9. The aspect ratio tells us that

$$\Delta_1 = 2 \times \frac{4}{3} = \frac{8}{3}.$$

The global coordinates of $(1/2, 1/2)$ are

$$x_1 = 0 + \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3} \qquad x_2 = 0 + \frac{1}{2} \cdot 2 = 1.$$

10. Each coordinate will follow similarly, so let's work out the details for x_1. First, construct the ratio that defines the relationship between a coordinate u_1 in the NDC system and coordinate x_1 in the viewport

$$\frac{u_1 - (-1)}{1 - (-1)} = \frac{x_1 - \min_1}{\max_1 - \min_1}.$$

Solve for x_1, and the equations for x_2 follow similarly, namely

$$x_1 = \frac{(\max_1 - \min_1)}{2}(u_1 + 1) + \min_1,$$

$$x_2 = \frac{(\max_2 - \min_2)}{2}(u_2 + 1) + \min_2.$$

Chapter 2   Here and There: Points and Vectors in 2D

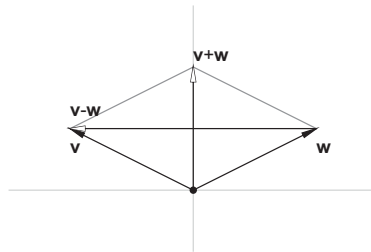$\star$**1.** Figure C.2 illustrates.

Figure C.2.
Parallelogram rule: the sum and difference of two vectors forms the diagonals of the parallelogram defined by them.

2. The vectors $\mathbf{v}$ and $\mathbf{w}$ form adjacent sides of a parallelogram. The vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ form the diagonals of this parallelogram, and an example is illustrated in Sketch 2.6.

⋆**3.** The geometrically meaningful expressions include the following.

      b: The expression forms a convex combination.

      c: A point and a vector forms a point.

      e: Any combination of vectors is allowed.

      f: Same reason as item e.

      g: Same reason as item e.

      h: The sum of the coefficients is one.

These are illustrated in Figure C.3 for

$$\mathbf{p} = \begin{bmatrix} 15 \\ 8 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 20 \\ 18 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$
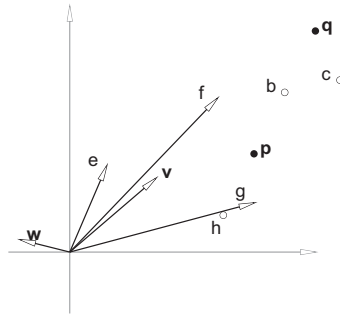

Figure C.3.
Geometrically meaningful expressions of points $\mathbf{p}, \mathbf{q}$ and vectors $\mathbf{v}, \mathbf{w}$. (Letters correspond to items in solution 3 for Chapter 2.)

4. The operations have the following results.

    (a) vector

    (b) point

    (c) point

    (d) vector

    (e) vector

    (f) point

5. The midpoint between $\mathbf{p}$ and $\mathbf{q}$ is

$$\mathbf{m} = \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}.$$

⋆**6.** An example is illustrated in Figure C.4 for the triangle

$$\mathbf{p}_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

where the point with barycentric coordinates $(1/2, 1/4, 1/4)$ is

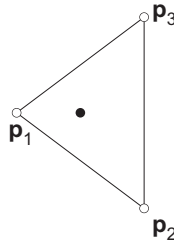$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Figure C.4.
Barycentric coordinates: the point with barycentric coordinates $(1/2, 1/4, 1/4)$.

⋆**7.** A line.

8. A triangle.

9. The length of the vector

$$\mathbf{v} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

is

$$\|\mathbf{v}\| = \sqrt{(-4)^2 + (-3)^2} = 5.$$

⋆**10.** The magnitude of the vector $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{3^2 + (-3)^2} = \sqrt{18} \approx 4.2.$$

⋆**11.** The length of $-2\mathbf{v}$ is $|2|\|\mathbf{v}\| = 20$.

⋆**12.** The distance between $\mathbf{p}$ and $\mathbf{q}$ is $\sqrt{61} \approx 7.8$.

13. The distance between $\mathbf{p}$ and $\mathbf{q}$ is 1.

14. A unit vector has length one.

15. The normalized vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} -4/5 \\ -3/5 \end{bmatrix}.$$

⋆16. The normalized vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}.$$

17. The barycentric coordinates are $(1-t)$ and t such that $\mathbf{r} = (1-t)\mathbf{p}+t\mathbf{q}$. We determine the location of $\mathbf{r}$ relative to $\mathbf{p}$ and $\mathbf{q}$ by calculating $l_1 = \|\mathbf{r} - \mathbf{p}\| = 2\sqrt{2}$ and $l_2 = \|\mathbf{q} - \mathbf{r}\| = 4\sqrt{2}$. The barycentric coordinates must sum to one, so we need the total length $l_3 = l_1 + l_2 = 6\sqrt{2}$. Then the barycentric coordinates are $t = l_1/l_3 = 1/3$ and $(1-t) = 2/3$. Check that this is correct:

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 7 \\ 7 \end{bmatrix}.$$

⋆18. The barycentric coordinates are $(1-t)$ and t such that $\mathbf{r} = (1-t)\mathbf{p}+t\mathbf{q}$. We determine the location of $\mathbf{r}$ relative to $\mathbf{p}$ and $\mathbf{q}$ by calculating $l_1 = \|\mathbf{r} - \mathbf{p}\| = 4\sqrt{2}$ and $l_2 = \|\mathbf{q} - \mathbf{r}\| = 2\sqrt{2}$. The barycentric coordinates must sum to one, so we need the total length $l_3 = l_1 + l_2 = 6\sqrt{2}$. Then the barycentric coordinates are $t = l_1/l_3 = 2/3$ and $(1-t) = 1/3$. Check that this is correct:

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 7 \\ 7 \end{bmatrix}.$$

19. No, they are linearly dependent.

⋆20. The area of the parallelogram is zero since $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

21. Yes, $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$ since they are linearly independent.

⋆22. The linear combination is $\mathbf{u} = 2\mathbf{v}_1 + 1\mathbf{v}_2$.

⋆23. These properties are easily shown when starting with the knowledge that scalar multiplication is symmetric and distributive: $ab = ba$ and $a(b + c) = ab + ac$ for scalars a, b, c.

Symmetric property:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$
$$u_1 v_1 + u_2 v_2 = v_1 u_1 + v_2 u_2,$$

which follows from the symmetric property of scalar multiplication.

Homogeneous property:

$$\mathbf{v} \cdot (s\mathbf{w}) = s(\mathbf{v} \cdot \mathbf{w})$$
$$v_1 s w_1 + v_2 s w_2 = s(v_1 w_1 + v_2 w_2),$$

which again follows from the symmetric property of scalar multiplication.

Distributive property:

$$(\mathbf{v} + \mathbf{w})\mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$$
$$(v_1 + w_1)u_1 + (v_2 + w_2)u_2 = v_1 u_1 + v_2 u_2 + w_1 u_1 + w_2 u_2,$$

and by the distributive and symmetric property of scalar multiplication, we see these are equal.

We can see that $\mathbf{v} \cdot \mathbf{v} > 0$ if $\mathbf{v} \neq \mathbf{0}$ since

$$\mathbf{v} \cdot \mathbf{v} = v_1 v_1 + v_2 v_2 = (v_1)^2 + (v_2)^2$$

is positive for $v_1 \neq 0$ and $v_2 \neq 0$. In fact, only when $\mathbf{v}$ is the zero vector is $(v_1)^2 + (v_2)^2 = 0$.

24. The dot product, $\mathbf{v} \cdot \mathbf{w} = 5 \times 0 + 4 \times 1 = 4$. Scalar product is another name for dot product and the dot product has the symmetry property, therefore $\mathbf{w} \cdot \mathbf{v} = 4$.

25. The angle between the vectors $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ is $90°$ by inspection. Sketch it and this will be clear. Additionally, notice $5 \times 3 + 5 \times -3 = 0$.

⋆26. The cosine of the angle is computed as

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{20}{(5\sqrt{2}) \times 4} = \frac{1}{\sqrt{2}} \approx 0.707107.$$

(It is handy to remember that $\cos \theta = 1/\sqrt{2}$ corresponds to $\theta = 45°$.) Since $\cos \theta > 0$, we know that the angle is less than $90°$.

27. The angles fall into the following categories: θ_1 is obtuse, θ_2 is a right angle, and θ_3 is acute.

28. The orthogonal projection $\mathbf{u}$ of $\mathbf{w}$ onto $\mathbf{v}$ is determined by (2.21), or specifically,

$$\mathbf{u} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}}{\sqrt{2}^2} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}.$$

Draw a sketch to verify. Therefore, the $\mathbf{u}^\perp$ that completes the orthogonal decomposition of $\mathbf{w}$ is

$$\mathbf{u}^\perp = \mathbf{w} - \mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}.$$

Add $\mathbf{u}^\perp$ to your sketch to verify.

⋆29. In words: we must find the orthogonal projection of $\mathbf{w}$ onto $\mathbf{v}$. First notice that $\|\mathbf{v}\| = 1$, thus

$$\mathbf{u} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|} \mathbf{v} = 2\sqrt{2}\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Next, we find

$$\mathbf{u}^\perp = \mathbf{w} - \mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

⋆30. No, the Cauchy-Schwartz inequality states that

$$(\mathbf{v} \cdot \mathbf{w})^2 \leq \|\mathbf{v}\|^2 \|\mathbf{w}\|^2.$$

31. Equality of the Cauchy-Schwartz inequality holds when $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent. A simple example:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

The two sides of Cauchy-Schwartz are then $(\mathbf{v} \cdot \mathbf{w})^2 = 9$ and $\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = 1^2 \times 3^2 = 9$.

32. No, the triangle inequality states that $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

⋆33. When $\mathbf{w} = k\mathbf{v}$ for $k \geq 0$, we have equality in the triangle inequality. Example:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

then $\|\mathbf{v} + \mathbf{w}\| = 3$ and $\|\mathbf{v}\| + \|\mathbf{w}\| = 1 + 2 = 3$.

Chapter 3   Lining Up: 2D Lines

1. The line is defined by the equation $\mathbf{l}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$, thus

$$\mathbf{l}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

We can check that

$$\mathbf{l}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \times \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{p},$$

$$\mathbf{l}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{q}.$$

⋆2. The line $\mathbf{l}(t)$ in the linear interpolation form is $\mathbf{l}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$, or

$$\mathbf{l}(t) = (1 - t) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

We can check that

$$\mathbf{l}(0) = 1 \times \mathbf{p} + 0 \times \mathbf{q} = \mathbf{p},$$
$$\mathbf{l}(1) = 0 \times \mathbf{p} + 1 \times \mathbf{q} = \mathbf{q}.$$

3. The parameter value $t = 2$ is outside of $[0, 1]$, therefore $\mathbf{l}(2)$ is not formed from a convex combination.