Chapter 2

Sequences

2.1 Convergence of Sequences

2. If not then there is an $\epsilon > 0$ such that the interval $(t - \epsilon, t]$ contains no element of S. But then $t - \epsilon/2$ is an upper bound for S, and that contradicts the hypothesis that t is the least upper bound of S.

3. If the sequence does not converge to α then, for some $\epsilon > 0$, it is the case that no element of the sequence enters the interval $(\alpha - \epsilon, \alpha + \epsilon)$. But that would imply that no subsequence has a subsequence that converges to α. Contradiction.

4. First of all notice that the Cauchy criterion can be stated as

$$\lim_{N,M\to\infty} |a_N - a_M| = 0. \tag{$*$}$$

Indeed, $(*)$ is equivalent to saying that, for any $\epsilon > 0$, there exists N_ϵ such that if $N, M > N_\epsilon$ then

$$|a_N - a_M| < \epsilon.$$

Seeking a contradiction suppose that $(*)$ does not hold for the sequence in the statement of the exercise. Then there exist $\{N_j, M_j\}_{j=1}^{\infty}$ such that

$$|a_{N_j} - a_{M_j}| > \delta$$

for some $\delta > 0$. We can arrange for this double sequence to satisfy

$$N_{j+1} > M_j$$

for all j. Let $k > 1/\delta$. Then

$$\begin{aligned} 1 < k \cdot \delta &< |a_{N_1} - a_{M_1}| + |a_{N_2} - a_{M_2}| + \cdots + \\ &\quad |a_{N_k} - a_{M_k}| \\ &= |a_{N_1} - a_{N_1+1} + a_{N_1+1} - \cdots - a_{M_1}| + \cdots + \\ &\quad |a_{N_k} - a_{N_k+1} + a_{N_k+1} - \cdots - a_{M_k}| \\ &\leq |a_{N_1} - a_{N_1+1}| + |a_{N_1+1} - a_{N_1+2}| + \cdots + \\ &\quad |a_{M_1-1} - a_{M_1}| + \cdots + |a_{M_k-1} - a_{M_k}| \\ &\leq |a_{N_1} - a_{N_1+1}| + \cdots + |a_{M_k-1} - a_{M_k}| \\ &\leq 1, \end{aligned}$$

a contradiction.

6. The answer is no. We can even construct a sequence with arbitrarily long repetitive strings and with subsequences that converges to *any* real number α. Indeed, order $\mathbb{Q}$ into a sequence $\{q_n\}$. Consider the following sequence

$$\{q_1, q_2, q_2, q_1, q_1, q_1, q_2, q_2, q_2, q_2, q_3, q_3, q_3, q_3, q_3, q_1, q_1, q_1, q_1, q_1, q_1, \cdots\}.$$

In this way we have repeated each rational number infinitely many times, and with arbitrarily long strings. From the above sequence we can find subsequences that converge to any real number.

7. If it is not the case that β_j tend to infinity then there is a positive integer N such that $|\beta_j| \leq N$ for every j. But then there is a subsequence β_{j_k} that converges to some β_0 that is less than or equal to N in absolute value. It follows then that the α_j are bounded and they have a subsubsequence $\alpha_{j_{k_\ell}}$ that converge to some α_0. It follows then that $\alpha_{j_{k_\ell}}/\beta_{j_{k_\ell}} \to \alpha_0/\beta_0 = \alpha$. So α is rational, and that is a contradiction.

10. Write

$$\mathrm{Tan}^{-1}(1) = \int_0^1 \frac{1}{1+t^2}\, dt$$

or

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+t^2}\, dt\,.$$

Now we may approximate the integral on the right by the Riemann sum

$$\sum_{j=1}^{k} \frac{1}{1+[j/k]^2} \cdot \frac{1}{k} \,.$$

2.2 Subsequences

1. Clearly any increasing sequence $\{a_j\}$ that is bounded above is bounded. By Bolzano-Weierstrass it has a convergent subsequence $\{a_{j_k}\}$. But the same argument shows that any subsequence has a convergent subsequence with the same limit α. By Exercise 3 of the last section, the full sequence converges to α. In fact α is simply the least upper bound of the sequence.

3. For any positive integer set

$$\phi(n) = n - k\pi,$$

where k is the (unique) integer such that

$$k\pi < n < (k+1)\pi.$$

By the pigeonhole principle, the set of all $\phi(n)$ will contain arbitrarily small elements. So $\{\phi(n)\}$ is dense in $[0, \pi]$.

By calculus we know that $\cos x$ is one-to-one on $[0, \pi]$ with values in $[-1, 1]$. Let $\cos^{-1}$ be the inverse. For $\alpha \in [-1, 1]$ we have $\cos^{-1}\alpha \in [0, \pi]$. Thus there exists a sequence

$$\phi(j_k) \longrightarrow \cos^{-1}\alpha \text{ as } k \to \infty.$$

By the continuity of the cosine function,

$$\cos(j_k) = \cos(\phi(j_k)) \longrightarrow \alpha \quad \text{as} \quad k \to \infty \,.$$

4. This problem is equivalent to Exercise 3 above.

6. Certainly Proposition 2.13 shows that the b_j converge to some limit β (see also Exercise 1 above). But the limit of the b_j is also the $\liminf$ of the original sequence a_j. It follows then that there is a subsequence of the a_j that converges to β.

7. Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers. Now consider the sequence

$$q_1, q_1, q_2, q_1, q_2, q_3, q_1, q_2, q_3, q_4, \ldots . \qquad (*)$$

If α is any real number, then let $r_1, r_2, \ldots$ be a sequence of rationals that converges to α (say the decimal expansion of α). Then we may find a subsequence of $(*)$ that is identical to $r_1, r_2, \ldots$. That does the job.

9. If instead $\{a_j\}$ converges then $\{a_j\}$ is bounded. This contradicts the property of having a subsequence that diverges to $\pm\infty$.

2.3 Limsup and Liminf

1. Consider the sequence

$$0, 1, 2, 3, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots .$$

The the supremum of this set of numbers is 3, while the limsup is 0. A similar example applies to the inf and liminf.

2. Let $\alpha \equiv \limsup a_j$ and $\beta \equiv \liminf a_j$. Let $A_j = \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}$ and $B_j = \inf\{a_j, a_{j+1}, a_{j+2}, \ldots\}$. Then

$$\begin{aligned} \sup\{1/a_j, 1/a_{j+1}, 1/a_{j+2}, \cdots\} &= 1/\inf\{a_j, a_{j+1}, a_{j+2}, \cdots\} \\ &= 1/B_j. \end{aligned}$$

Thus $\limsup 1/a_j = 1/\beta$.

Analogously one shows that $\liminf_{j\to\infty} 1/a_j = 1/\alpha$.

4. We write

$$(|\sin j|)^{\sin j} = e^{\sin j \log|\sin j|}.$$

Now we look at the function

$$f(x) = x \log|x|$$

when $x \in [-1, 1]$. We have

$$f'(x) = \log|x| + 1.$$

Thus f has a maximun when $x = -e^{-1}$ and a minimum when $x = e^{-1}$. Moreover

$$f(-e^{-1}) = e^{-1}$$

and

$$f(e^{-1}) = -e^{-1}.$$

We know that $\{\sin j\}$ is dense in $[-1, 1]$. Thus there exist sequences $\{j_k\}$ and $\{j_\ell\}$ such that $\sin j_k \to 1/e$ and $\sin j_\ell \to -1/e$. Then

$$\limsup_{j\to\infty} |\sin j|^{\sin j} = e^{1/e}$$

and

$$\liminf_{j\to\infty} |\sin j|^{\sin j} = e^{-1/e}.$$

6. Let $\alpha = \liminf a_j = \limsup a_j$. Seeking a contradiction suppose that $\{a_j\}$ does not converge. Then there exist $\epsilon > 0$ and a subsequence $\{a_{j_k}\}$ such that for all k

$$|a_{j_k} - \alpha| > \epsilon.$$

Let $\beta = \limsup a_{j_k} (\neq \alpha)$ and $a_{j_{k_\ell}}$ be a subsequence such that $\lim_{\ell\to\infty} a_{j_{k_\ell}} = \beta$. But $\{a_{j_{k_\ell}}\}$ is a subsequence of the original sequence. By Corollary 2.33,

$$\liminf a_j \le \lim_{\ell\to\infty} a_{j_{k_\ell}} \le \limsup a_j$$

and by the Pinching Principle

$$\lim_{\ell\to\infty} a_{j_{k_l}} = \alpha.$$

This contradiction shows that $\{a_j\}$ converges to α.

7. Let

$$\alpha_j = b - 1/2^j \qquad \text{if } j \text{ is even}$$

and

$$\alpha_j = a + 1/2^j \qquad \text{if } j \text{is odd}.$$

Then it is clear that the limsup is b and the liminf is a.

8. The complex numbers cannot be ordered (see Exercise 8 in Section 1.2). So the concepts of limsup and liminf make no sense.

9. Consider the sequence $\{a_j + b_j\} \equiv \{c_j\}$. Let $\alpha = \limsup_{j\to\infty} a_j$, $\beta = \limsup_{j\to\infty} b_j$ and $\gamma = \limsup_{j\to\infty} c_j$. Then there exist subsequences $\{a_{j_k}\}, \{b_{j_l}\}$, and $\{c_{j_m}\}$ such that

$$\alpha = \lim_{k\to\infty} a_{j_k},$$

$$\beta = \lim_{l\to\infty} b_{j_l},$$

$$\gamma = \lim_{m\to\infty} c_{j_m}.$$

Then

$$\begin{aligned} \gamma &= \lim_{m\to\infty} a_{j_m} + b_{jm} \\ &= \limsup_{m\to\infty} a_{j_m} + \limsup_{m\to\infty} b_{j_m} \\ &\leq \alpha + \beta. \end{aligned}$$

It can be proved in the same fashion that

$$\liminf(a_j + b_j) \geq \liminf a_j + \liminf b_j.$$

When dealing with $\limsup(a_j \cdot b_j)$ we have to be careful of the signs. If a_j and b_j are all non-negative numbers, then

$$\begin{aligned} \limsup(a_j \cdot b_j) &= \lim_{k\to\infty} (a_{j_k} \cdot b_{j_k}) \\ &= \lim_{k\to\infty} a_{j_k} \cdot \lim_{k\to\infty} b_{j_k} \\ &\leq \alpha \cdot \beta. \end{aligned}$$

Notice that in the inequality we have used that fact that all the quantities involved are non-negative ($x_1 < y_1$ and $x_2 < y_2$ implies $x_1 \cdot x_2 \leq y_1 \cdot y_2$ only if x_1, x_2, y_1, y_2 are non-negative). Using this comment, it is easy to construct sequences $\{a_j\}$ and $\{b_j\}$ of negative numbers for which

$$\limsup(a_j \cdot b_j) > \limsup a_j \cdot \limsup b_j.$$

2.4 Some Special Sequences

1. Let $r = p/q = m/n$ be two representations of the rational number r. Recall that for any real α, the number α^r is defined as the real number β for which

$$\alpha^m = \beta^n.$$

Let β' satisfy

$$\alpha^p = \beta'^q.$$

We want to show that $\beta = \beta'$. we have

$$\begin{aligned}\beta^{n\cdot q} &= \alpha^{m\cdot q}\\ &= \alpha^{p\cdot n}\\ &= \beta'^{q\cdot n}.\end{aligned}$$

By the uniqueness of the $(n \cdot q)^{th}$ root of a real number it follows that

$$\beta = \beta',$$

proving the desired equality. The second equality follows in the same way. Let

$$\alpha = \gamma^n.$$

Then

$$\alpha^m = \gamma^{n\cdot m}.$$

Therefore, if we take the n^{th} root on both sides of the above inequality, we obtain

$$\gamma^m = (\alpha^m)^{1/n}.$$

Recall that γ is the n^{th} root of α. Then we find that

$$(\alpha^{1/n})^m = (\alpha^m)^{1/n}.$$

Using similar arguments, one can show that for all real numbers α and β and $q \in \mathbb{Q}$

$$(\alpha \cdot \beta)^q = \alpha^q \cdot \beta^q.$$

Finally, let α, β, and γ be positive real numbers. Then

$$\begin{aligned}
(\alpha\cdot\beta)^\gamma &= \sup\{(\alpha\cdot\beta)^q : q\in\mathbb{Q}, q\le\gamma\} \\
&= \sup\{\alpha^q\beta^q : q\in\mathbb{Q}, q\le\gamma\} \\
&= \sup\{\alpha^q : q\in\mathbb{Q}, q\le\gamma\}\cdot\sup\{\beta^q : q\in\mathbb{Q},\ q\le\gamma\} \\
&= \alpha^\gamma\cdot\beta^\gamma.
\end{aligned}$$

2. It suffices to notice that, for any fixed x,

$$\begin{aligned}
\lim_{j\to\infty}\left(1+\frac{x}{j}\right)^j &= \lim_{j\to\infty}\left\{\left(1+\frac{x}{j}\right)^{j/x}\right\}^x \\
&= \left\{\lim_{j/x\to\infty}\left\{1+\frac{x}{j}\right\}^{j/x}\right\}^x \\
&= e^x.
\end{aligned}$$

4. Write

$$\begin{aligned}
\frac{j^j}{(2j)!} &= \frac{j\cdots j}{1\cdots j\cdot j+1\cdots 2j} \\
&\le \frac{1}{1\cdots j} \\
&= \frac{1}{j!}.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{j\to\infty}\frac{j^j}{(2j)!} &\le \lim_{j\to\infty}\frac{1}{j!} \\
&= 0.
\end{aligned}$$

6. We write $F(x) = a_0 + a_1x + a_2x^2 + \cdots$. Here the a_j's are the terms of the Fibonacci sequence and the letter x denotes an unspecified variable. What is curious here is that we do not care about what x is. We intend to manipulate the function F in such a fashion that we will be able to solve for the coefficients a_j. Just think of $F(x)$ as a polynomial with a *lot* of coefficients.

Notice that

$$xF(x) = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots$$

and

$$x^2F(x) = a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \cdots .$$

Thus, grouping like powers of x, we see that

$$\begin{aligned} & F(x) - xF(x) - x^2F(x) \\ = \; & a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 \\ & \quad +(a_3 - a_2 - a_1)x^3 + (a_4 - a_3 - a_2)x^4 + \cdots . \end{aligned}$$

But the basic property that defines the Fibonacci sequence is that $a_2 - a_1 - a_0 = 0$, $a_3 - a_2 - a_1 = 0$, etc. Thus our equation simplifies drastically to

$$F(x) - xF(x) - x^2F(x) = a_0 + (a_1 - a_0)x.$$

We also know that $a_0 = a_1 = 1$. Thus the equation becomes

$$(1 - x - x^2)F(x) = 1$$

or

$$F(x) = \frac{1}{1 - x - x^2}. \tag{$*$}$$

It is convenient to factor the denominator as follows:

$$F(x) = \frac{1}{\left[1 - \frac{-2}{1-\sqrt{5}}x\right] \cdot \left[1 - \frac{-2}{1+\sqrt{5}}x\right]}$$

(just simplify the right hand side to see that it equals ($*$).

A little more algebraic manipulation yields that

$$F(x) = \frac{5+\sqrt{5}}{10}\left[\frac{1}{1 + \frac{2}{1-\sqrt{5}}x}\right] + \frac{5-\sqrt{5}}{10}\left[\frac{1}{1 + \frac{2}{1+\sqrt{5}}x}\right].$$

Now we want to apply the formula for the sum of a geometric series to each of the fractions in brackets ([ ]). For the first fraction, we think of $-\frac{2}{1-\sqrt{5}}x$ as λ. Thus the first expression in brackets equals

$$\sum_{j=0}^{\infty} \left(-\frac{2}{1-\sqrt{5}}x\right)^j .$$

Likewise the second sum equals

$$\sum_{j=0}^{\infty}\left(-\frac{2}{1+\sqrt{5}}x\right)^j .$$

All told, we find that

$$F(x)=\frac{5+\sqrt{5}}{10}\sum_{j=0}^{\infty}\left(-\frac{2}{1-\sqrt{5}}x\right)^j+\frac{5-\sqrt{5}}{10}\sum_{j=0}^{\infty}\left(-\frac{2}{1+\sqrt{5}}x\right)^j .$$

Grouping terms with like powers of x, we finally conclude that

$$F(x)=\sum_{j=0}^{\infty}\left[\frac{5+\sqrt{5}}{10}\left(-\frac{2}{1-\sqrt{5}}x\right)^j+\frac{5-\sqrt{5}}{10}\left(-\frac{2}{1+\sqrt{5}}x\right)^j\right]x^j.$$

But we began our solution of this problem with the formula

$$F(x)=a_0+a_1x+a_2x^2+\cdots .$$

The two different formulas for $F(x)$ must agree. In particular, the coefficients of the different powers of x must match up. We conclude that

$$a_j=\frac{5+\sqrt{5}}{10}\left(-\frac{2}{1-\sqrt{5}}\right)^j+\frac{5-\sqrt{5}}{10}\left(-\frac{2}{1+\sqrt{5}}\right)^j .$$

We rewrite

$$\frac{5+\sqrt{5}}{10}=\frac{1}{\sqrt{5}}\cdot\frac{1+\sqrt{5}}{2} \qquad\qquad \frac{5-\sqrt{5}}{10}=-\frac{1}{\sqrt{5}}\cdot\frac{1-\sqrt{5}}{2}$$

and

$$-\frac{2}{1-\sqrt{5}}=\frac{1+\sqrt{5}}{2} \qquad\qquad -\frac{2}{1+\sqrt{5}}=\frac{1-\sqrt{5}}{2}.$$

Making these four substitutions into our formula for a_j, and doing a few algebraic simplifications, yields

$$a_j=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^j-\left(\frac{1-\sqrt{5}}{2}\right)^j}{\sqrt{5}}$$

as desired.

9. We see that

$$(a^b)^c = \sup\{(a^b)^\gamma : \gamma < c \text{ and } \gamma \text{ rational}\}$$

$$= \sup\{(\sup\{a^\beta : \beta < b \text{ and } \beta \text{ rational}\})^\gamma : \gamma < c \text{ and } \gamma \text{ rational}\}$$

$$= \sup\{a^{\beta\gamma} : \beta < b \text{ and } \beta \text{ rational}, \gamma < c \text{ and } \gamma \text{ rational}\}\,.$$

But this last is a^{bc}.

The proof of the other identity is similar.