FUNDAMENTALS OF OPEN CHANNEL FLOW

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<<Slides relating to Chapter 2 Energy>>

Remember fluid mechanics?

- Name two important equations you learned…
  - Continuity:

$$Q = A \cdot v$$

  - Bernoulli:

$$E = \frac{p}{\gamma} + \frac{v^2}{2g} + y$$

How is that we can measure energy in units of feet ([=] length)?

- In physics I learned E [=] ML^2T^{-2}

$$\text{Total Energy} = \text{Kinetic Energy} + \text{Potential Energy}$$

$$E = \frac{1}{2}mv^2 + mgh$$

- Divide by mg

$$E' = \frac{v^2}{2g} + h$$

- Call $E' \rightarrow E$ and $h \rightarrow y$

The basic energy equation for open channel flow

$$E = \frac{v^2}{2g} + y$$

- or, actually we usually equate energy between an upstream location (1) and downstream location (2):

$$\frac{v_1^2}{2g} + y_1 = \frac{v_2^2}{2g} + y_2 + h_L$$

- where h_L is headloss between 1 and 2

What's happened to the pressure term from Bernoulli's equation?

$$\frac{v_1^2}{2g} + y_1 = \frac{v_2^2}{2g} + y_2 + h_L$$

- **Pressure isn't zero. It's simply that we are studying *open* channels and therefore the pressure at location 1 and 2 is the same – atmospheric pressure.**

Analogy to Pipe Flow – the Dual Role of Flow Depth

- << to be drawn on whiteboard >>

"Specific" or "Unit" Discharge

- Discharge is Q [=] L^3/T
- Consider a rectangular channel of width, w.
  - Specific or unit discharge (q) is the discharge per unit width of channel:

$$q = \frac{Q}{w}$$

- Specific or unit discharge is q [=] L^2/T
- q is ***ONLY*** defined in the context of a rectangular channel

Alternative Expression for Specific Energy

- By continuity:

$$Q = A \cdot v = y \cdot w \cdot v$$

- Therefore:

$$q = y \cdot v$$

- and:

$$v = \frac{q}{y}$$

Alternative Expression for Specific Energy

- **Substituting in specific energy equation:**

$$E = \frac{v^2}{2g} + y = \frac{q^2}{2gy^2} + y$$

- **This second form of the equation is:**
  - ***ONLY*** **defined for rectangular section**
  - **Useful because it expresses E as a function of y only (all other terms are constants)**

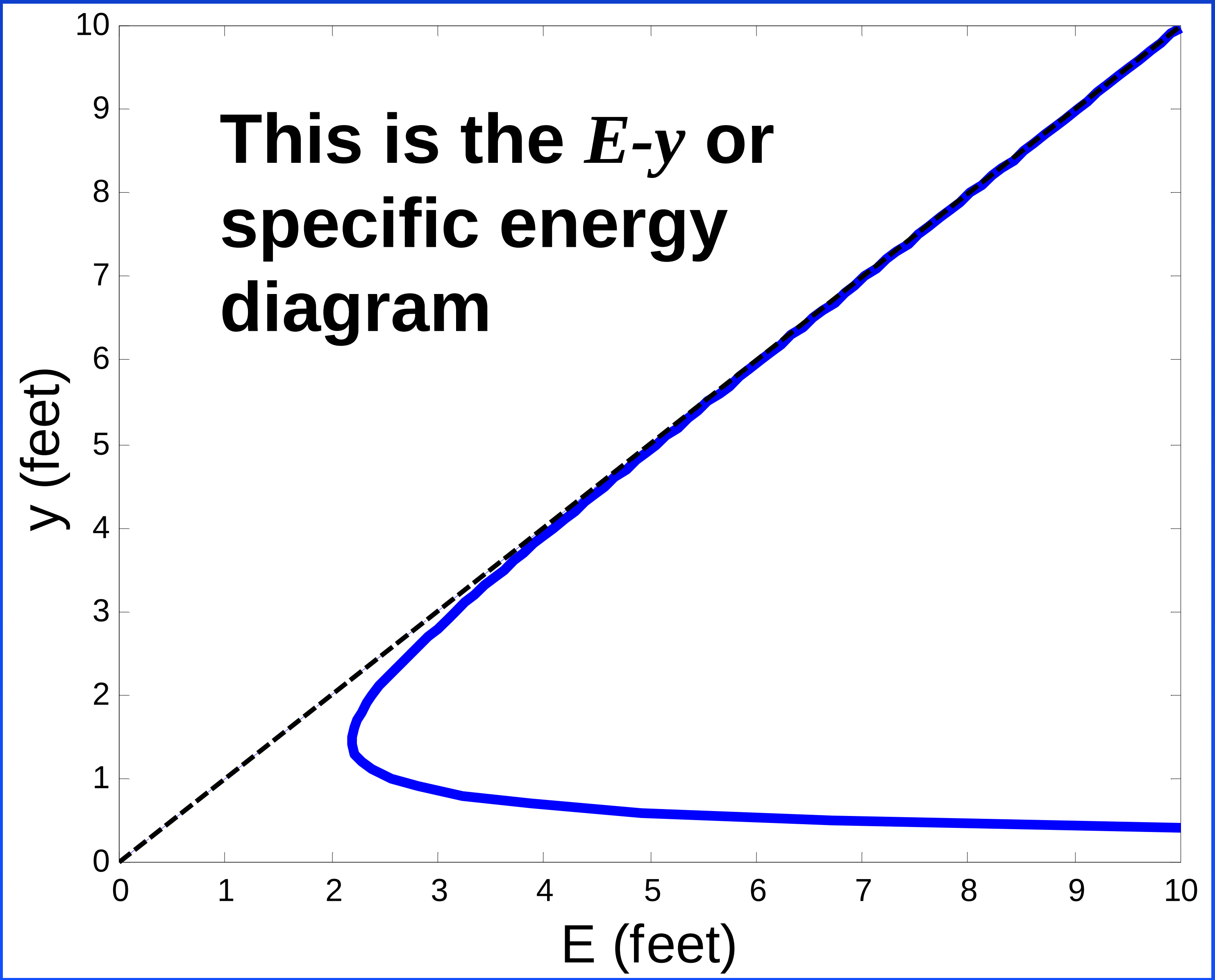
Example 1:

- Let Q = 10 ft^3/s, w = 1 foot

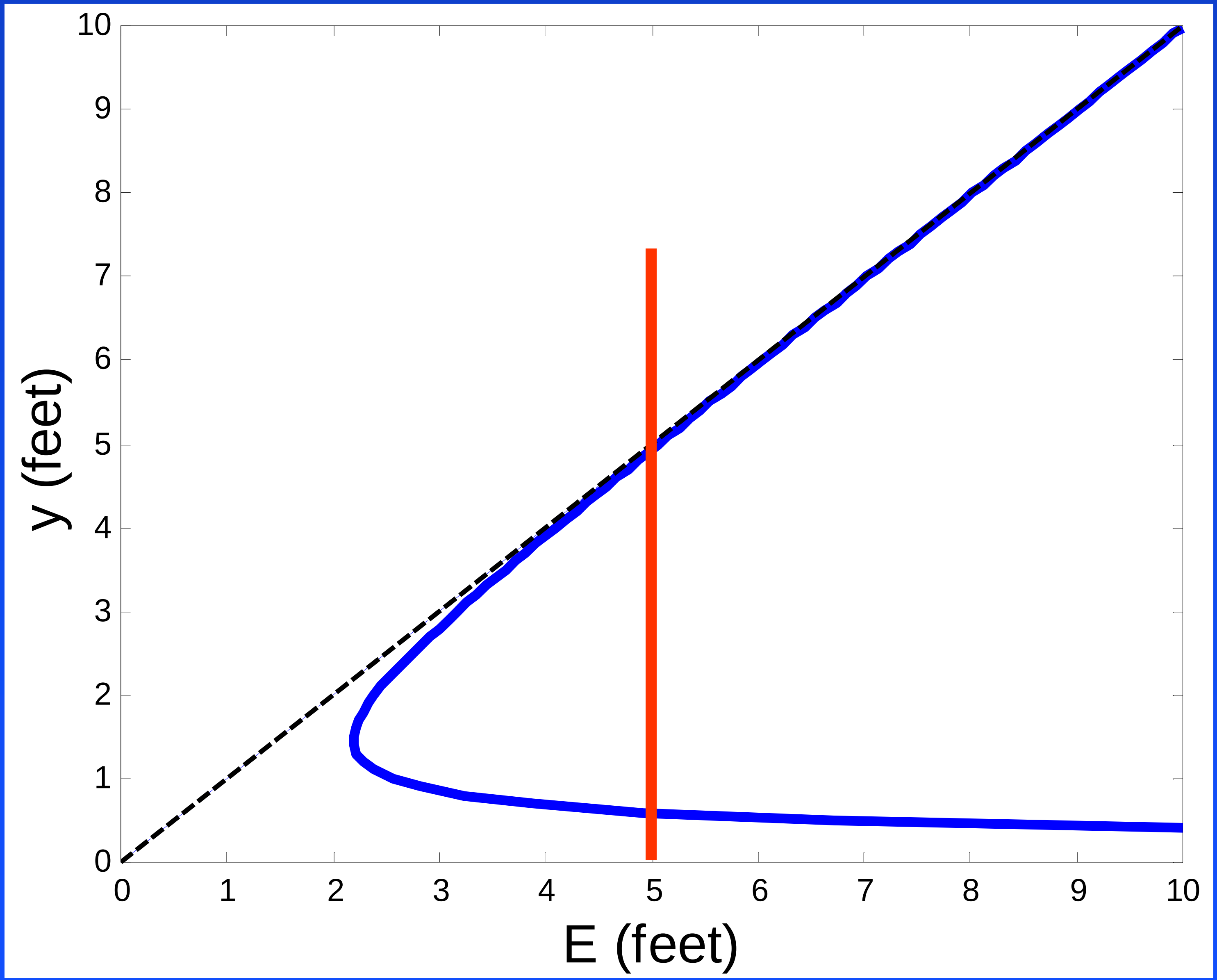
$$E = \frac{q^2}{2gy^2} + y$$

- So q = 10/1 = 10 ft^2/s
- We could let y vary over a range of values (say 0.4 feet to 10 feet). The resulting relationship between E and y looks like:

Example 1(cont.):



Example 1(cont.): Consider E=5 feet



Example 1(cont.):

- How can there be two depths that possess the same energy?
- E is sum of kinetic and energy sources.
  - One depth has high kinetic, low potential energy
  - One depth has high potential, low kinetic energy
  - Which is which?
- How do we find these two depths?

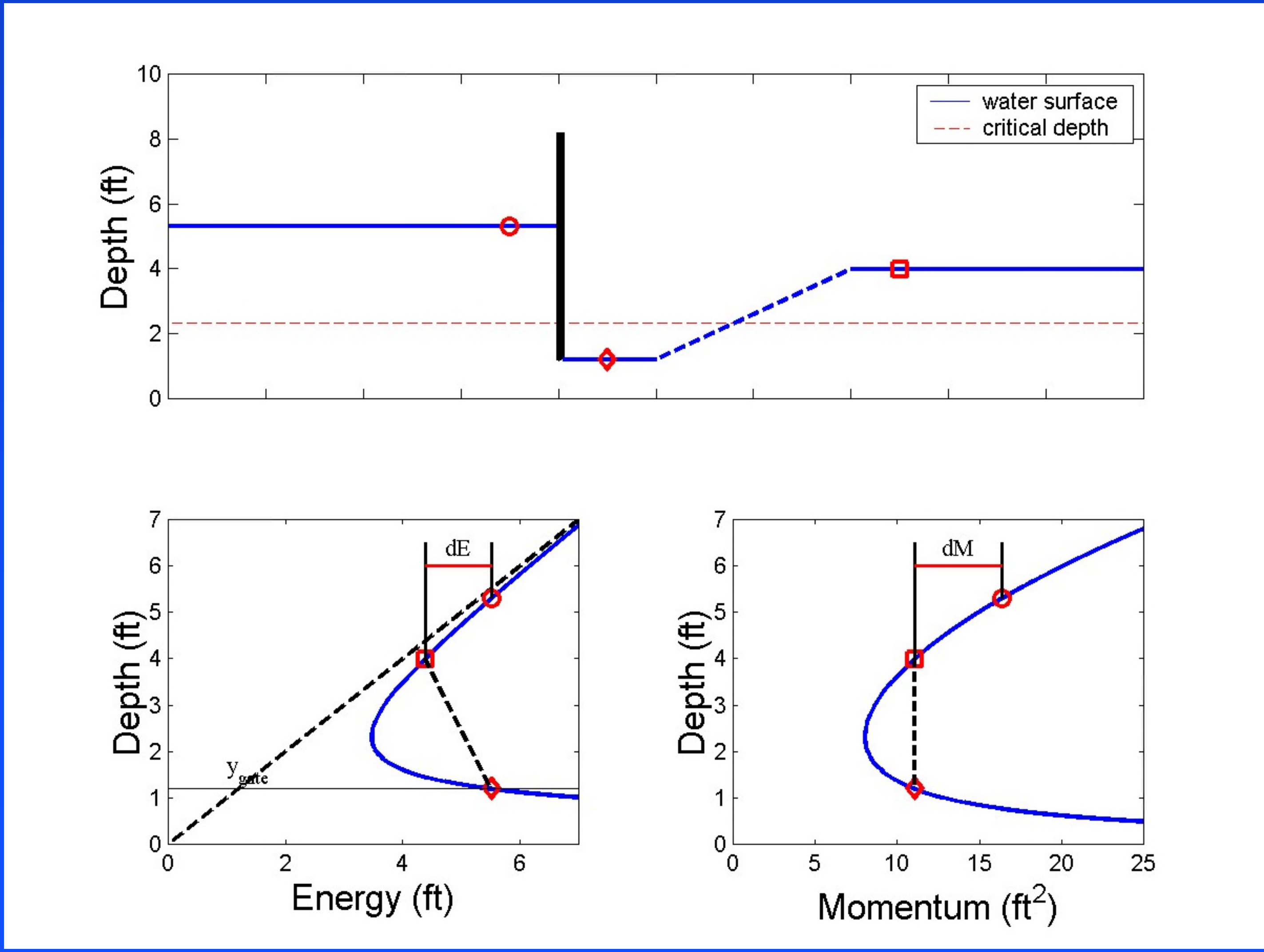
Example 1 (cont): Take-away facts

- There are 2 depths (called "alternate depths") which have the same energy for a given discharge.
- Depths can be calculated by trial and error
- For a ***RECTANGULAR*** channel if we know one depth, y_1, we can calculate the other, y_2:

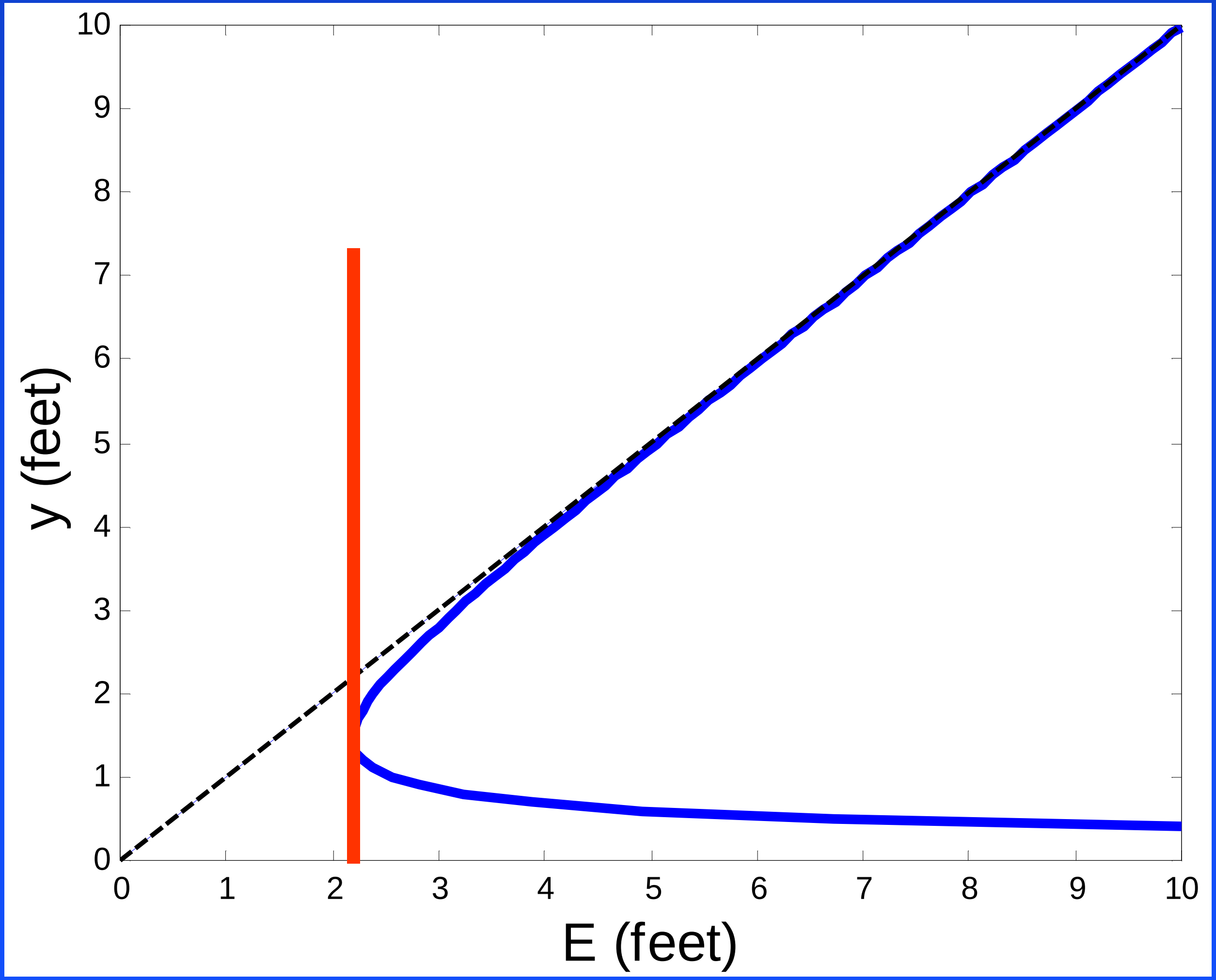
$$y_2 = \frac{2y_1}{-1+\sqrt{1+\frac{8gy_1^3}{q^2}}}$$

Example 1: Relevant Animation (available only at Scholar Site)

- **Sluice gate followed by a hydraulic jump**



Example 2: What is the smallest energy that can be associated with q=10 ft^2/s?



Example 2 (cont): What is the smallest energy that can be associated with q = 10 ft^2/s?

- << to be derived on whiteboard >>

Example 2 (cont): Take-away facts

- Critical depth in a rectangular channel:

$$y_c = \left(\frac{q^2}{g}\right)^{1/3}$$

- Critical energy in a rectangular channel:

$$E_c = 1.5 y_c$$

- Definition of Froude number in a rectangular channel:

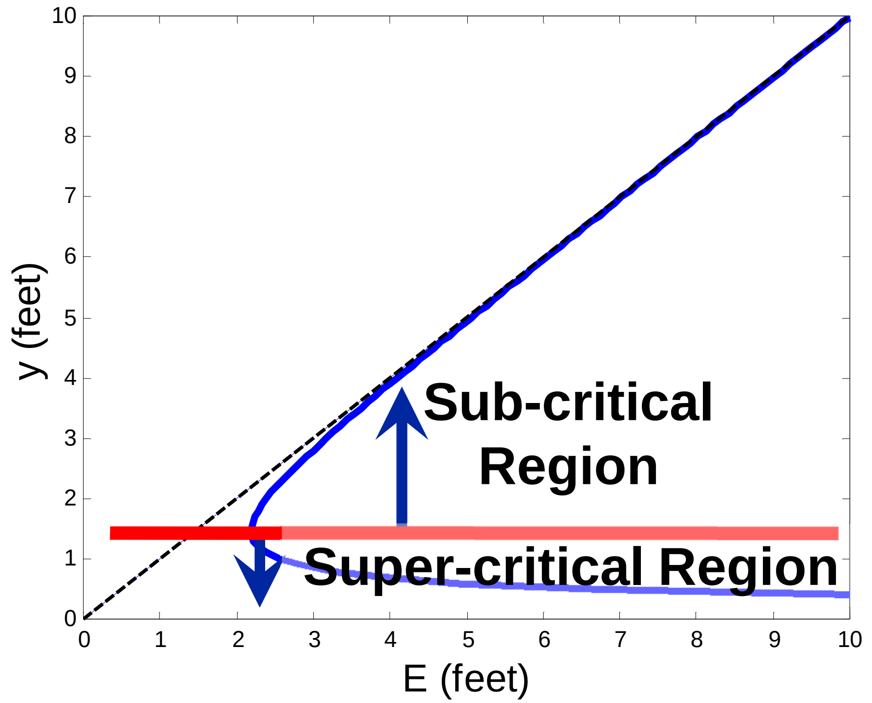
$$F_r = \frac{v}{\sqrt{gy}}$$

Example 2 (cont): Take-away facts

- Froude number at critical depth

$$\left(F_r\right)_c = \frac{v}{\sqrt{g y_c}} = 1$$

- Regions of super-critical or sub-critical flow.



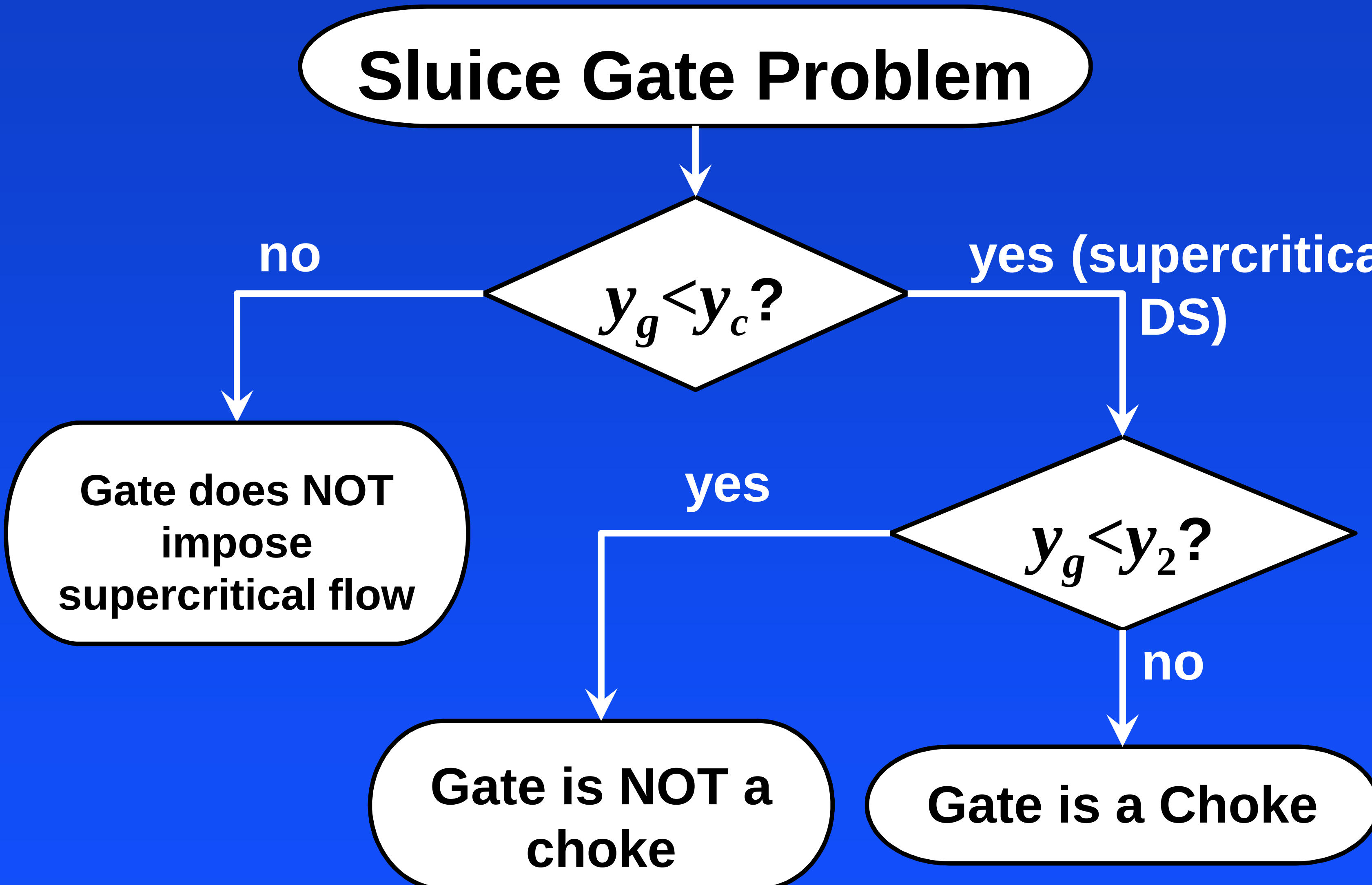
Example 3: Sluice Gate

- Settings:
  - q = 10.0 ft^2/s
  - Incoming flow depth is y_1=4.94 feet
  - Gate opening, y_g is set to y_g=1.30 feet
- Fact: Energy is conserved at a sluice gate.
- What is depth downstream of gate?
- << to be derived on whiteboard >>

Example 3 (cont): Take-away facts

- Flow accessibility:
  - Super-critical flow downstream ***ONLY*** if gate opening is less than or equal to critical depth.
  - Depth downstream is alternate to depth upstream (generally less than gate opening).
- What if $y_g > y_c$?
- What if $y_g < y_2$? (Example 4)

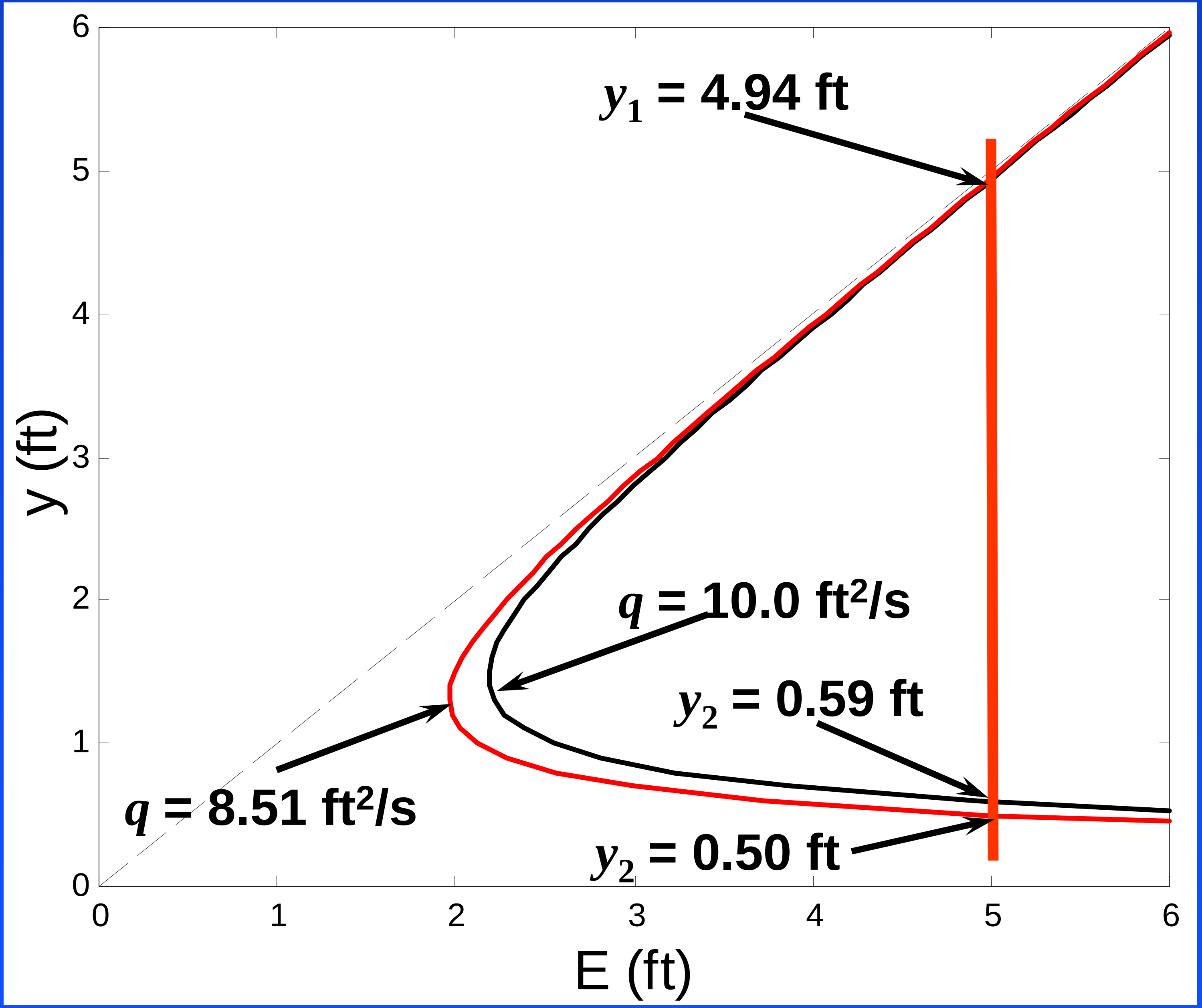
Sluice Gate Problems: A flow chart



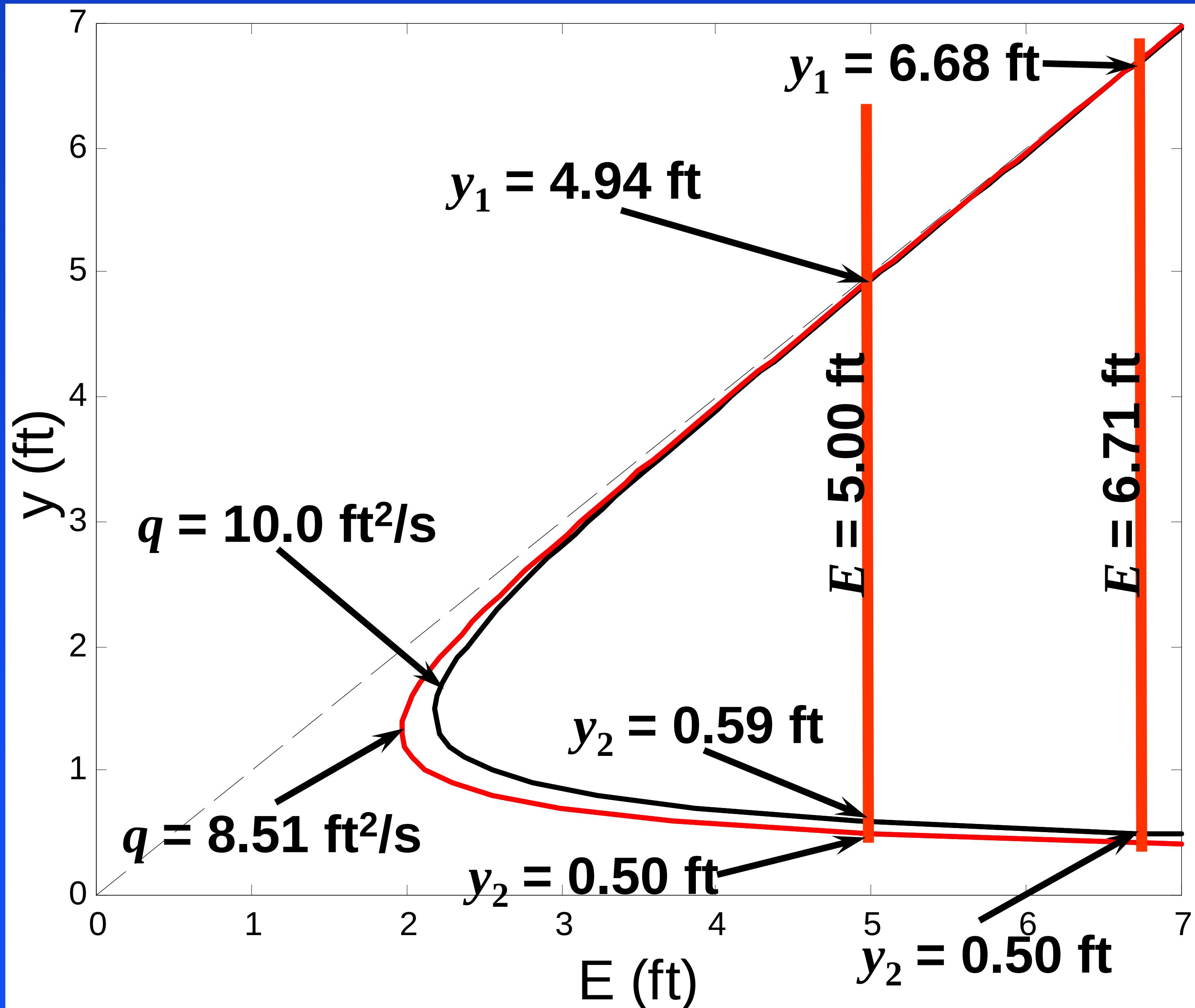
Example 4: Sluice Gate

- Settings:
  - q = 10.0 ft^2/s
  - Incoming flow depth is y_1=4.94 feet
  - Gate opening, y_g is set to y_g=0.50 feet
- Recall, alternate depth to y_1=4.94 feet is y_2 = 0.59 feet (so y_g<y_2)
- What is depth downstream of gate? Are any other depths changed?
- With E=5.0 feet, at the instant y_g=0.50 feet what is unit discharge?
- << to be derived on whiteboard >>

Example 4 (cont): E-y diagram for choke



Example 4 (cont): E-y diagram for choke



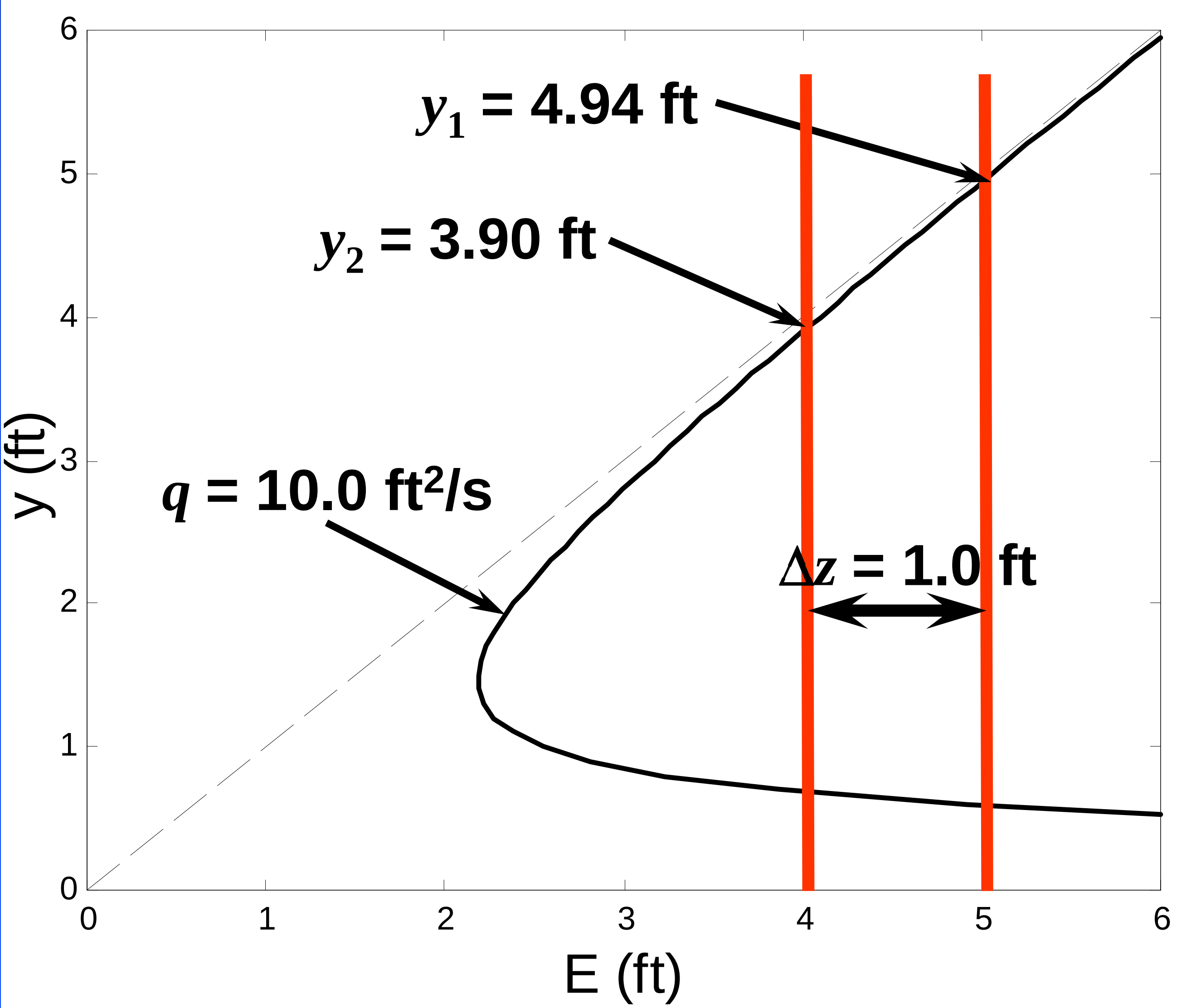
Example 4 (cont): Take-away facts

- This example shows the sluice gate acting as a “choke”. The gate is requiring the flow to have more energy than is initially present to pass the full discharge.
- The choke changes the steady-state values of both y_1 and y_2
- There is a temporary transient condition, for which we can determine the initial passed discharge based on critical energy considerations.

Example 5: Step Problem (sub-critical flow upstream of step)

- Settings:
  - q = 10.0 ft^2/s
  - Incoming flow depth is y_1=4.94 feet (E=5.0 feet)
- A short, smooth upward step (Δz=1.0 foot) is encountered.
- What is the depth of flow beyond the step?
- What is Froude number before/after the step?
- << to be derived on whiteboard >>

Example 5 (cont): Upstream and Downstream depths for an upward step



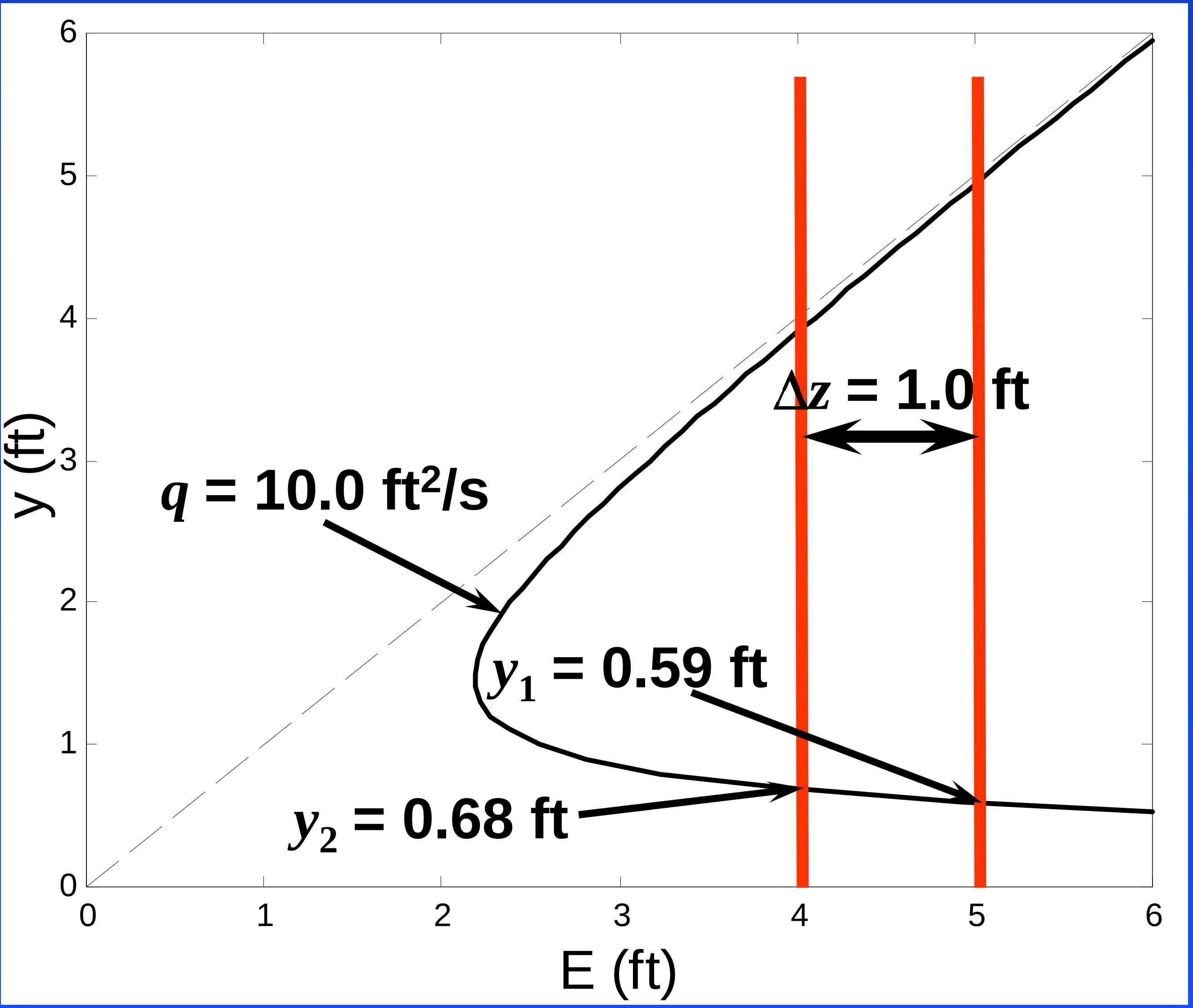
Example 5 (cont): Take-away facts

- Step acts as an energy “tax”.
- Depth of flow downstream of step is smaller than upstream of step.
- Net water surface elevation shifts downwards!
- Flow is driven closer to critical conditions.

Example 6: Step Problem (super-critical flow upstream of step)

- Settings:
  - q = 10.0 ft^2/s
  - Incoming flow depth is y_1=0.59 feet (E=5.0 feet)
- A short, smooth upward step (Δz=1.0 foot) is encountered.
- What is the depth of flow beyond the step?
- What is Froude number before/after the step?
- << to be derived on whiteboard >>

Example 6 (cont): Upstream and Downstream depths for an upward step



Example 6 (cont): Take-away facts

- Step again acts as an energy “tax”.
- Depth of flow downstream of step is greater than upstream of step.
- Flow is driven closer to critical conditions.

Example 7: Step Problem – a "negative" step

- **Settings:**
  - q = 10.0 ft^2/s
  - Incoming flow depth is y_1=3.9 feet (E=4.0 feet)
- **A short, smooth *downward* step (Δz=-1.0 foot) is encountered.**
- **What is the depth of flow beyond the step?**
- **What is Froude number before/after the step?**
- **<< to be derived on whiteboard >>**

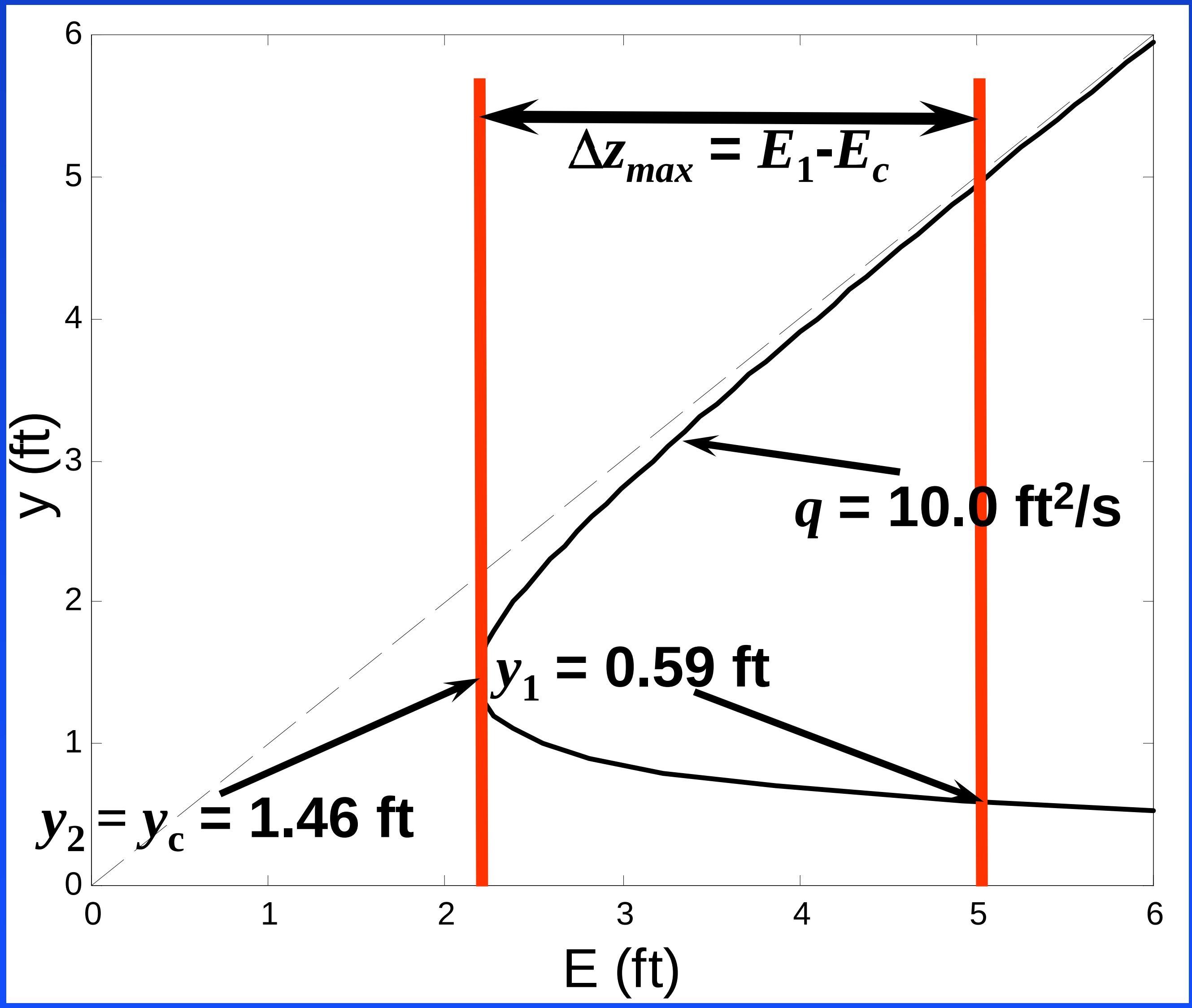
Example 7 (cont): Take-away facts

- Step again acts as an energy “gift”.
- For sub-critical upstream depth, depth of flow downstream of negative step is greater than upstream of step.
- For super-critical upstream depth, depth of flow downstream of negative step is less than upstream of step.
- Flow is driven further away from critical conditions.
- Can assemble examples 5 & 7 or 6 & 7 into steps of finite length.

Example 8: Maximum step – a step as a choke (sub-critical upstream)

- Settings:
  - q = 10.0 ft^2/s
  - Incoming flow depth is y_1=4.94 feet (E=5.0 feet)
- What is the biggest step possible with flow as specified?
- What if step is bigger than that?
- << to be derived on whiteboard >>

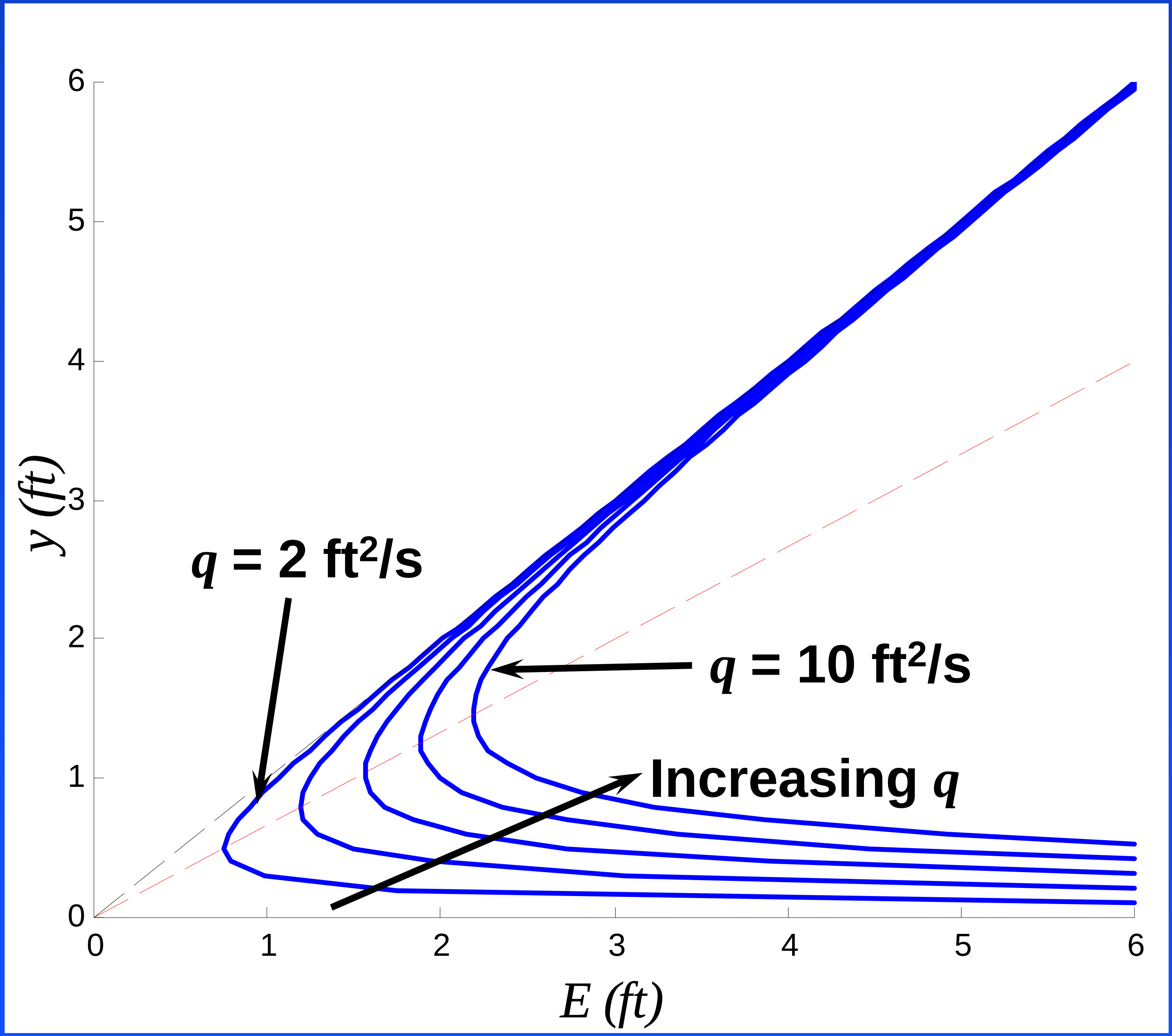
Example 8 (cont): Maximum step size



Example 8 (cont): Take-away facts

- Biggest step corresponds to step that brings flow to the brink of critical conditions.
- A larger step acts as a choke.
  - There is a temporary transient condition, for which we can determine the initial passed discharge based on critical energy considerations.

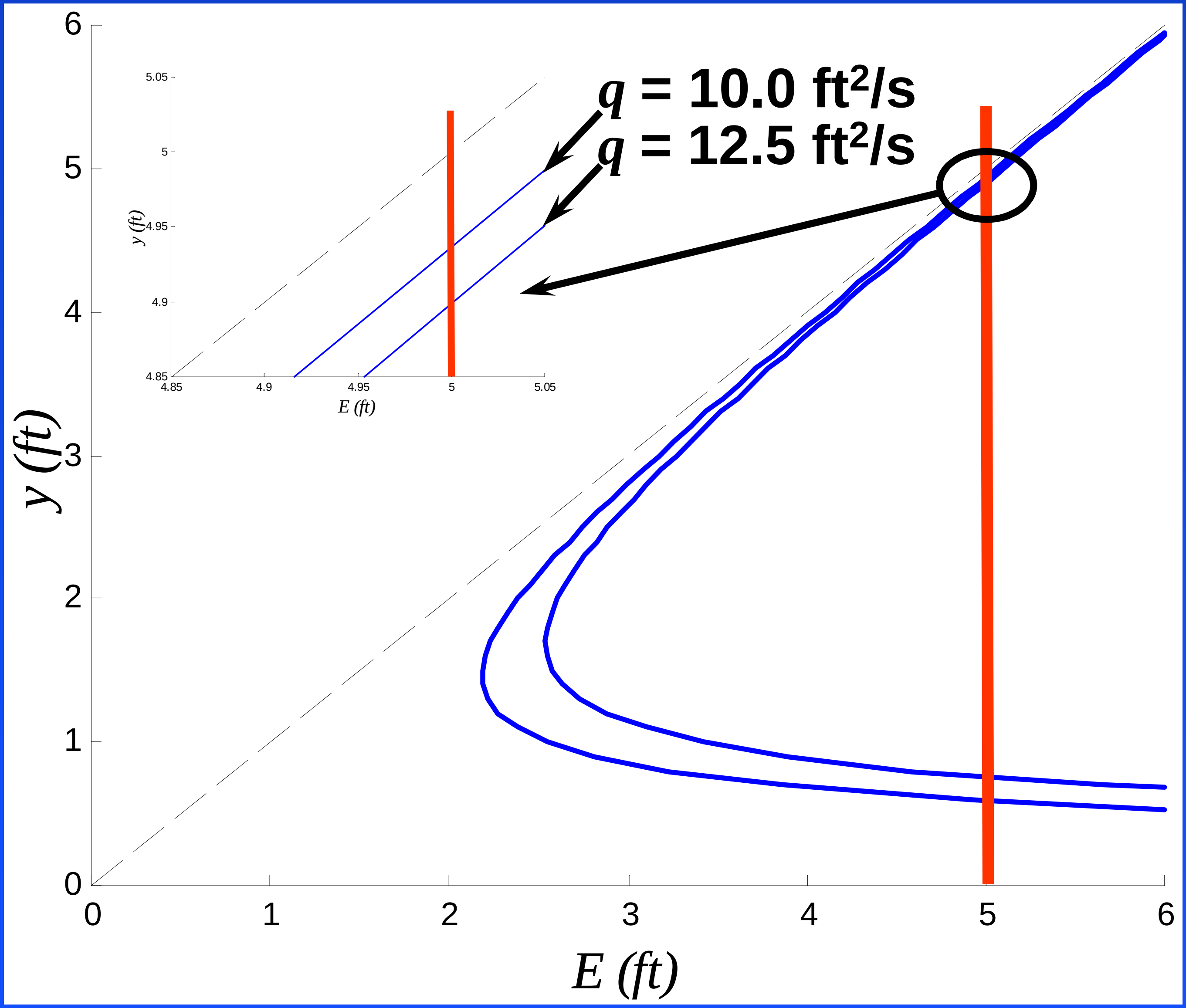
A comment on q and the E-y diagram



Example 9: A constriction (sub-critical)

- Settings:
  - Q = 100. ft^3/s
  - w_1= 10.0 ft, q_1 = 10.0 ft^2/s
  - Constriction: w_2= 8.0 ft, q_2 = 12.5 ft^2/s
  - Incoming flow depth is y_1=4.94 feet (E=5.0 feet)
- What is the depth of flow in the constriction?
- Interpret effect of constriction on E-y diagram
- << to be derived on whiteboard >>

Example 9 (cont): *E*-*y* interpretation

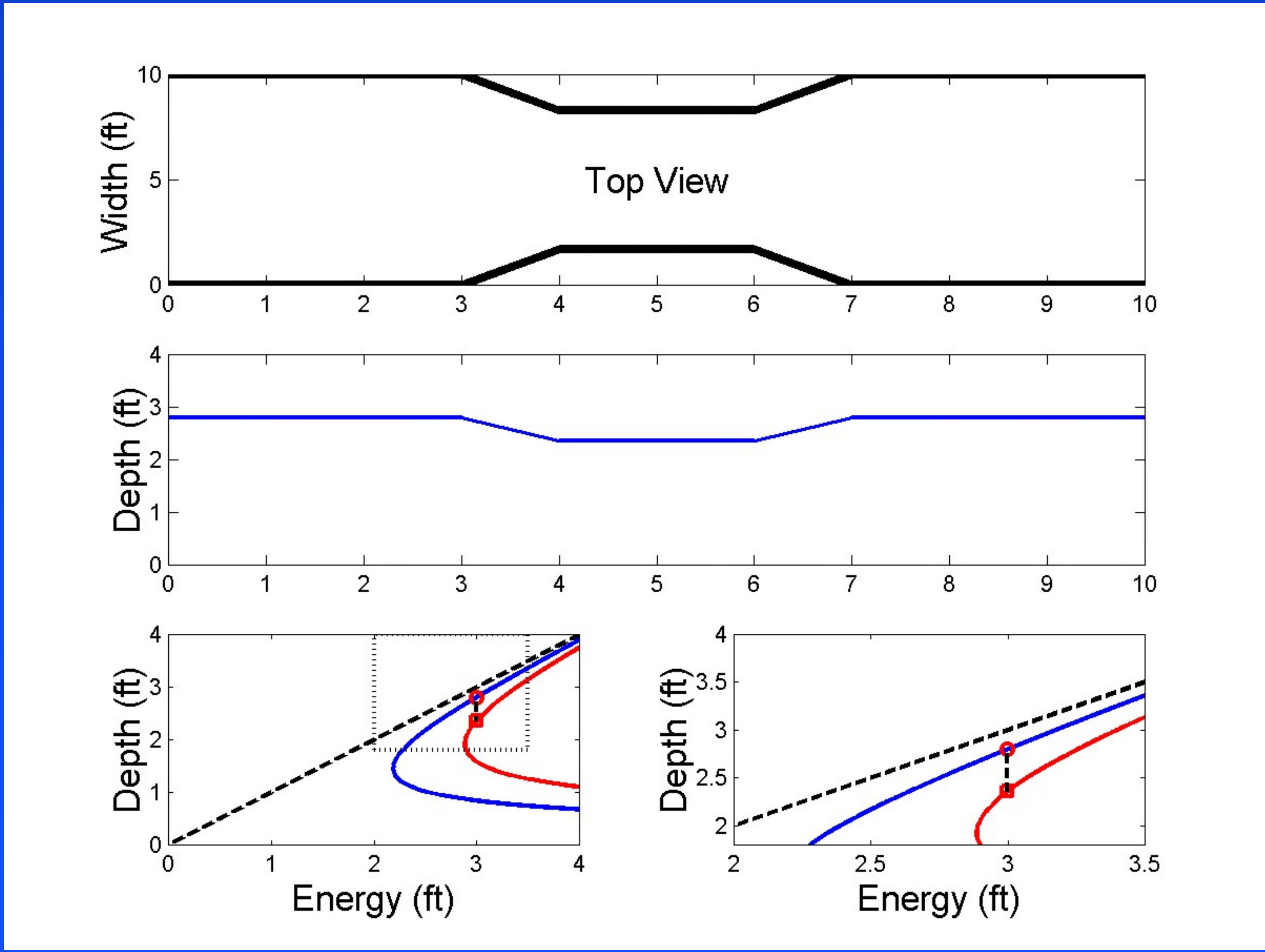


Example 9 (cont): Take-away facts

- Constriction has the effect of changing the "q-curve" which applies to the flow.
  - Since flowing sub-critically, depths shifts downwards to corresponding higher q-curve.
- Energy is conserved so shift on E-y diagram is in vertical direction.
- Constriction moves flow closer to critical flow conditions.

Example 9: Relevant Animation (available only at Scholar Site)

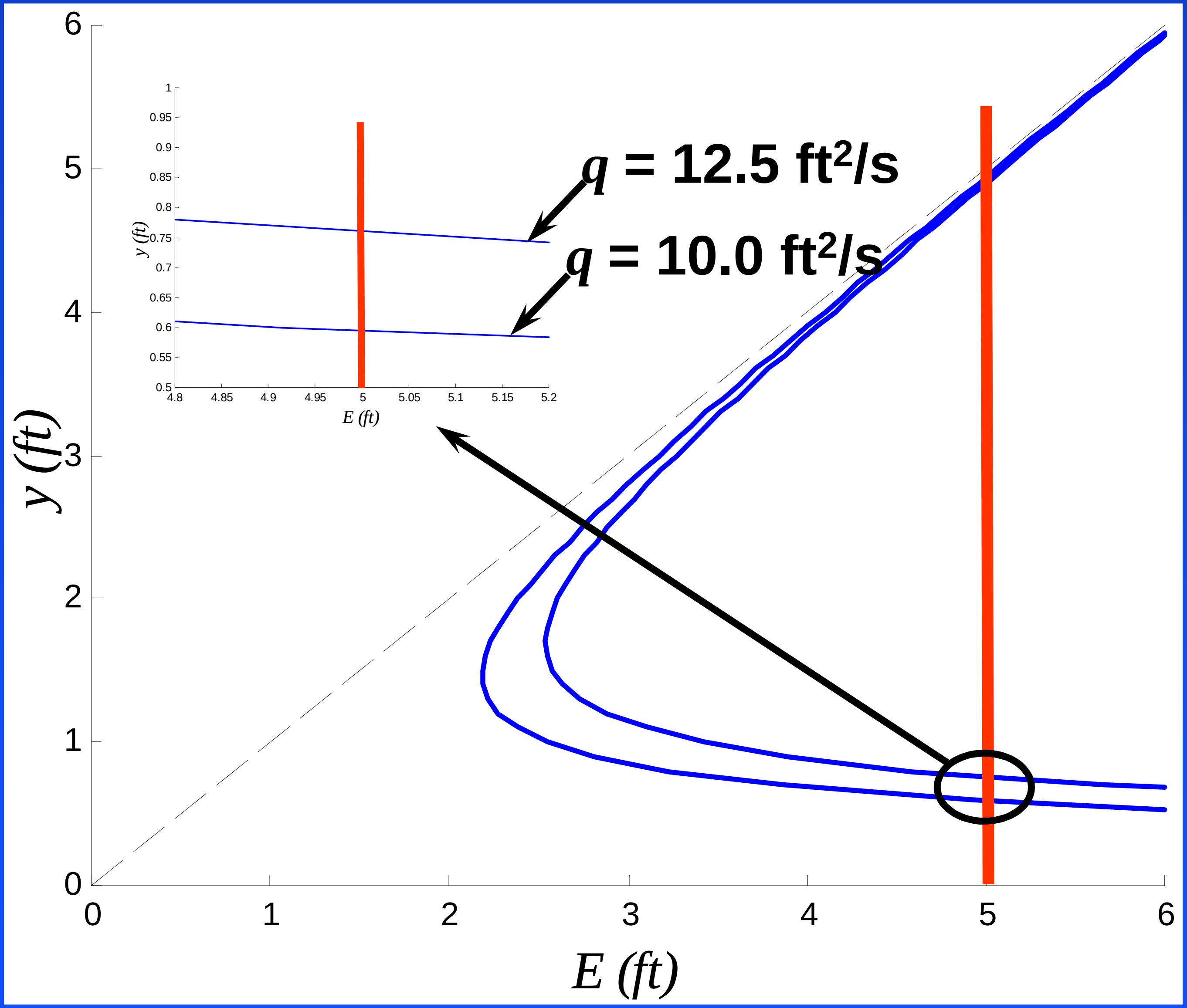
- Steady-state constriction animation



Example 10: A constriction (super-critical)

- Settings:
  - Q = 100. ft^3/s
  - w_1= 10.0 ft, q_1 = 10.0 ft^2/s
  - Constriction: w_2= 8.0 ft, q_2 = 12.5 ft^2/s
  - Incoming flow depth is y_1=0.59 feet (E=5.0 feet)
- What is the depth of flow in the constriction?
- Interpret effect of constriction on E-y diagram
- << to be derived on whiteboard >>

Example 10 (cont): E-y interpretation



Example 10 (cont): Take-away facts

- Constriction has the effect of changing the "q-curve" which applies to the flow.
  - Since flowing super-critically, depths shifts upwards to corresponding higher q-curve.
- Energy is conserved so shift on E-y diagram is in vertical direction.
- Constriction moves flow closer to critical flow conditions.

Example 11: A “negative” constriction (sub- and super-critical)

- Settings:
  - Q = 100. ft^3/s
  - w_1= 10.0 ft, q_1 = 10.0 ft^2/s
  - Constriction: w_2= 12.0 ft, q_2 = 8.33 ft^2/s
  - Incoming flow depth 1a is y_{1a}=4.94 feet (E=5.0 feet – sub-critical)
  - Incoming flow depth 1b is y_{1b}=0.59 feet (E=5.0 feet – super-critical)
- What is the depth of flow in the “constriction”?
- << to be derived on whiteboard >>

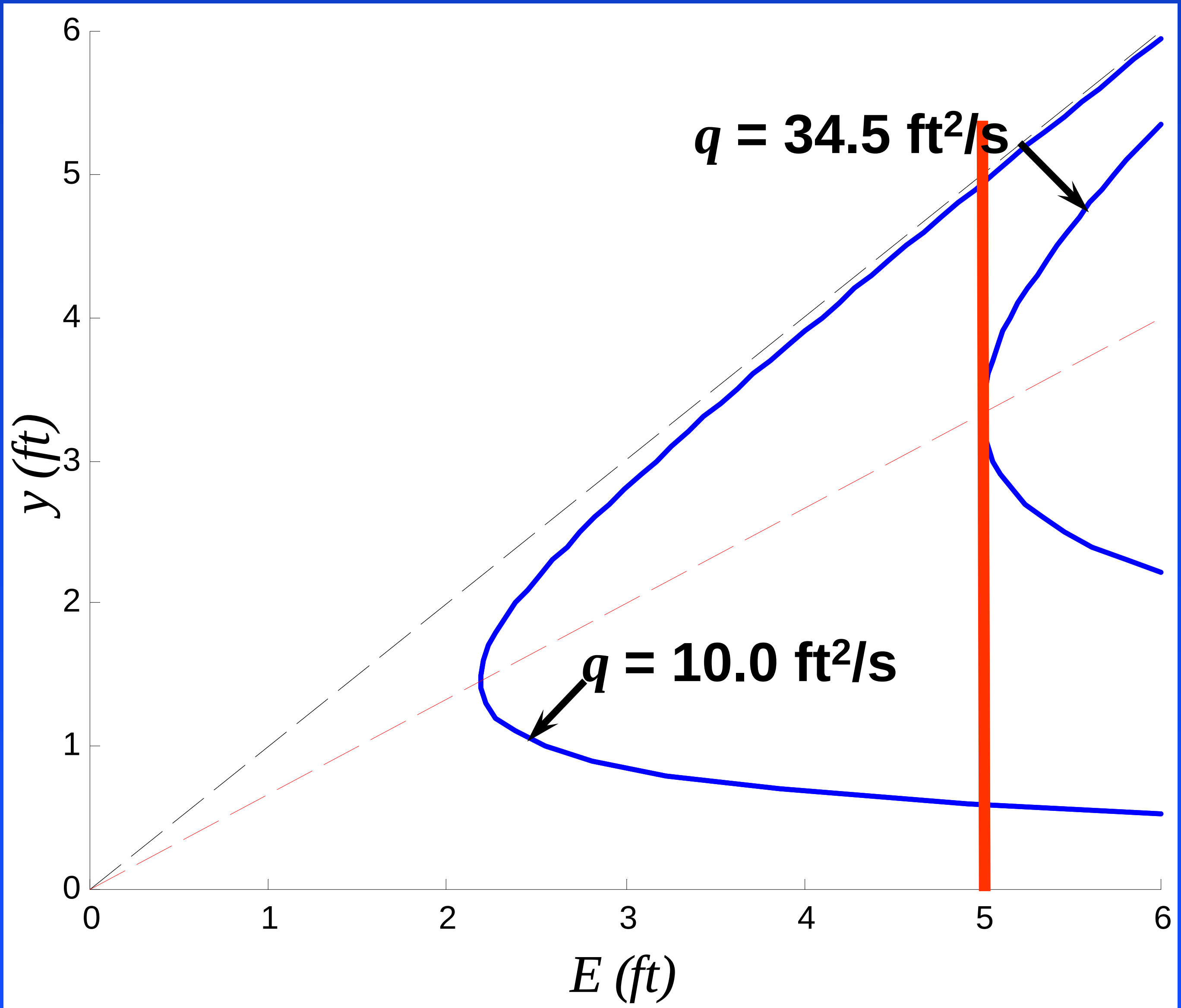
Example 11 (cont): Take-away facts

- "Negative" constriction has the effect of changing the "q-curve" which applies to the flow.
- Expansion drives depths in opposite direction from constriction
  - Sub-critical flow depth increases
  - Super-critical flow depth decreases
  - Froude #'s move further away from critical flow conditions.

Example 12: Maximum constriction – a constriction as a choke (sub-critical upstream)

- Settings:
  - q = 10.0 ft^2/s
  - Incoming flow depth is y_1=4.94 feet (E=5.0 feet)
- What is the biggest constriction possible with flow as specified?
- What if constriction is bigger than that?
- << to be derived on whiteboard >>

Example 12 (cont): *E*-*y* interpretation



Example 12 (cont): Take-away facts

- Width that drives flow to critical conditions determined from observing the y_c=2/3E_c.
- To think about: Repeat same theme as Examples 4 and 8 where constriction is instantaneously applied (say, new width is 2.50 feet).
  - What is flow (Q) initially?
  - What is final upstream depth?
  - What if flow upstream of constriction is super-critical?

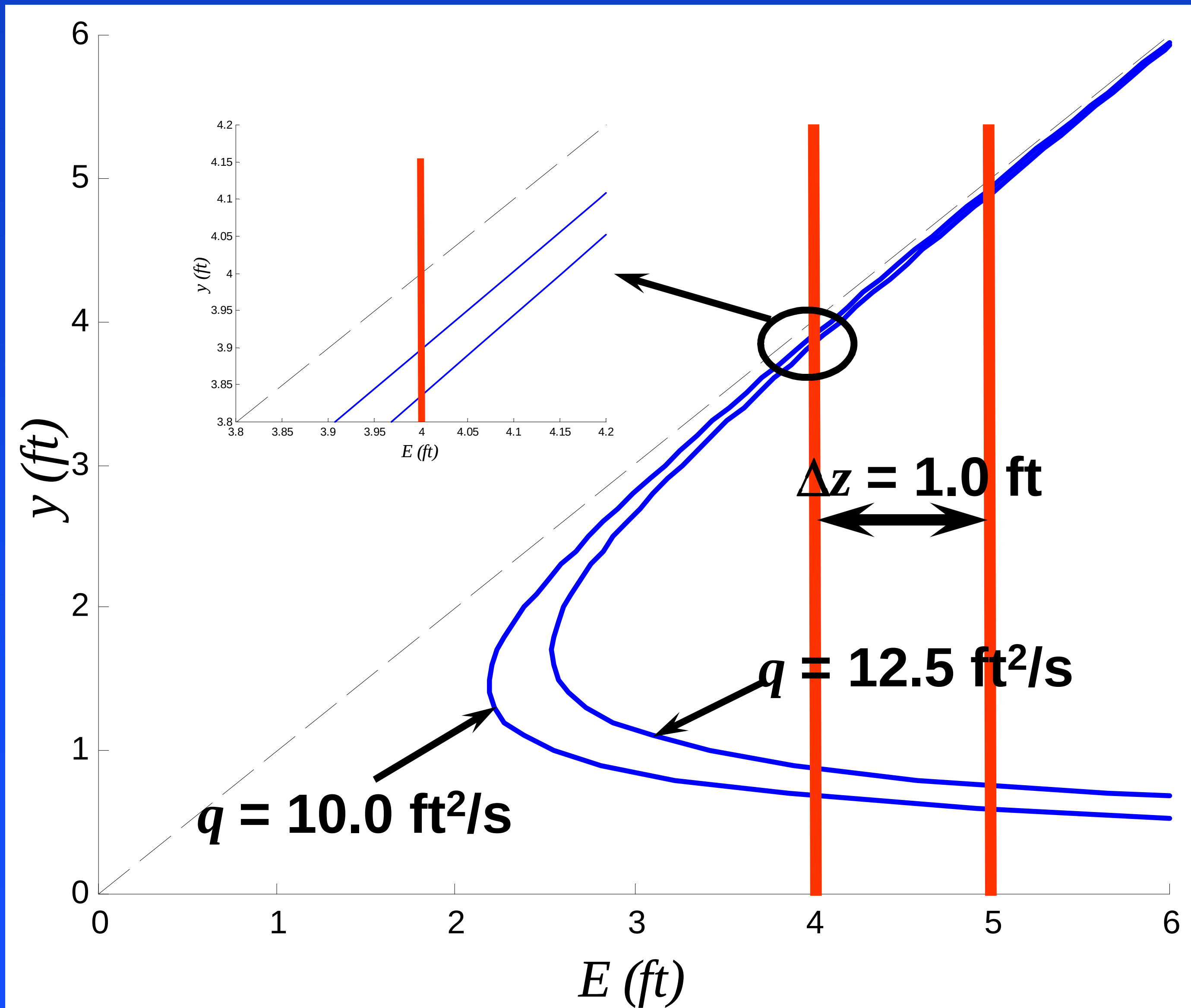
Example 12 (cont): Take-away facts

- Width that drives flow to critical conditions determined from observing the $y_c = 2/3E_c$.
- To think about: Repeat same theme as Examples 4 and 8 where constriction is instantaneously applied (say, new width is 2.50 feet).
  - What is flow (Q) initially?
  - What is final upstream depth?
  - What if flow upstream of constriction is super-critical?

Example 13: Simultaneous step and constriction

- Settings:
  - Q = 100. ft^3/s
  - Incoming flow depth is y_1=4.94 feet (E=5.0 feet)
  - Upward step Δz=1.0 foot
  - Constriction: w_1 = 10 feet, w_2 = 8 feet
- Find E_2, y_2, Fr_2
- Interpret on E-y diagram
- << to be derived on whiteboard >>

Example 13 (cont): *E*-*y* interpretation



Example 13 (cont): Take-away facts

- Problem is just a simple composite (super-position) of the individual step and constriction problems.
- Since both upward step and constriction individually move flow towards critical flow, the pair acting together move it even more towards critical flow
- Permutations on this theme:
  - Constriction causes choke, how much of downwards step is needed?
  - Upwards step causes choke, how much of an expansion is needed?

Occurrence of Critical Flow

- **Previously, we derived:**

$$\frac{dE}{dy} = 1 - \frac{q^2}{gy^3} = 1 - F_r^{\,2}$$

- **Now, let's consider total head (*H*) measured along longitudinal flow direction**

$$H = E + z = y + \frac{v^2}{2g} + z$$

- **Let's take *dH/dx*: <<see whiteboard>>**

Occurrence of Critical Flow: Take-away facts

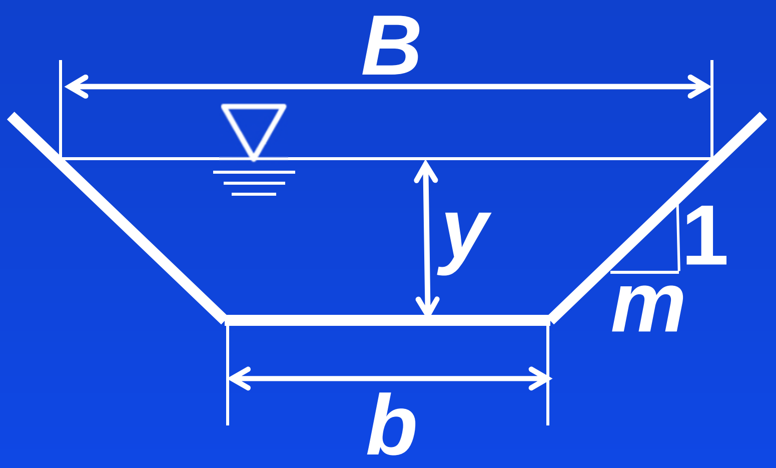
- Regardless of cross-sectional shape:

$$\frac{dH}{dx} = \frac{dE}{dx} + \frac{dz}{dx} = \left(\frac{dy}{dx}\right) \cdot \left(1 - F_r^{\,2}\right) + \frac{dz}{dx}$$

- Can be invoked to identify locations where critical flow occurs (e.g. outfall to steep slope)
- Can also be used to determine qualitative value of depth of flow under varying circumstances (esp. step problems)

Non-Rectangular Geometry – obvious but…

- Trapezoidal Cross-Section



$$A = by + my^2$$

$$B = b + 2my$$

- << Sample calculation on whiteboard >>

Using Circular Cross-Section Properties Handout

d = diameter

y = depth of flow

A = water area

P = wetter perimeter

R = hydraulic radius

T = top width

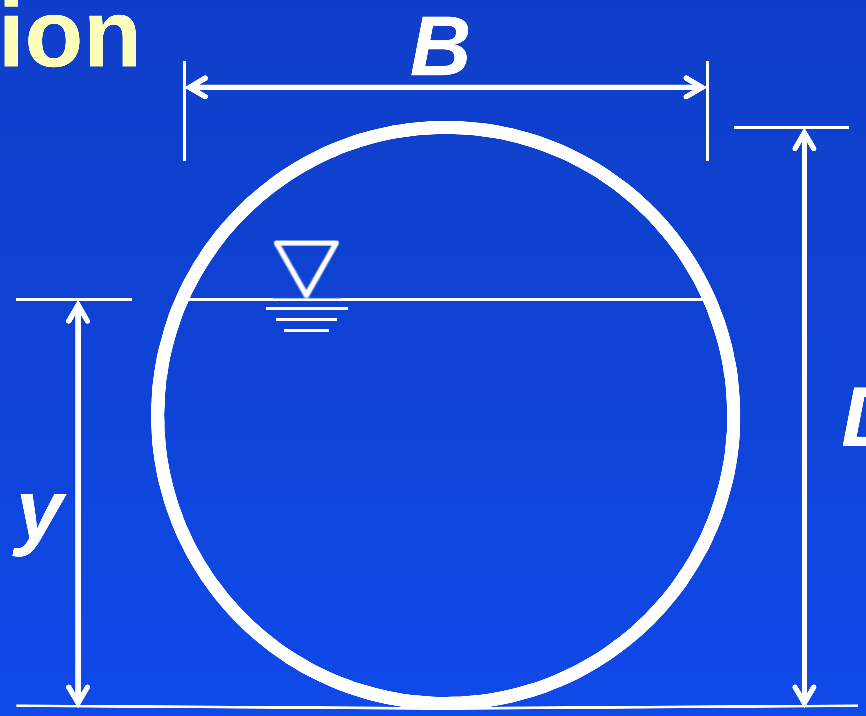
D = hydraulic depth

z = $AD^{0.5}$= section factor for critical-flow computation

$\frac{y}{d}$	$\frac{A}{d^2}$	$\frac{P}{d}$	$\frac{R}{d}$	$\frac{T}{d}$	$\frac{D}{d}$	$\frac{z}{d^{2.5}}$	$\frac{AR^{2/3}}{d^{8/3}}$
0.33	0.2260	1.2239	0.1848	0.9404	0.2404	0.1107	0.0736
0.34	0.2355	1.2451	0.1891	0.9474	0.2486	0.1172	0.0776
0.35	0.2450	1.2661	0.1935	0.9539	0.2568	0.1241	0.0820
0.36	0.2546	1.2870	0.1978	0.9600	0.2652	0.1310	0.0864
0.37	0.2642	1.3078	0.2020	0.9656	0.2736	0.1381	0.0909
0.38	0.2739	1.3284	0.2061	0.9708	0.2822	0.1453	0.0955
0.39	0.2836	1.3490	0.2102	0.9755	0.2908	0.1528	0.1020
0.40	0.2934	1.3694	0.2142	0.9798	0.2994	0.1603	0.1050
0.41	0.3032	1.3898	0.2181	0.9837	0.3082	0.1682	0.1100

Circular Cross-Section

- Calculate factor, y/d
- Look up A/d^2, T/d



$$A = \left(\frac{A}{d^2}\right) \cdot D^2$$

$$B = \left(\frac{T}{d}\right) \cdot D$$

From Circular Cross-Section Properties Handout

- << Sample calculation on whiteboard >>

Figure 2-24
(y_c)

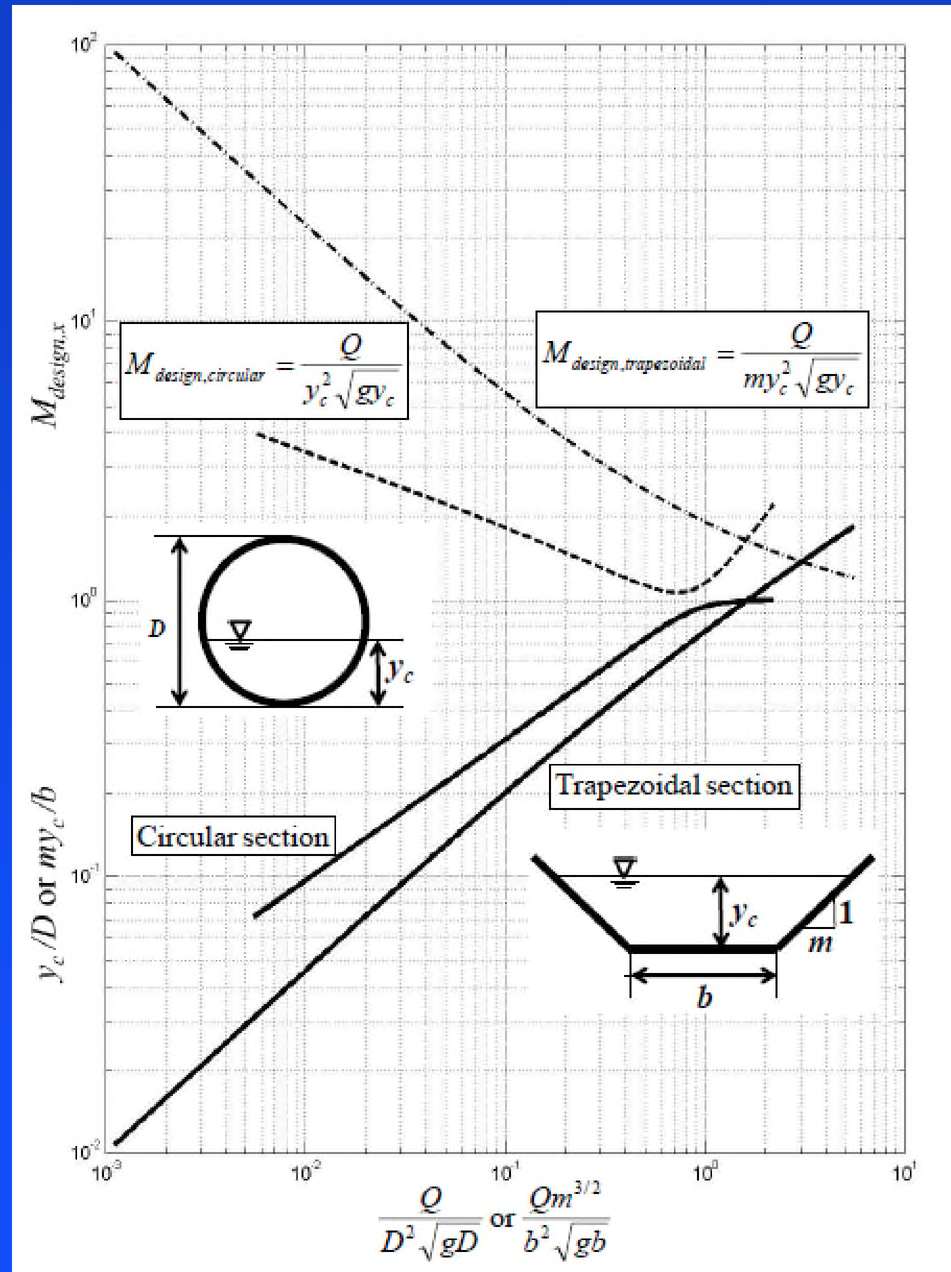
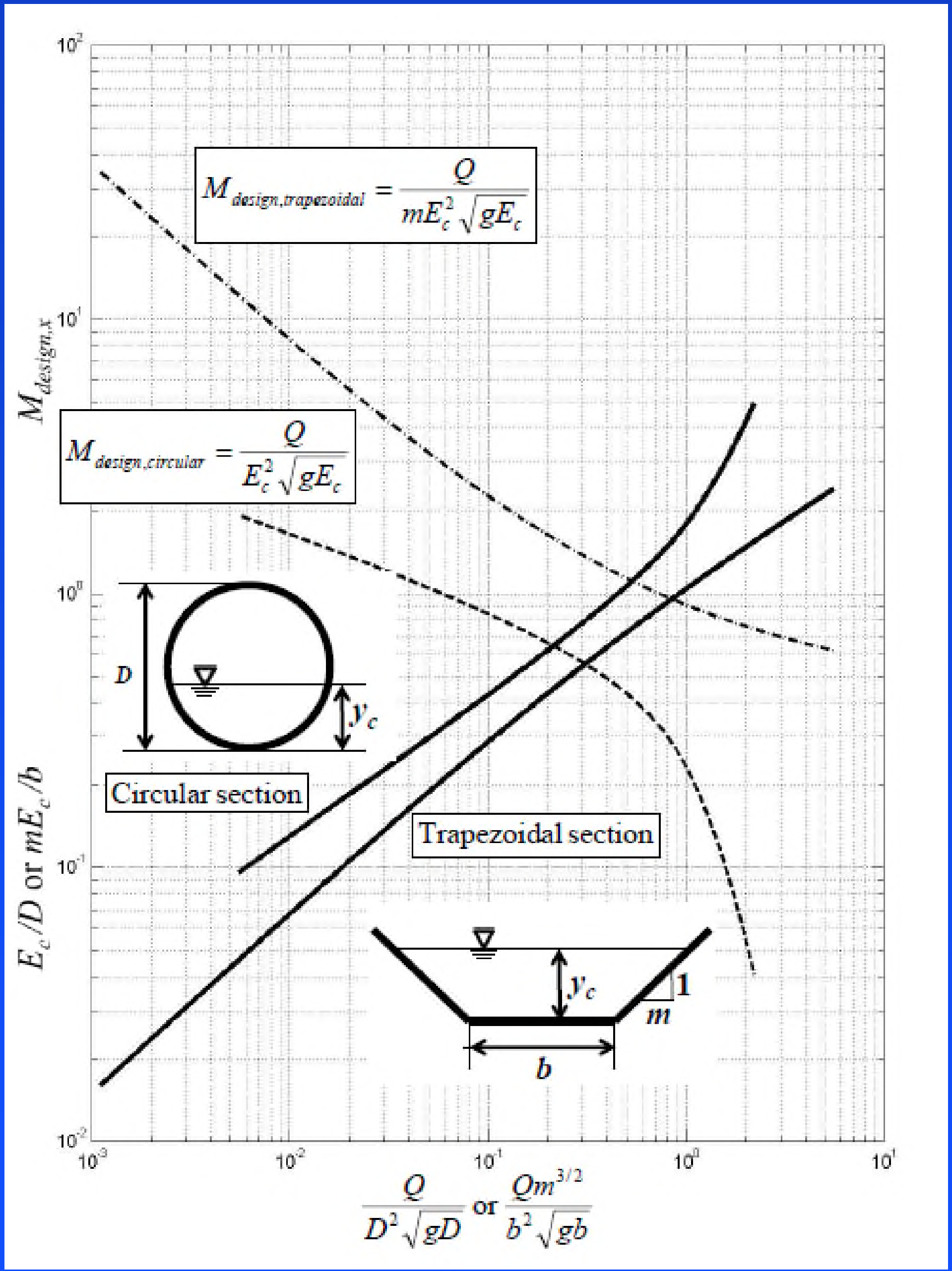


Figure 2-25
(E_c)



Example 14: Using Figures 2-24 and 2-25

- Settings:
  - Trapezoidal Channel, b = 5 feet, m = 2
  - Outfall to steep slope
  - Lake level away from outfall is 2 feet higher than outfall invert
- Find Q
- << to be derived on whiteboard >>

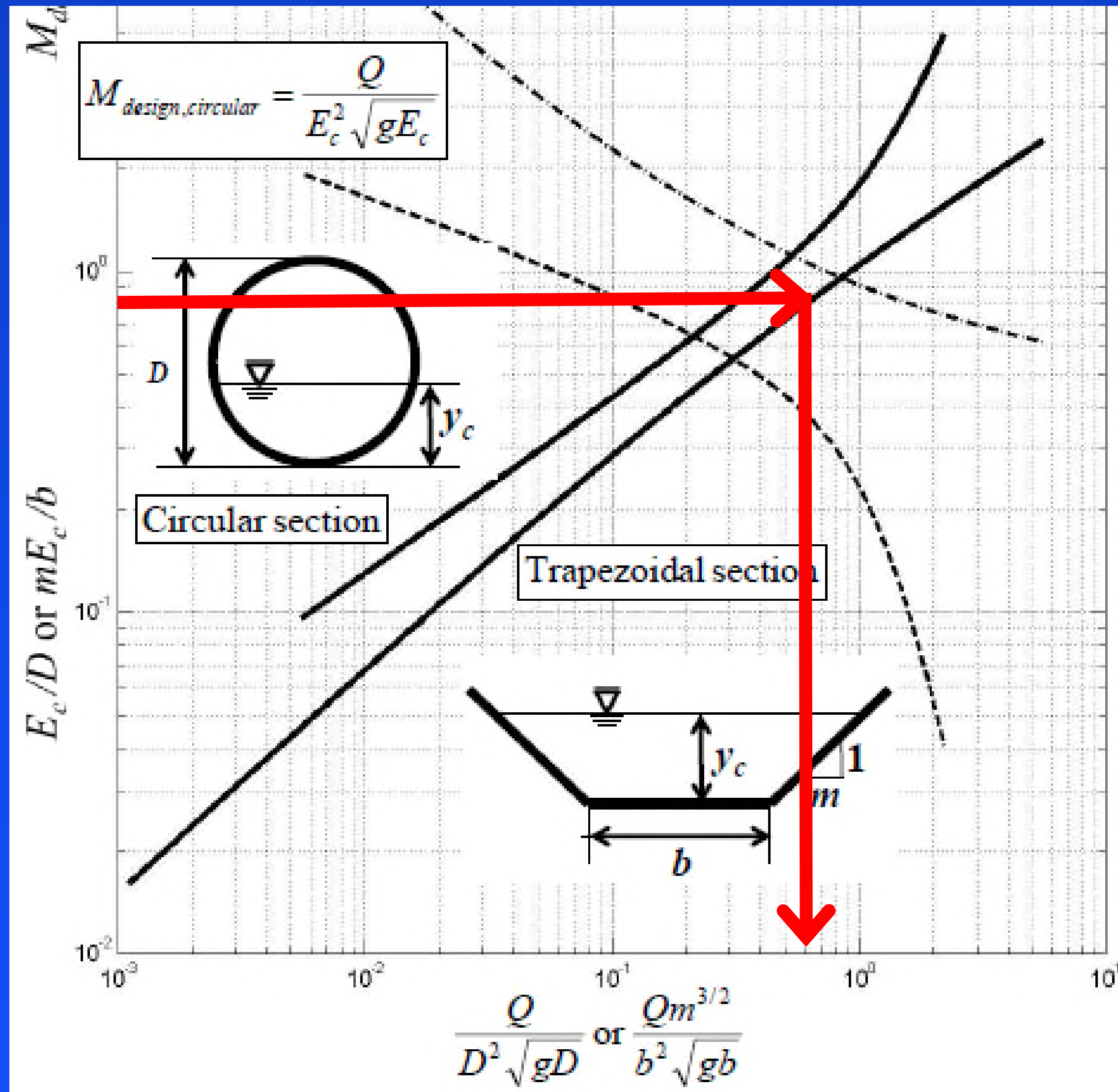
$$\frac{mE_c}{b} = \frac{2 \cdot 2}{5} = 0.8$$

$$Z = 0.6$$

$$Q = \frac{Zb^2\sqrt{gb}}{m^{3/2}}$$

$$Q = \frac{(0.6)(5)^2\sqrt{32.2(5)}}{2^{3/2}}$$

$$Q = 67\ \frac{\text{ft}^3}{\text{s}}$$



Solving Example 14 w/o Figure 2-25

- Re-arrange Specific Energy equation, solving for Q:

$$Q = \sqrt{\left(E_c - y\right) \cdot (2gA^2)}$$

- Note that A is $f(y)$
- Build spreadsheet with trial y values, $A(y)$, and resultant Q (from above eq'n)
- Choose maximum resultant Q

Solving Example 14 w/o Figure 2-25

- Maximum Q value
- Note that solution points to 68 ft^3/s (Figure 2-24 gave us 67 ft^3/s)
- Looks like we can really only read Figure 2-25 to one significant figure
- So Figure 2-25 answer is more accurately, 70 ft^3/s

y	A	Q
1.35	10.40	67.25
1.36	10.50	67.40
1.37	10.60	67.54
1.38	10.71	67.67
1.39	10.81	67.78
1.40	10.92	67.88
1.41	11.03	67.97
1.42	11.13	68.04
1.43	11.24	68.10
1.44	11.35	68.14
1.45	11.46	68.17
1.46	11.5[illegible]	68.19
1.47	11.67	68.19
1.48	11.78	68.17
1.49	11.89	68.14
1.50	12.00	68.09
1.51	12.11	68.03
1.52	12.22	67.95
1.53	12.33	67.85

Example 15: Using Figures 2-24 and 2-25

- Settings:
  - Trapezoidal Channel, b = ? feet, m = 2.0
  - Outfall to steep slope
  - Depth of flow at outfall is 3.0 feet at outfall invert
  - Discharge is 260 ft^3/s
- Find b
- Which Figure?
- << to be derived on whiteboard >>

$$M_1 = \frac{Q}{m y_c^2 \sqrt{g y_c}} = 1.47, \text{ use } 1.5$$

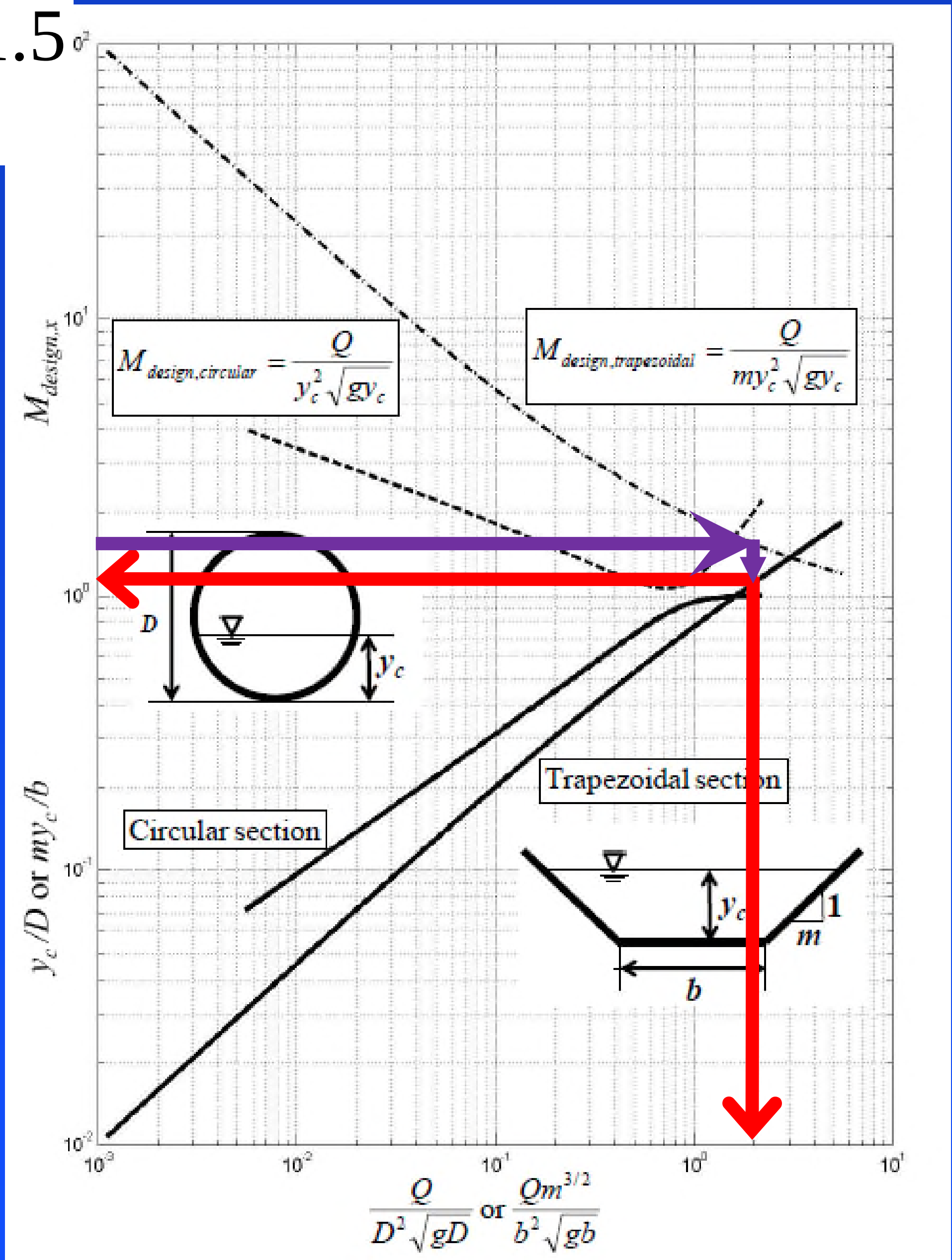
$$\frac{m y_c}{b} = \frac{2 \cdot 3}{b} = 1.1$$

$$\therefore b = 5.5$$

$$b = \left(\frac{Q m^{3/2}}{Z \sqrt{g}} \right)^{2/5}$$

$$b = \left(\frac{260 \cdot 2^{3/2}}{2 \sqrt{32.2}} \right)^{2/5}$$

$$\therefore b = 5.3$$



More Precision: Solving Example 15 w/o Figure 2-24

$$F_r = \frac{Q}{A\sqrt{g \cdot \left(\frac{A}{B}\right)}}$$

- Use definition of Froude # and known trapezoidal geometry. Vary b.
- Compare to $b = 5.3$ ft.

b	B	A	Fr
4.9	16.9	32.7	1.007319
4.91	16.91	32.73	1.006232
4.92	16.92	32.76	1.005147
4.93	16.93	32.79	1.004065
4.94	16.94	32.82	1.002985
4.95	16.95	32.85	1.001907
4.96	16.96	32.88	1.000831
4.97	16.97	32.91	0.999757
4.98	16.98	32.94	0.998686
4.99	16.99	32.97	0.997617
5	17	33	0.99655
5.01	17.01	33.03	0.995485
5.02	17.02	33.06	0.994422
5.03	17.03	33.09	0.993362
5.04	17.04	33.12	0.992304
5.05	17.05	33.15	0.991248
5.06	17.06	33.18	0.990194
5.3	17.3	33.9	0.965537

Review of Approach to Solving Critical Flow in Circular or Trapezoidal Channel

- Identify Figure. Does the problem pertain to critical *depth* or *energy*?
- Identify what problem pertains to:
  1. Discharge?
  2. Depth or Energy?
  3. Geometry (D or b)?
- If high precision is needed, spreadsheet approach may be indicated.

Non-Rectangular Channel Relationships – Discharge

- Previously (unit discharge [=] ft^2/s):

$$q = \frac{Q}{b}$$

- Now (discharge [=] ft^3/s):

$$Q$$

Non-Rectangular Channel Relationships – Specific Energy

- **Previously:**

$$E = \frac{q^2}{2gy^2} + y$$

- **Now:**

$$E = \frac{Q^2}{2gA^2} + y$$

Always good:

$$E = \frac{v^2}{2g} + y$$

Non-Rectangular Channel Relationships – Critical Depth

- **Previously:**

$$y_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}}$$

- **Now (use Figure 2-24):**

$$y_c = f_1(Q, m, b) \text{ or } f_2(Q, D)$$

Non-Rectangular Channel Relationships – Critical Energy

- **Previously:**

$$E_c = \frac{3}{2} y_c$$

- **Now (use Figure 2-25):**

$$y_c = f_3(Q, m, b) \text{ or } f_4(Q, D)$$

Non-Rectangular Channel Relationships – Froude Number

- **Previously:**

$$F_r = \frac{v}{\sqrt{gy}} = \frac{q}{y\sqrt{gy}}$$

- **Now:**

$$F_r = \frac{v}{\sqrt{g \cdot \left(\frac{A}{B}\right)}} = \frac{Q}{A\sqrt{g \cdot \left(\frac{A}{B}\right)}}$$

Non-Rectangular Channel Relationships – Alternate Depth

- **Previously:**

$$y_2 = \frac{2y_1}{-1 + \sqrt{1 + \frac{8gy_1^3}{q^2}}}$$

- **Now:**

$$\text{N/A : use trial \& error or Goal Seek in Excel}$$

Reminder: "Summary..." under "Help Files and Useful Links"

TABLE 2.6 Summary Table of Energy-Related Open Channel Flow Relationships

Quantity	*Rectangular Section*	*Irregular Section (e.g., Trapezoidal, Circular)*
Discharge	$q = \frac{Q}{b}$	Q
Critical depth	$y_c = \left(\frac{q^2}{g}\right)^{1/3}$	Figure 2.24 or use Goal Seek
Froude number	$F_r = \frac{v}{\sqrt{gy}}$	$F_r = \frac{v}{\sqrt{g\left(\frac{A}{B}\right)}}$
Energy equation	$E = y + \frac{v^2}{2g}$	$E = y + \frac{v^2}{2g}$
	$E = y + \frac{q^2}{2gy^2}$	$E = y + \frac{Q^2}{2gA^2}$
Critical energy	$E_c = \frac{3}{2}y_c$	Figure 2.25 or use Goal Seek
Alternate depths	$y_2 = \frac{2y_1}{-1+\sqrt{1+\frac{8gy_1^3}{q^2}}}$	Use Goal Seek

Brief Digression: Missing Discussion about Critical Flow

- "Information" travel and shallow wave ripple propagation

$$v_w = \sqrt{gy}$$