1=1, c_2=\frac{2}{3}$, and the solution is $u(t)=\cos t+\frac{2}{3}\sin t-\frac{1}{3}\sin 2t$.

9. $\ddot{u}+4\dot{u}+3u=4e^{-t}$, $u(0)=0$, $\dot{u}(0)=-1$

Solution

Characteristic values are $\lambda=-1,-3$ so that $u_h=c_1e^{-t}+c_2e^{-3t}$. Because e^{-t} coincides with an independent solution we pick $u_p=Kte^{-t}$ and insert into the ODE to get $K=2$, hence $u_p=2te^{-t}$. A general solution is $u=c_1e^{-t}+c_2e^{-3t}+2te^{-t}$. By initial conditions, $c_1=-\frac{3}{2}, c_2=\frac{3}{2}$ and the solution is $u=-\frac{3}{2}e^{-t}+\frac{3}{2}e^{-3t}+2te^{-t}$.

10. $2\ddot{y}+3\dot{y}+y=0$, $y(0)=0$, $\dot{y}(0)=\frac{1}{2}$

Solution

Characteristic values are $\lambda=-1,-\frac{1}{2}$ so that $y_h=c_1e^{-t/2}+c_2e^{-t}$. Since the ODE is homogeneous, $y(t)=y_h(t)$. By initial conditions, $c_1=1, c_2=-1$, and the solution is $y(t)=e^{-t/2}-e^{-t}$.

In Problems 11–14 write the expression in the form $D\sin(\omega t+\phi)$.

11. $\cos t+3\sin t$

Solution

Write $\cos t+3\sin t=D\sin(t+\phi)=D\sin t\cos\phi+D\cos t\sin\phi$ and compare the two sides to find

$$\begin{matrix} D\sin\phi=1 \\ D\cos\phi=3 \end{matrix} \overset{D=\sqrt{10}}{\Rightarrow} \begin{matrix} \sin\phi>0 \\ \cos\phi>0 \end{matrix} \Rightarrow \tan\phi=\frac{1}{3} \overset{\text{1st quadrant}}{\Rightarrow} \phi=0.3218 \text{ rad}$$

Therefore, $\cos t+3\sin t=\sqrt{10}\sin(t+0.3218)$.

12. $\cos 2t-\sin 2t$

Solution

Write $\cos 2t-\sin 2t=D\sin(2t+\phi)=D\sin 2t\cos\phi+D\cos 2t\sin\phi$. Comparing the two sides,

$$\begin{matrix} D\sin\phi=1 \\ D\cos\phi=-1 \end{matrix} \overset{D=\sqrt{2}}{\Rightarrow} \begin{matrix} \sin\phi>0 \\ \cos\phi<0 \end{matrix} \Rightarrow \tan\phi=-1 \overset{\text{2nd quadrant}}{\Rightarrow} \phi=2.3562 \text{ rad}$$

Therefore $\cos 2t-\sin 2t=\sqrt{2}\sin(2t+2.3562)$.

13. $-\sin 2t-\frac{1}{2}\cos 2t$

Solution

Write $-\sin 2t-\frac{1}{2}\cos 2t=D\sin(2t+\phi)=D\sin 2t\cos\phi+D\cos 2t\sin\phi$. Comparing the two sides,

$$\begin{array}{l} D\sin\phi = -\frac{1}{2} \\ D\cos\phi = -1 \end{array} \quad \overset{D=\frac{1}{2}\sqrt{5}}{\Rightarrow} \quad \begin{array}{l} \sin\phi < 0 \\ \cos\phi < 0 \end{array} \quad \Rightarrow \quad \tan\phi = \tfrac{1}{2} \quad \overset{\text{3rd quadrant}}{\Rightarrow} \quad \phi = -2.6779 \text{ rad}$$

Therefore, $-\sin 2t - \frac{1}{2}\cos 2t = \frac{1}{2}\sqrt{5}\sin(2t - 2.6779)$.

14. $3\sin\omega t - \cos\omega t$

Solution

Expand $3\sin\omega t - \cos\omega t = D\sin(\omega t + \phi) = D\sin\omega t\cos\phi + D\cos\omega t\sin\phi$. Comparison gives

$$\begin{array}{l} D\sin\phi = -1 \\ D\cos\phi = 3 \end{array} \quad \overset{D=\sqrt{10}}{\Rightarrow} \quad \begin{array}{l} \sin\phi < 0 \\ \cos\phi > 0 \end{array} \quad \Rightarrow \quad \tan\phi = -\tfrac{1}{3} \quad \overset{\text{4th quadrant}}{\Rightarrow} \quad \phi = -0.3218 \text{ rad}$$

Therefore $3\sin\omega t - \cos\omega t = \sqrt{10}\sin(\omega t - 0.3218)$.

In Problems 15–16 write the expression in the form $D\cos(\omega t + \phi)$.

15. $\frac{2}{3}\cos t - \sin t$

Solution

Write $\frac{2}{3}\cos t - \sin t = D\cos(t + \phi) = D\cos t\cos\phi - D\sin t\sin\phi$ and compare the two sides to find

$$\begin{array}{l} D\sin\phi = 1 \\ D\cos\phi = \frac{2}{3} \end{array} \quad \overset{D=\frac{1}{3}\sqrt{13}}{\Rightarrow} \quad \begin{array}{l} \sin\phi > 0 \\ \cos\phi > 0 \end{array} \quad \Rightarrow \quad \tan\phi = \tfrac{3}{2} \quad \overset{\text{1st quadrant}}{\Rightarrow} \quad \phi = 0.9828 \text{ rad}$$

Therefore, $\frac{2}{3}\cos t - \sin t = \frac{1}{3}\sqrt{13}\cos(t + 0.9828)$.

16. $4\cos t + 3\sin t$

Solution

Write $4\cos t + 3\sin t = D\cos(t + \phi) = D\cos t\cos\phi - D\sin t\sin\phi$ and compare the two sides to find

$$\begin{array}{l} D\sin\phi = -3 \\ D\cos\phi = 4 \end{array} \quad \overset{D=5}{\Rightarrow} \quad \begin{array}{l} \sin\phi < 0 \\ \cos\phi > 0 \end{array} \quad \Rightarrow \quad \tan\phi = -\tfrac{3}{4} \quad \overset{\text{4th quadrant}}{\Rightarrow} \quad \phi = -0.6435 \text{ rad}$$

Therefore, $4\cos t + 3\sin t = 5\cos(t - 0.6435)$.

Problem Set 2.3

In Problems 1–8,

(a) Find the Laplace transform of the given function. Use Table 2.2 when applicable.

(b) Confirm the result in MATLAB.

1. e^{at-b}, $a, b = const$

Solution

(a) $\mathscr{L}\left\{e^{at-b}\right\} = \mathscr{L}\left\{e^{at}e^{-b}\right\} \overset{\text{linearity}}{=} \mathscr{L}\left\{e^{at}\right\}e^{-b} = \dfrac{e^{-b}}{s-a}$

(b)

```
>> syms a b t
>> laplace(exp(a*t-b))
```

```
ans =
-1/(exp(b)*(a - s))
```

2. $\frac{2}{3}t^2 - 1$

Solution

(*a*) $\mathscr{L}\left\{\frac{2}{3}t^2 - 1\right\} = \dfrac{2}{3}\dfrac{2}{s^3} - \dfrac{1}{s} = \dfrac{4}{3s^3} - \dfrac{1}{s}$

(*b*)

```
>> syms t
>> laplace(2/3*t^2-1)
ans =
4/(3*s^3) - 1/s
```

3. $\sin(\omega t + \phi)$, $\omega, \phi = const$

Solution

(*a*) Using trigonometric expansion, we find

$$\begin{aligned}\mathscr{L}\left\{\sin(\omega t + \phi)\right\} &= \mathscr{L}\left\{\sin \omega t \cos\phi + \cos\omega t \sin\phi\right\} = \mathscr{L}\left\{\sin\omega t\right\}\cos\phi + \mathscr{L}\left\{\cos\omega t\right\}\sin\phi \\ &= \frac{\omega}{s^2+\omega^2}\cos\phi + \frac{s}{s^2+\omega^2}\sin\phi = \frac{\omega\cos\phi + s\sin\phi}{s^2+\omega^2}\end{aligned}$$

(*b*)

```
>> syms w t p
>> laplace(sin(w*t+p))

ans =

(w*cos(p) + s*sin(p))/(s^2 + w^2)
```

4. $\cos(\omega t - \phi)$, $\omega, \phi = const$

Solution

(*a*) Using trigonometric expansion, we find

$$\begin{aligned}\mathscr{L}\{\cos(\omega t - \phi)\} &= \mathscr{L}\{\cos\omega t\}\cos\phi + \mathscr{L}\{\sin\omega t\}\sin\phi \\ &= \frac{s}{s^2+\omega^2}\cos\phi + \frac{\omega}{s^2+\omega^2}\sin\phi = \frac{s\cos\phi + \omega\sin\phi}{s^2+\omega^2}\end{aligned}$$

(*b*)

```
>> syms w t p
>> laplace(cos(w*t-p))

ans =

(s*cos(p) + w*sin(p))/(s^2 + w^2)
```

5. $\cos^2 t$

Solution

(*a*) Noting that $\cos^2 t = \frac{1}{2}(1+\cos 2t)$, we find

$$\mathscr{L}\left\{\cos^2 t\right\} = \frac{1}{2}\mathscr{L}\{1+\cos 2t\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] \overset{\text{Simplify}}{=} \frac{s^2+2}{s(s^2+4)}$$

(*b*)
```
>> syms t
>> laplace(cos(t)^2)
ans =
 (s^2 + 2)/(s*(s^2 + 4))
```

6. $t\cos t$

Solution

(*a*) Following Eq. (2.16),

$$\mathscr{L}\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}$$

(*b*)
```
>> syms t
>> simple(laplace(t*cos(t)))
ans =
 (s^2 - 1)/(s^2 + 1)^2
```

7. $t^2\sin\omega t$

Solution

(*a*) Following the general form of Eq. (2.16) with $g(t)=\sin\omega t$ and $n=2$,

$$\mathscr{L}\left\{t^2\sin\omega t\right\} = \frac{d^2}{ds^2}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega(3s^2-\omega^2)}{(s^2+\omega^2)^3}$$

(*b*)
```
>> syms t w
>> simple(laplace(t^2*sin(w*t)))
ans =
 (2*w*(3*s^2 - w^2))/(s^2 + w^2)^3
```

8. $t\sinh t$

Solution

(*a*) Comparing with Eq. (2.16), we have $g(t)=\sinh t$ so that

$$G(s) = \mathscr{L}\{\sinh t\} = \frac{1}{2}\mathscr{L}\left\{e^t - e^{-t}\right\} = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] \overset{\text{Simplify}}{=} \frac{1}{s^2-1}$$

Then,

$$\mathscr{L}\left\{t\sinh t\right\} = -\frac{d}{ds}\left(\frac{1}{s^2-1}\right) = \frac{2s}{(s^2-1)^2}$$

(*b*)
```
>> syms t
>> laplace(t*sinh(t))
ans =
 (2*s)/(s^2 - 1)^2
```

In Problems 9–12,

(*a*) Express the signal in terms of unit-step functions.

(*b*) Find the Laplace transform of the expression in (*a*) using the shift on $t-$axis.

9. $g(t) = \begin{cases} -1 & \text{if} \quad 0<t<1 \\ 1 & \text{if} \quad 1<t<2 \\ 0 & \text{otherwise} \end{cases}$

Solution

(a) $g(t) = -u(t) + 2u(t-1) - u(t-2)$

(b) $G(s) = \dfrac{-1+2e^{-s} - e^{-2s}}{s} = \dfrac{-(1-e^{-s})^2}{s}$

10. $g(t) = \begin{cases} t & \text{if} \quad 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

Solution

(a) We construct $g(t)$ using the strategy outlined in Figure PS2-3 No10, resulting in

$$g(t) = tu(t) - tu(t-1)$$

(b) We will take the Laplace transform term-by-term. For the 1st term $\mathscr{L}\{tu(t)\} = \dfrac{1}{s^2}$. For the 2nd term, comparison with Eq. (2.18) reveals $f(t-1) = t$, which implies $f(t) = t+1$. Therefore $F(s) = \dfrac{1}{s^2} + \dfrac{1}{s}$ and $\mathscr{L}\{tu(t-1)\} = \left(\dfrac{1}{s^2} + \dfrac{1}{s}\right)e^{-s}$. Combining the two results, we find

$$G(s) = \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s}$$

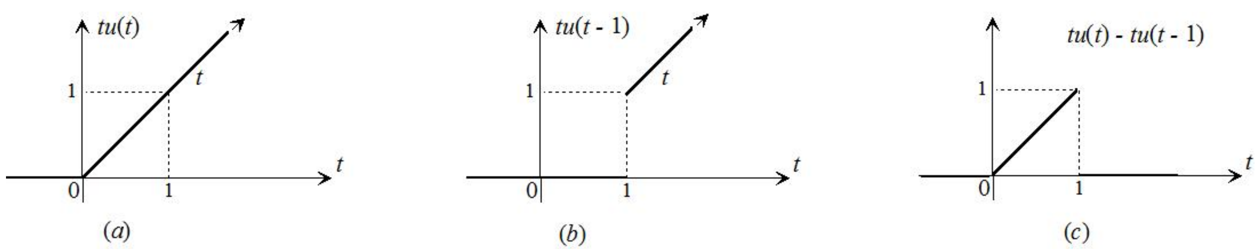

Figure PS2-3 No10

11. $g(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ t & \text{if} \quad 0 < t < 1 \\ 1 & \text{if} \quad t > 1 \end{cases}$

Solution

(a) $g(t) = tu(t) - tu(t-1) + u(t-1) \overset{\text{simplify}}{=} tu(t) - (t-1)u(t-1)$

(b) $G(s) = \dfrac{1}{s^2} - \dfrac{e^{-s}}{s^2} = \dfrac{1-e^{-s}}{s^2}$

12. $g(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1-t & \text{if} \quad 0 < t < 1 \\ 0 & \text{if} \quad t > 1 \end{cases}$

Solution

(a) Construct $g(t)$ using the strategy shown in Figure PS2-3 No12, leading to $g(t) = (1-t)u(t) - (1-t)u(t-1)$.

(b) Rewrite the expression obtained in (a) as $g(t) = u(t) - tu(t) + (t-1)u(t-1)$. Then

$$G(s)=\frac{1}{s}-\frac{1}{s^2}+\frac{e^{-s}}{s^2}$$

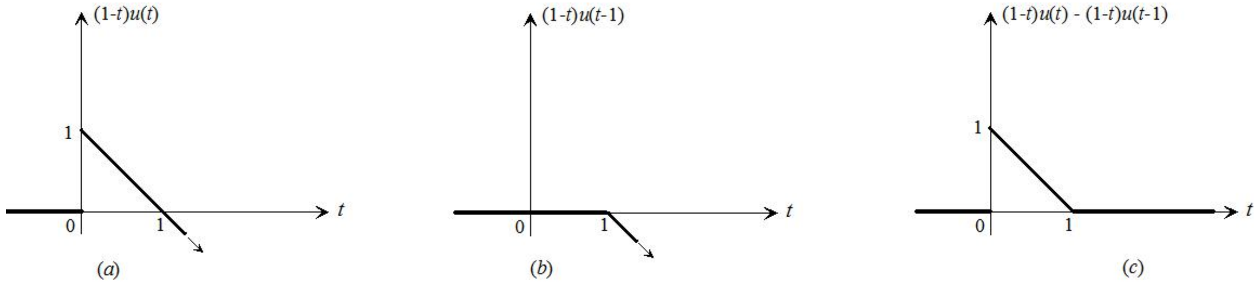

Figure PS2-3 No12

In Problems 13–16 find the Laplace transform of each periodic function whose definition in one period is given.

13. $f(t)=\begin{cases}1 & \text{if} \quad 0<t<1\\ -1 & \text{if} \quad 1<t<2\end{cases}$

Solution

Noting $P=2$, we find

$$F(s)=\frac{1}{1-e^{-2s}}\left\{\int_0^1 e^{-st}dt-\int_1^2 e^{-st}dt\right\}=\frac{1-2e^{-s}+e^{-2s}}{s(1-e^{-2s})}=\frac{(1-e^{-s})^2}{s(1-e^{-2s})}\overset{\text{simplify}}{=}\frac{1-e^{-s}}{s(1+e^{-s})}$$

14. $f(t)=2(1-t),\ 0<t<1$

Solution

The period is $P=1$. Using the description of $f(t)$, we have

$$F(s)=\frac{1}{1-e^{-s}}\int_0^1 e^{-st}\,2(1-t)dt=\frac{2(e^{-s}+s-1)}{s^2(1-e^{-s})}$$

15. $f(t)=\begin{cases}t & \text{if} \quad 0<t<1\\ 1-t & \text{if} \quad 1<t<2\end{cases}$

Solution

The period is $P=2$. Using the description of $f(t)$, we have

$$F(s)=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt$$

But

$$\int_0^2 e^{-st}f(t)dt=\int_0^1 e^{-st}tdt+\int_1^2 e^{-st}(1-t)dt=\frac{e^{-s}(e^{-s}-1)}{s}+\frac{(1-e^{-s})^2}{s^2}$$

Therefore,

$$F(s)=\frac{e^{-s}(e^{-s}-1)}{s(1-e^{-2s})}+\frac{(1-e^{-s})^2}{s^2(1-e^{-2s})}=\frac{1-(s+1)e^{-s}}{s^2(1+e^{-s})}$$

16. $f(t) = \begin{cases} 1 & \text{if} \quad 0 < t < 1 \\ 2-t & \text{if} \quad 1 < t < 2 \end{cases}$

Solution

The period is $P = 2$. Using the description of $f(t)$, we have

$$F(s) = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)dt = \frac{1}{1-e^{-2s}}\left\{\int_0^1 e^{-st}dt + \int_1^2 e^{-st}(2-t)dt\right\} = \frac{1}{s(1-e^{-2s})} + \frac{e^{-s}(e^{-s}-1)}{s^2(1-e^{-2s})}$$

This simplifies to

$$F(s) = \frac{1}{s(1-e^{-2s})} - \frac{e^{-s}}{s^2(1+e^{-s})}$$

17. Find the Laplace transform of the periodic function $f(t)$ in Figure 2.14.

Solution

The period is $P = 1$. Using the description of $f(t)$, we have

$$F(s) = \frac{1}{1-e^{-s}}\int_0^1 e^{-st} f(t)dt = \frac{1}{1-e^{-s}}\left\{\int_0^{1/2} e^{-st}dt - \int_{1/2}^1 e^{-st}dt\right\} = \frac{\left(1-e^{-s/2}\right)^2}{s\left(1-e^{-s}\right)} = \frac{1-e^{-s/2}}{s\left(1+e^{-s/2}\right)}$$

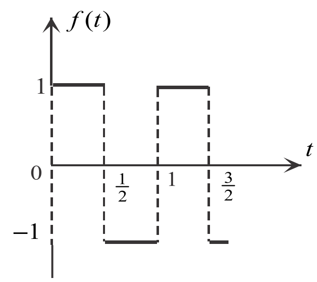

Figure 2.14

18. Find the Laplace transform of the periodic function $f(t)$ in Figure 2.15.

Solution

The period is $P = 2b$. Using the description of $f(t)$, we have

$$F(s) = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)dt = \frac{1}{1-e^{-2bs}}\left\{\int_0^b e^{-st}tdt + \int_b^{2b} e^{-st}(2b-t)dt\right\} = \frac{(1-e^{-bs})^2}{s^2(1-e^{-2bs})}$$

This can be rewritten as

$$F(s) = \frac{1-e^{-bs}}{s^2(1+e^{-bs})} = \frac{1}{s^2}\cdot\frac{e^{bs/2}-e^{-bs/2}}{e^{bs/2}+e^{-bs/2}} = \frac{1}{s^2}\tanh\frac{bs}{2}$$

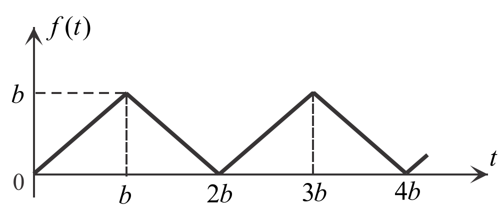

Figure 2.15

In Problems 19–24,

(a) Find the inverse Laplace transform using the partial-fraction expansion method.

(b) Repeat in MATLAB.

19. $\dfrac{3s+4}{s(s+1)}$

Solution

(a) Expand as

$$\frac{3s+4}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{(A+B)s+A}{s(s+1)} \quad \Rightarrow \quad \begin{array}{l} A+B=3 \\ A=4 \end{array} \quad \Rightarrow \quad \begin{array}{l} A=4 \\ B=-1 \end{array}$$

Therefore,

$$\frac{3s+4}{s(s+1)} = \frac{4}{s} - \frac{1}{s+1} \quad \overset{\mathcal{L}^{-1}}{\Rightarrow} \quad 4 - e^{-t}$$

(b)

```
>> syms s
>> ilaplace((3*s+4)/s/(s+1))

ans =

4 - 1/exp(t)
```

20. $\dfrac{3s^2+2s+2}{(s^2+1)(s+2)}$

Solution

(a) Partial-fraction expansion leads to

$$\frac{3s^2+2s+2}{(s^2+1)(s+2)} = \frac{As+B}{s^2+1} + \frac{C}{s+2} = \frac{(A+C)s^2+(2A+B)s+2B+C}{(s^2+1)(s+2)} \quad \Rightarrow \quad \begin{array}{r} A+C=3 \\ 2A+B=2 \\ 2B+C=2 \end{array}$$

Simultaneous solution gives $A=1, B=0, C=2$ so that

$$\frac{3s^2+2s+2}{(s^2+1)(s+2)} = \frac{s}{s^2+1} + \frac{2}{s+2} \quad \overset{\mathcal{L}^{-1}}{\Rightarrow} \quad \cos t + 2e^{-2t}$$

(b)

```
>> syms s
>> ilaplace((3*s^2+2*s+2)/(s^2+1)/(s+2))
ans =
2/exp(2*t) + cos(t)
```

21. $\dfrac{s+10}{s(s^2+2s+5)}$

Solution

(a) Using partial fractions,

$$\frac{s+10}{s(s^2+2s+5)}=\frac{A}{s}+\frac{Bs+C}{s^2+2s+5}=\frac{(A+B)s^2+(2A+C)s+5A}{s(s^2+2s+5)}$$

Subsequently,

$$\begin{aligned} A+B&=0 \\ 2A+C&=1 \\ 5A&=10 \end{aligned} \quad \overset{\text{solve}}{\Rightarrow} \quad \begin{aligned} A&=2 \\ B&=-2 \\ C&=-3 \end{aligned}$$

Then

$$\frac{s+10}{s(s^2+2s+5)}=\frac{2}{s}-\frac{2s+3}{(s+1)^2+2^2}=\frac{2}{s}-\frac{2(s+1)+1}{(s+1)^2+2^2} \quad \overset{\mathscr{L}^{-1}}{\Rightarrow} \quad 2-2e^{-t}\cos 2t-\frac{1}{2}e^{-t}\sin 2t$$

(*b*)

```
>> syms s
ilaplace((s+10)/s/(s^2+2*s+5))
ans =
2 - (2*(cos(2*t) + sin(2*t)/4))/exp(t)
```

22. $\dfrac{4s+5}{s^2(s^2+4s+5)}$

Solution

(*a*) Forming partial fractions,

$$\frac{4s+5}{s^2(s^2+4s+5)}=\frac{A}{s^2}+\frac{B}{s}+\frac{Cs+D}{s^2+4s+5}=\frac{(B+C)s^3+(A+4B+D)s^2+(4A+5B)s+5A}{s^2(s^2+4s+5)}$$

Then

$$\begin{aligned} B+C&=0 \\ A+4B+D&=0 \\ 4A+5B&=4 \\ 5A&=5 \end{aligned} \quad \overset{\text{solve}}{\Rightarrow} \quad \begin{aligned} A&=1 \\ B&=0 \\ C&=0 \\ D&=-1 \end{aligned}$$

Therefore

$$\frac{4s+5}{s^2(s^2+4s+5)}=\frac{1}{s^2}-\frac{1}{(s+2)^2+1^2} \quad \overset{\mathscr{L}^{-1}}{\Rightarrow} \quad t-e^{-2t}\sin t$$

(*b*)

```
>> syms s
>> ilaplace((4*s+5)/s^2/(s^2+4*s+5))

ans =

t-exp(-2*t)*sin(t)
```

23. $\dfrac{s-8}{s(s+2)^2}$

Solution

(*a*) Forming partial fractions,

$$\frac{s-8}{s(s+2)^2}\equiv\frac{A}{s}+\frac{A_2}{(s+2)^2}+\frac{A_1}{s+2}=\frac{A(s+2)^2+A_2s+A_1s(s+2)}{s(s+2)^2}=\frac{\left(A+A_1\right)s^2+\left(4A+A_2+2A_1\right)s+4A}{s(s+2)^2}$$

Then,

$$\begin{cases} A + A_1 = 0 \\ 4A + A_2 + 2A_1 = 1 \\ 4A = -8 \end{cases} \quad \Rightarrow \quad \begin{array}{l} A = -2 \\ A_2 = 5 \\ A_1 = 2 \end{array}$$

Therefore,

$$\frac{s-8}{s(s+2)^2} \equiv \frac{-2}{s} + \frac{5}{(s+2)^2} + \frac{2}{s+2} \quad \overset{\mathscr{L}^{-1}}{\Rightarrow} \quad -2 + 5te^{-2t} + 2e^{-2t}$$

(*b*)

```
>> syms s
ilaplace((s-8)/s/(s+2)^2)

ans =

2/exp(2*t) + (5*t)/exp(2*t) - 2
```

24. $\dfrac{s^2+s-1}{(s+3)(s^2+2s+2)}$

Solution

(*a*) Partial-fraction expansion gives

$$\frac{s^2+s-1}{(s+3)(s^2+2s+2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+2} = \frac{(A+B)s^2 + (2A+3B+C)s + 2A+3C}{(s+3)(s^2+2s+2)}$$

Then

$$\begin{array}{c} A + B = 1 \\ 2A + 3B + C = 1 \\ 2A + 3C = -1 \end{array} \quad \Rightarrow \quad \begin{array}{l} A = 1 \\ B = 0 \\ C = -1 \end{array}$$

Finally,

$$\frac{s^2+s-1}{(s+3)(s^2+2s+2)} = \frac{1}{s+3} - \frac{1}{(s+1)^2+1^2} \quad \overset{\mathscr{L}^{-1}}{\Rightarrow} \quad e^{-3t} - e^{-t}\sin t$$

(*b*)

```
>> syms s
>> ilaplace((s^2+s-1)/(s+3)/(s^2+2*s+2))

ans =

-exp(-t)*sin(t)+exp(-3*t)
```

In Problems 25–30,

(*a*) Solve the initial-value problem.

(*b*) Confirm the result in MATLAB.

25. $\dot{x} + 2x = 2u(t) - u(t-1),\ x(0) = 0$

Solution

(*a*) Taking the Laplace transform and using the zero initial condition, yields

$$(s+2)X(s) = \frac{2}{s} - \frac{e^{-s}}{s} \qquad \Rightarrow \qquad X(s) = \frac{2}{s(s+2)} - \frac{1}{s(s+2)}e^{-s}$$

Let $G(s) = \dfrac{1}{s(s+2)}$ so that by partial-fraction expansion we have $g(t) = \frac{1}{2}(1-e^{-2t})$. Next,

$$x(t) = \mathscr{L}^{-1}\left\{\frac{2}{s(s+2)}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s(s+2)}e^{-s}\right\} = 2\mathscr{L}^{-1}\left\{G(s)\right\} - \mathscr{L}^{-1}\left\{e^{-s}G(s)\right\}$$

Using the shift on the $t-$axis for the second term, we find $x(t) = 2g(t) - g(t-1)u(t-1)$. But $g(t)$ was defined earlier, hence

$$x(t) = 1 - e^{-2t} - \tfrac{1}{2}\left[1 - e^{-2(t-1)}\right]u(t-1)$$

(*b*)

```
>> x = dsolve('Dx+2*x=2*heaviside(t)-heaviside(t-1)','x(0)=0')

x =

(heaviside(t)*(exp(2*t) - 1) - (heaviside(t - 1)*(exp(2*t) -
   exp(2)))/2)/exp(2*t)
```

26. $\ddot{x} + 2\dot{x} + x = g(t),\ x(0) = 0,\ \dot{x}(0) = 0,\ g(t) = \begin{cases} 1 & \text{if} \quad 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

Solution

(*a*) We first write $g(t) = u(t-1) - u(t-2)$ so that $G(s) = \dfrac{e^{-s} - e^{-2s}}{s}$. Laplace transformation of the ODE and using the zero initial conditions, yields

$$(s+1)^2 X(s) = \frac{e^{-s} - e^{-2s}}{s} \qquad \Rightarrow \qquad X(s) = \frac{1}{s(s+1)^2}e^{-s} - \frac{1}{s(s+1)^2}e^{-2s}$$

Let $H(s) = \dfrac{1}{s(s+1)^2}$ so that by partial-fraction expansion we have $h(t) = 1 - (t+1)e^{-t}$. Next,

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{s(s+1)^2}e^{-s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s(s+1)^2}e^{-2s}\right\} = \mathscr{L}^{-1}\left\{e^{-s}H(s)\right\} - \mathscr{L}^{-1}\left\{e^{-2s}H(s)\right\}$$

By the shift on the $t-$axis, we find $x(t) = h(t-1)u(t-1) - h(t-2)u(t-2)$. Using the expression for $h(t)$ defined earlier, we have

$$x(t) = \left[1 - te^{-(t-1)}\right]u(t-1) - \left[1 - (t-1)e^{-(t-2)}\right]u(t-2)$$

(*b*)

```
>> x = simple(dsolve('D2x+2*Dx+x=heaviside(t-1)-heaviside(t-2)','x(0)=0,
   Dx(0)=0'))

x =

heaviside(t - 1) - heaviside(t - 2) - heaviside(t - 2)*exp(2 - t) -
   t*heaviside(t - 1)*exp(1 - t) + t*heaviside(t - 2)*exp(2 - t)
```

27. $3\ddot{x}+\dot{x}=e^{-t}$, $x(0)=0$, $\dot{x}(0)=\frac{1}{3}$

Solution

(a) Laplace transformation of the ODE and using the initial conditions leads to

$$(3s^2+s)X(s)-1=\frac{1}{s+1} \qquad \Rightarrow \qquad X(s)=\frac{s+2}{s(3s+1)(s+1)}$$

Finally, using partial-fraction expansion, we find

$$x(t)=\mathscr{L}^{-1}\left\{\frac{s+2}{s(3s+1)(s+1)}\right\}=\frac{1}{2}e^{-t}-\frac{5}{2}e^{-t/3}+2$$

(b)

```
>> x = simple(dsolve('3*D2x + Dx = exp(-t)','x(0)=0, Dx(0)=1/3'))

x =

1/(2*exp(t)) - 5/(2*exp(t/3)) + 2
```

28. $\ddot{x}+9x=\sin t$, $x(0)=1$, $\dot{x}(0)=0$

Solution

(a) Laplace transform of the ODE and using the initial conditions, we find

$$s^2X(s)-s+9X(s)=\frac{1}{s^2+1} \qquad \Rightarrow \qquad X(s)=\frac{1}{(s^2+1)(s^2+9)}+\frac{s}{s^2+9}$$

It is then readily seen that $x(t)=\frac{1}{8}\sin t-\frac{1}{24}\sin 3t+\cos 3t$.

(b)

```
>> x = simple(dsolve('D2x + 9*x = sin(t)','x(0)=1,Dx(0)=0'))


x =

cos(3*t) + sin(t)^3/6
```

This expression simplifies to agree with Part (a).

29. $\ddot{x}+\dot{x}-2x=e^{t}$, $x(0)=0,$ $\dot{x}(0)=1$

Solution

(a) Taking the Laplace transform of the ODE and using the given initial conditions, yields

$$(s^2+s-2)X(s)=\frac{1}{s-1}+1 \qquad \Rightarrow \qquad X(s)=\frac{s}{(s-1)^2(s+2)}$$

By partial-fraction expansion we find $x(t)=\frac{1}{3}te^{t}-\frac{2}{9}e^{-2t}+\frac{2}{9}e^{t}$.

(b)

```
>> x = simple(dsolve('D2x + Dx - 2*x = exp(t)','x(0)=0,Dx(0)=1'))
```

```
x =

(2*exp(t))/9 - 2/(9*exp(2*t)) + (t*exp(t))/3
```

30. $\ddot{x}+3\dot{x}=1,\ x(0)=2,\ \dot{x}(0)=0$

Solution

(a) Taking the Laplace transform of the ODE and using the given initial conditions, yields

$$(s^2+3s)X(s)=\frac{1}{s}+2s+6 \qquad \Rightarrow \qquad X(s)=\frac{1}{s^2(s+3)}+\frac{2}{s}$$

By partial-fraction expansion we find $x(t)=\frac{17}{9}+\frac{1}{3}t+\frac{1}{9}e^{-3t}$.

(b)

```
>> x = simple(dsolve('D2x + 3*Dx = 1','x(0)=2,Dx(0)=0'))

x =

t/3 + 1/(9*exp(3*t)) + 17/9
```

In Problems 31–36 decide whether the final-value theorem is applicable, and if so, find x_{ss}.

31. $X(s)=\dfrac{1}{2s(s+3)}$

Solution

The poles are at $0,-3$ hence the FVT applies.

$$x_{ss}=\lim_{s\to 0}\left\{sX(s)\right\}=\lim_{s\to 0}\left\{\frac{1}{2(s+3)}\right\}=\frac{1}{6}$$

32. $X(s)=\dfrac{s+2}{(s+4)(s^2+4s+5)}$

Solution

The poles are at $-4,-2\pm j$ hence the FVT applies.

$$x_{ss}=\lim_{s\to 0}\left\{sX(s)\right\}=\lim_{s\to 0}\left\{\frac{s(s+2)}{(s+4)(s^2+4s+5)}\right\}=0$$

33. $X(s)=\dfrac{s+1}{s^2(s+3)(s+2)}$

Solution

The poles are at $0,0,-3,-2$. Since the pole at the origin is repeated, the FVT does not apply.

34. $X(s)=\dfrac{s+3}{(s+2)^2(s+1)}$

Solution

The poles are at $-2,-2,-1$, hence the FVT applies.

$$x_{ss} = \lim_{s\to 0}\{sX(s)\} = \lim_{s\to 0}\left\{\frac{s(s+3)}{(s+2)^2(s+1)}\right\} = 0$$

35. $X(s) = \dfrac{s^2+1}{s(s+1)^2}$

Solution

The poles are at $0,-1,-1$ hence the FVT applies.

$$x_{ss} = \lim_{s\to 0}\{sX(s)\} = \lim_{s\to 0}\left\{\frac{s^2+1}{(s+1)^2}\right\} = 1$$

36. $X(s) = \dfrac{s+\frac{1}{2}}{s(s^2+s+1)}$

Solution

The poles are at $0,-\frac{1}{2}\pm\frac{\sqrt{3}}{2}j$ hence the FVT applies.

$$x_{ss} = \lim_{s\to 0}\{sX(s)\} = \lim_{s\to 0}\left\{\frac{s+\frac{1}{2}}{s^2+s+1}\right\} = \frac{1}{2}$$

In Problems 37–40 evaluate $x(0^+)$ using the initial-value theorem.

37. $X(s) = \dfrac{s^2+1}{s(2s^2+s+3)}$

Solution

$$x(0^+) = \lim_{s\to \infty}\{sX(s)\} = \lim_{s\to \infty}\left\{\frac{s^2+1}{2s^2+s+3}\right\} = \frac{1}{2}$$

38. $X(s) = \dfrac{3s+2}{(s+1)(s+2)^2}$

Solution

$$x(0^+) = \lim_{s\to \infty}\{sX(s)\} = \lim_{s\to \infty}\left\{\frac{s(3s+2)}{(s+1)(s+2)^2}\right\} = 0$$

39. $X(s) = \dfrac{s(s+4)}{(s+1)(s+2)(s+3)}$

Solution

$$x(0^+) = \lim_{s\to \infty}\{sX(s)\} = \lim_{s\to \infty}\left\{\frac{s^2(s+4)}{(s+1)(s+2)(s+3)}\right\} = 1$$

40. $X(s) = \dfrac{s^2+2}{(3s+1)(s^2+9)}$

Solution

$$x(0^+) = \lim_{s\to \infty}\{sX(s)\} = \lim_{s\to \infty}\left\{\frac{s(s^2+2)}{(3s+1)(s^2+9)}\right\} = \frac{1}{3}$$

Review Problems

In Problems 1–4 perform the operations and express the result in rectangular form.

1. $\dfrac{(1-3j)^2}{2+j}$

Solution

$$\frac{(1-3j)^2}{2+j}=\frac{-8-6j}{2+j}\times\frac{2-j}{2-j}=\frac{-22-4j}{5}=-\frac{22}{5}-\frac{4}{5}j$$

2. $\dfrac{1+\frac{1}{3}j}{2j(3-2j)}$

Solution

$$\frac{1+\frac{1}{3}j}{2j(3-2j)}=\frac{1+\frac{1}{3}j}{4+6j}\times\frac{4-6j}{4-6j}=\frac{6-\frac{14}{3}j}{52}=\frac{3}{26}-\frac{7}{78}j$$

3. $(0.6-0.8j)^5$

Solution

Since $0.6-0.8j=e^{-0.9273j}$, we have

$$(0.6-0.8j)^5=\left(e^{-0.9273j}\right)^5=e^{-4.6365j}=-0.0758+0.9971j$$

4. $\dfrac{(1+3j)^4}{(3+j)^3}$

Solution

$$\frac{(1+3j)^4}{(3+j)^3}=\frac{\left(\sqrt{10}e^{1.2490j}\right)^4}{\left(\sqrt{10}e^{0.3218j}\right)^3}=\frac{\sqrt{10}e^{4.9962j}}{e^{0.9653j}}=\sqrt{10}e^{4.0309j}=-1.9921-2.4559j$$

5. Find all possible values of $\left(-3-\frac{3}{2}j\right)^{1/3}$.

Solution

Let $z=-3-\frac{3}{2}j=\frac{3\sqrt{5}}{2}e^{-2.6779j}$ so that $r=\frac{3\sqrt{5}}{2}$ and $\theta=-2.6779$ rad. With $n=3$, Eq. (2.9) yields

$$\sqrt[3]{z}=\sqrt[3]{r}\left[\cos\frac{\theta+2k\pi}{3}+j\sin\frac{\theta+2k\pi}{3}\right]\ ,\quad k=0,1,2$$

Substituting for r and θ, we find the three roots as

$$0.9390-1.1656j\ ,\quad 0.5400+1.3961j\ ,\quad -1.4790-0.2305j$$

6. Find all possible values of $\sqrt{1-0.3j}$.

Solution

Let $z=1-0.3j=1.0440e^{-0.2915j}$ so that $r=1.0440$ and $\theta=-0.2915$ rad. With $n=2$, Eq. (2.9) gives

$$\sqrt{z}=\sqrt{r}\left[\cos\frac{\theta+2k\pi}{2}+j\sin\frac{\theta+2k\pi}{2}\right]\ ,\quad k=0,1$$

Substituting for r and θ, we find the two roots as $1.0109-0.1484j,\ -1.0109+0.1484j$.

7. Solve the initial-value problem $3\dot{y}+y=2t$, $y(0)=-4$.

Solution

Rewrite in standard form as $\dot{y}+\frac{1}{3}y=\frac{2}{3}t$ so that by Eq. (2.12),

$$y(t)=e^{-t/3}\left[\int e^{t/3}\,\tfrac{2}{3}t\ dt+c\right]=e^{-t/3}\left[2e^{t/3}(t-3)+c\right]=2(t-3)+ce^{-t/3}$$

Initial condition gives $y(0)=-6+c=-4$ so that $c=2$. Therefore, $y(t)=2(t-3)+2e^{-t/3}$.

8. Express $\cos t-2\sin t$ in the form $D\cos(t+\phi)$ for suitable amplitude D and phase ϕ .

Solution

Expand $D\cos(t+\phi)=D\cos t\cos\phi-D\sin t\sin\phi$. Comparing with $\cos t-2\sin t$, we find

$$\begin{aligned} D\cos\phi &= 1 \\ D\sin\phi &= 2 \end{aligned} \quad \overset{\phi \text{ is in the 1st quadrant}}{\Rightarrow} \quad \tan\phi=2 \quad \Rightarrow \quad \phi=1.1071 \text{ rad}$$

The amplitude is $D=\sqrt{1^2+2^2}=\sqrt{5}$. Therefore $\cos t-2\sin t=\sqrt{5}\cos(t+1.1071)$.

9. Solve the initial-value problem $\ddot{x}+3\dot{x}=3e^{-3t}$, $x(0)=0,\ \dot{x}(0)=2$ using
(a) the method of undetermined coefficients,
(b) Laplace transformation.

Solution

(a) The characteristic equation $\lambda^2+3\lambda=0$ yields .. so that $x_h(t)=A+Be^{-3t}$. Because the function on the right side of the ODE matches a homogeneous solution, pick $x_p=kte^{-3t}$ and insert into the ODE to find $-3K=3$, hence $K=-1$ and $x_p=-te^{-3t}$. Therefore

$$x(t)=A+Be^{-3t}-te^{-3t}$$

Using the initial conditions, we find $A=1$ and $B=-1$, and $x(t)=1-e^{-3t}-te^{-3t}=1-(t+1)e^{-3t}$.
(b) Taking the Laplace transform of the ODE and using the initial conditions, we arrive at

$$s^2X(s)-2+3sX(s)=\frac{3}{s+3} \quad \Rightarrow \quad X(s)=\frac{2s+9}{s(s+3)^2}$$

By partial-fraction expansion,

$$X(s)=\frac{2s+9}{s(s+3)^2}\equiv\frac{A}{s}+\frac{B}{s+3}+\frac{C}{(s+3)^2}=\frac{(A+B)s^2+(6A+3B+C)s+9A}{s(s+3)^2}$$

Equating the coefficients of like powers of s yields $A=1, B=-1, C=-1$ so that

$$X(s)=\frac{1}{s}+\frac{-1}{s+3}+\frac{-1}{(s+3)^2} \quad \overset{\mathscr{L}^{-1}}{\Rightarrow} \quad x(t)=1-e^{-3t}-te^{-3t}$$

10. Solve the initial-value problem $\ddot{x}+2\dot{x}=\delta(t-1),\ x(0)=0,\ \dot{x}(0)=1$.

Solution

Taking the Laplace transform of the ODE and using the initial conditions, we find

$$s^2X(s)-1+2sX(s)=e^{-s} \quad \Rightarrow \quad X(s)=\frac{1}{s(s+2)}e^{-s}+\frac{1}{s(s+2)}$$

Let $G(s)=\dfrac{1}{s(s+2)}$ so that $g(t)=\frac{1}{2}(1-e^{-2t})$. Then, using the shift on the $t-$axis, we have

$$x(t)=\mathscr{L}^{-1}\left\{G(s)e^{-s}\right\}+\mathscr{L}^{-1}\left\{G(s)\right\}=g(t-1)u(t-1)+g(t)$$

Taking into account the description of $g(t)$ given earlier, we find the solution as

$$x(t)=\tfrac{1}{2}\left[1-e^{-2(t-1)}\right]u(t-1)+\tfrac{1}{2}(1-e^{-2t})$$

11. Find the Laplace transform of the periodic function in Figure 2.16.

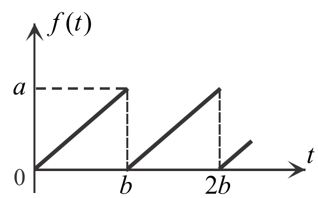

Figure 2.16 Problem 11.

Solution

The period is $P=b$. Using the description of $f(t)$, we have

$$F(s)=\frac{1}{1-e^{-bs}}\int_0^b e^{-st}f(t)dt=\frac{1}{1-e^{-bs}}\int_0^b e^{-st}\frac{a}{b}tdt$$

But

$$\int_0^b e^{-st}\frac{a}{b}tdt=-\frac{ae^{-bs}}{s}-\frac{a\left(e^{-bs}-1\right)}{bs^2}$$

Therefore,

$$F(s)=-\frac{ae^{-bs}}{s\left(1-e^{-bs}\right)}+\frac{a}{bs^2}=\frac{a}{s}\left[\frac{1}{bs}-\frac{1}{e^{bs}-1}\right]$$

12. Find the Laplace transform of the periodic function whose definition in one period is

$$f(t)=t^2,\ 0<t<1$$

Solution

The period is $P=1$. Using the description of $f(t)$, we have

$$F(s)=\frac{1}{1-e^{-s}}\int_0^1 e^{-st}t^2dt=\frac{2-e^{-s}(s^2+2s+2)}{s^3(1-e^{-s})}$$

13. Evaluate the convolution $u(t-a)*t$.

Solution

$$u(t-a)*t=\int_a^t (t-\tau)d\tau = \tfrac{1}{2}(a-t)^2$$

14. Find the convolution $u(t-1)*e^{-t}$.

Solution

$$u(t-1)*e^{-t} = \int_1^t e^{-(t-\tau)}d\tau = 1-e^{1-t}$$

15. Using partial-fraction expansion, find

$$\mathscr{L}^{-1}\left\{\frac{2s^2+1}{s^2(4s^2+1)}\right\}$$

Solution

Partial-fraction expansion results in

$$\frac{2s^2+1}{s^2(4s^2+1)} \equiv \frac{1}{s^2}-\frac{\frac{1}{2}}{s^2+\frac{1}{4}}$$

Therefore $\mathscr{L}^{-1}\left\{\dfrac{2s^2+1}{s^2(4s^2+1)}\right\} = t-\sin(t/2)$.

16. Using convolution, find

$$\mathscr{L}^{-1}\left\{\frac{2s}{(s+1)(s^2+1)}\right\}$$

Solution

Writing the transform function as the product of $\dfrac{2}{s+1}$ and $\dfrac{s}{s^2+1}$, and noting that their inverse Laplace transforms are $2e^{-t}$ and $\cos t$, we have

$$\mathscr{L}^{-1}\left\{\frac{2s}{(s+1)(s^2+1)}\right\} = \int_0^t 2e^{-(t-\tau)}\cos\tau d\tau = 2e^{-t}\int_0^t e^{\tau}\cos\tau d\tau$$

But

$$\int_0^t e^{\tau}\cos\tau d\tau = \left[\tfrac{1}{2}e^{\tau}(\cos\tau+\sin\tau)\right]_0^t = \tfrac{1}{2}\left[e^t(\cos t+\sin t)-1\right]$$

Therefore

$$\mathscr{L}^{-1}\left\{\frac{2s}{(s+1)(s^2+1)}\right\} = 2e^{-t}\,\tfrac{1}{2}\left[e^t(\cos t+\sin t)-1\right] = \cos t+\sin t-e^{-t}$$

17. Consider

$$X(s)=\frac{1}{s(s+1)^2}$$

(*a*) Using the final-value theorem, if applicable, evaluate x_{ss} .

(*b*) Confirm the result of Part (*a*) by evaluating $\lim_{t\to\infty}\{x(t)\}$.

Solution

(a) Poles of $X(s)$ are at $0, -1, -1$ so that the FVT is applicable:

$$x_{ss} = \lim_{s\to 0}\{sX(s)\} = \lim_{s\to 0}\frac{1}{(s+1)^2} = 1$$

(b) Since $x(t) = 1-(t+1)e^{-t}$, we find $\lim_{t\to\infty} x(t) = 1$, confirming the earlier result.

18. Repeat Problem 17 for $X(s) = \dfrac{s+0.1}{s(s^2+0.2s+25.01)}$.

Solution

(a) Poles of $X(s)$ are at $0, -0.1 \pm 5j$ so that the FVT is applicable:

$$x_{ss} = \lim_{s\to 0}\{sX(s)\} = \lim_{s\to 0}\frac{s+0.1}{s^2+0.2s+25.01} = 0.0040$$

(b) Using either convolution or partial-fraction expansion, it is readily verified that

$$x(t) = 0.0040 - 0.0040e^{-0.1t}\cos 5t + 0.02e^{-0.1t}\sin 5t$$

The steady-state value is then calculated as $\lim_{t\to\infty} x(t) = 0.0040$, confirming Part ($a$).

19. Consider

$$X(s) = \frac{3s}{2(s^2+0.4s+1.04)}$$

(a) Using the initial-value theorem, evaluate $x(0^+)$.

(b) Confirm the result of Part (a) by evaluating $\lim_{t\to 0^+}\{x(t)\}$.

Solution

(a) $x(0^+) = \lim_{s\to\infty}\{sX(s)\} = \lim_{s\to\infty}\dfrac{3s^2}{2(s^2+0.4s+1.04)} = \dfrac{3}{2}$

(b) Rewrite

$$X(s) = \frac{3s}{2\left[(s+0.2)^2+1\right]} = \frac{3}{2}\left[\frac{s+0.2}{(s+0.2)^2+1} - 0.2\frac{1}{(s+0.2)^2+1}\right]$$

Therefore $x(t) = \frac{3}{2}\left[e^{-0.2t}\cos t - 0.2e^{-0.2t}\sin t\right]$ so that $\lim_{t\to 0^+}\{x(t)\} = \frac{3}{2}$, as asserted.

20. Assuming $X(s) = \dfrac{0.4s+0.3}{s(3s^2+1)}$, evaluate $\dot{x}(0^+)$ using the initial-value theorem.

Solution

$$\dot{x}(0^+) = \lim_{s\to\infty}\{s[sX(s)]\} = \lim_{s\to\infty}\frac{s^2(0.4s+0.3)}{s(3s^2+1)} = \frac{0.4}{3}$$