Chapter 2

Finding Roots of Real Single-Valued Functions

Bisection method

1. (a) $f(x) = x - 2\sin x$
   The first bisector $y = x$ and the function $y = 2\sin x$ intersect at 3 points with respective abscissas:

   $$root1 = 0,\ root2 > 0,\ root3 < 0$$

   Therefore root1=0 is an exact root of $f(x) = x - 2\sin x$, while $root2$ and $root3$ can be approximated by the bisection method.
   •$\frac{\pi}{2} < root2 < \frac{3\pi}{4}$, as $f(\frac{\pi}{2}) \times f(\frac{3\pi}{4}) < 0$

n	a_n	b_n	r_{n+1}	$f(r_{n+1})$
0	pi/2=1.5708	3pi/4=2.3562	1.9635	+
1	1.5708	1.9635	-1.7671	-
2	1.7671	1.9635	1.8653	-
3	1.8653	1.9635	1.9144	+
4	1.8653	1.9144	1.8899	-
5	1.8899	1.9144	1.9021	+
6	1.8899	1.9021	1.8960	+
7	1.8899	1.8960	1.8929	.

   The bisection method took 7 iterations to compute $root2 \approx 1.8960$ up to 3 decimals.(The 8^{th} confirms that the precision is reached). •$\frac{-3\pi}{4} < root3 < \frac{-\pi}{2}$, as $f(\frac{-3\pi}{4}) \times f(\frac{-\pi}{2}) < 0$

n	a_n	b_n	r_{n+1}	$f(r_{n+1})$
0	-3pi/4= -2.3562	-pi/2=-1.5708	-1.9635	-
1	-1.9635	-1.5708	-1.7671	+
2	-1.9635	-1.7671	-1.8653	+
3	-1.9635	-1.8653	-1.9144	-
4	-1.9144	-1.8653	-1.8899	+
5	-1.9144	-1.8899	-1.9021	-
6	- 1.9021	-1.8899	-1.8960	-
7	-1.8960	-1.8899	-1.8929	.

The bisection method took 7 iterations to compute $root3 \approx -1.8960$ up to 3 decimals.(The 8^{th} confirms that the precision is reached).

(b) $f(x) = x^3 - 2\sin x$
The cubic function $y = x^3$ and the function $y = 2\sin x$ intersect at 3 points with respective abscissas:

$$root1 = 0,\ root2 > 0,\ root3 = -root2 < 0.$$

Therefore $root1 = 0$ is an exact root of $f(x) = x^3 - 2\sin x$, while $root2$ and $root3$ can be approximated by the bisection method.
$1 < root2 < 1.5$, as $f(1) \times f(1.5) < 0$.
The same table as in (a) can be constructed to obtain the sequence of iterates:

$$\{1.2500, 1.1250, 1.1875, 1.2188, 1.2344, 1.2422, 1.2383, 1.2363\}.$$

Thus, the bisection method took 7 iterations to compute $root2 \approx 1.2363$ up to 3 decimals.(The 8^{th} confirms that the precision is reached).

(c) $f(x) = e^x - x^2 + 4x + 3$
The exponential function $y = e^x$ and the parabola $y = x^2 - 4x - 3$ intersect at 1 point with negative abcissa : $root < 0$. Therefore the function $f(x) = e^x - x^2 + 4x + 3$ has a unique negative root, with:
• $-1 < root < 0$, as $f(-1) \times f(0) < 0$

n	a_n	b_n	r_{n+1}	$f(r_{n+1})$
0	-1	0	-0.5	+
1	-1	-0.5	-0.75	-
2	-0.75	-0.5	-0.6250	+
3	-0.75	-0.6250	-0.6875	+
4	-0.75	- 0.6875	-0.7188	+
5	-0.75	- 0.7188	- 0.7344	+
6	- 0.75	- 0.7344	-0.7422	-
7	- 0.7422	-0.7188	-0.7383	+
8	- 0.7422	-0.7305	-0.7363	-
9	- 0.7364	-0.7305	- 0.7354	+
10	- 0.7364	-0.7335	- 0.7349	-
11	- 0.7349	-0.7335	- 0.7351	-
12	- 0.7349	-0.7335	- 0.7350	

The bisection method took 12 iterations to compute $root1 \approx -1.8960$ up to 3 decimals.(The 13^{th} confirms that the precision is reached).

(d) $f(x) = x^3 - 5x - x^2$
The cubic function $y = x^3 - 5x$ and the function $y = x^2$ intersect at 3

points with respective abscissas:

$$root1 = 0,\ root2 > 0,\ root3 < 0.$$

Therefore $root1 = 0$ is an exact root of $f(x) = x^3 - 5x - x^2$, while $root2$ and $root3$ can be approximated by the bisection method.
$2 < root2 < 3$, as $f(2) \times f(3) < 0$.
The same table as in (a) can be constructed to obtain the sequence of iterates approximating $root2$ up to 3 decimals:

$$\{2.5000, 2.7500, 2.8750, 2.8125, 2.7812, 2.7969, 2.7891, 2.7930\}.$$

Thus, the bisection method took 7 iterations to compute $root2 \approx 2.7930$ up to 3 decimals.(The 8^{th} confirms that the precision is reached).
$-2 < root3 < -1$, as $f(-2) \times f(-1) < 0$.
One obtains the sequence of iterates approximating $root3$ up to 3 decimals:

$$\{-1.5000, -1.7500, -1.8750, -1.8125, -1.7812, -1.7969, -1.7891, -1.7930\}.$$

Thus, the bisection method took 7 iterations to compute $root3 \approx -1.7930$ up to 3 decimals.(The 8^{th} confirms that the precision is reached).

2. (b) The graph of the function $f(x) = sinx - e^{-x}$ is shown in figure 2.1, and the approximation of the first positive root are shown in table 2.2.
   (a), (c) and (d), are solved with the following hints.
   For (a) one plots the functions x and $\tan x$ to find out that the infinite set of non-zero roots is: $\{\pm r_n\}$, with:

   $$(n-1)\pi < r_n < (n - \frac{1}{2})\pi,\ n = 1, 2, ...$$

   To apply the bisection method, for finding any of the r_n and since tan is not defined at $(n - \frac{1}{2})\frac{\pi}{2}$, we use as starting interval $((n-1)\pi, (n - \frac{1}{2})\pi) - \epsilon$, $\epsilon = \frac{1}{2}10^{-3} > 0$.
   The case (c) where $e^{-x} = \cos x$ is exactly similar to (b) as $\cos x = \sin(\pi/2 - x)$.
   Finally, (d) is similar to (a) on the basis that the roots are non-negative with the sequence of positive roots $\{r_n\}$ satisfying:

   $$(n-1)\frac{\pi}{2} < r_n < (n - \frac{1}{2})\frac{\pi}{2},\ n = 1, 2, ...$$

3. Based on the bisection method, the theoretical number of iterations to approximate a root up to 4 decimal figures is $k = 11$.

   (a) $f(x) = x^3 - e^x$

The computed sequence of iterations is:
$r_1 = 1.500, r_2 = 1.7500, r_3 = 1.8750, r_4 = 1.8125, r_5 = 1.8438,$
$r_6 = 1.8594,$
$r_7 = 1.8516, r_8 = 1.8555, r_9 = 1.8574, r_{10} = 1.8564, r_{11} = 1.8569.$

(b) $f(x) = x^2 - 4x + 4 - \ln x$
The computed sequence of iterations is:
$r_1 = 1.500, r_2 = 1.2500, r_3 = 1.3750, r_4 = 1.4375, r_5 = 1.4062,$
$r_6 = 1.4219,$
$r_7 = 1.4141, r_8 = 1.4102, r_9 = 1.4121, r_{10} = 1.4131, r_{11} = 1.4126.$

(c) $f(x) = x^3 + 4x^2 - 10$
The computed sequence of iterations is:
$r_1 = 1.500, r_2 = 1.2500, r_3 = 1.3750, r_4 = 1.3125, r_5 = 1.3438,$
$r_6 = 1.3594,$
$r_7 = 1.3672, r_8 = 1.3633, r_9 = 1.3652, r_{10} = 1.3643, r_{11} = 1.3647.$

(d) $f(x) = x^4 - x^3 - x - 1$
The computed sequence of iterations is:
$r_1 = 1.500, r_2 = 1.7500, r_3 = 1.6250, r_4 = 1.5625,$
$r_5 = 1.5938,$
$r_6 = 1.6094, r_7 = 1.6172, r_8 = 1.6211, r_9 = 1.6191, r_{10} = 1.6182, r_{11} = 1.6177.$

(e) $f(x) = x^5 - x^3 + 3$
The computed sequence of iterations is:
$r_1 = 1.500, r_2 = 1.7500, r_3 = 1.8750, r_4 = 1.9375, r_5 = 1.9688,$
$r_6 = 1.9844,$
$r_7 = 1.9922, r_8 = 1.9961, r_9 = 1.9980, r_{10} = 1.9990, r_{11} = 1.9995.$

(f) $f(x) = e^{-x} - \cos x$ The computed sequence of iterations is:
$r_1 = 1.500, r_2 = 1.2500, r_3 = 1.3750, r_4 = 1.3125, r_5 = 1.2812,$
$r_6 = 1.2969,$
$r_7 = 1.2891, r_8 = 1.2930, r_9 = 1.2910, r_{10} = 1.2920, r_{11} = 1.2925.$

(g) $f(x) = \ln(1 + x) - \frac{1}{x+1}$
The computed sequence of iterations is:
$r_1 = 1.500, r_2 = 1.7500, r_3 = 1.8750, r_4 = 1.9375, r_5 = 1.96888,$
$r_6 = 1.9844,$
$r_7 = 1.9922, r_8 = 1.9961, r_9 = 1.9980, r_{10} = 1.9990, r_{11} = 1.9995.$

4. Based on the bisection method, the theoretical number of iterations to approximate a root up to 5 decimal figures is $k = 15$.

(a) $f(x) = e^{-x} - 3x$

The computed sequence of iterations is:
$r_1 = 0.50000, 0 = 0.250000, r_3 = 0.375000, r_4 = 0.312500, r_5 = 0.281250$,
$r_6 = 0.265625, r_7 = 0.257812, r_8 = 0.253906$,
$r_9 = 0.255859, r_{10} = 0.256835, r_{11} = 0.257324, r_{12} = 0.257568, r_{13} = 0.257690, r_{14} = 0.257629, r_{15} = 0.257598$.

(b) $f(x) = e^x - 2$
The computed sequence of iterations is:
$r_1 = 0.500000, r_2 = 0.750000, r_3 = 0.625000, r_4 = 0.687500, r_5 = 0.718750$,
$r_6 = 0.703125, r_7 = 0.695312, r_8 = 0.691406$,
$r_9 = 0.693359, r_{10} = 0.692382, r_{11} = 0.692871, r_{12} = 0.693115, r_{13} = 0.693237, , r_{14} = 0.693176, r_{15} = 0.693145$.

(c) $f(x) = e^{-x} - x^2$
The computed sequence of iterations is:
$r_1 = 0.50000, r_2 = 0.750000, r_3 = 0.625000, r_4 = 0.687500$,
$r_5 = 0.718750, r_6 = 0.703125, r_7 = 0.710938, r_8 = 0.707031$,
$r_9 = 0.705078, r_{10} = 0.704102, r_{11} = 0.703613$,
$r_{12} = 0.703362, r_{13} = 0.703491, r_{14} = 0.703430$,
$r_{15} = 0.703461$.

(d) $f(x) = \cos x - x$
The computed sequence of iterations is:
$r_1 = 0.50000, r_2 = 0.750000, r_3 = 0.625000, r_4 = 0.687500$,
$r_5 = 0.718750, r_6 = 0.734375, r_7 = 0.742187, r_8 = 0.738281$,
$r_9 = 0.740234, r_{10} = 0.739257, r_{11} = 0.738769$,
$r_{12} = 0.739013, r_{13} = 0.739135, r_{14} = 0.739074$,
$r_{15} = 0.739105$.

(e) $f(x) = \cos x - \sqrt{x}$
The computed sequence of iterations is:
$r_1 = 0.500000, r_2 = 0.750000, r_3 = 0.625000, r_4 = 0.687500$,
$r_5 = 0.656250, r_6 = 0.640625, r_7 = 0.648437, r_8 = 0.644531$,
$r_9 = 0.642578, r_{10} = 0.641601, r_{11} = 0.642089, r_{12} = 0.641845$,
$r_{13} = 0.641723, r_{14} = 0.641662$,
$r_{15} = 0.641693$.

(f) $f(x) = e^x - 3x$
The computed sequence of iterations is:
$r_1 = 0.500000, r_2 = 0.750000, r_3 = 0.625000, r_4 = 0.562500$,
$r_5 = 0.593750, r_6 = 0.609375, r_7 = 0.617187, r_8 = 0.621093$,
$r_9 = 0.619140, r_{10} = 0.618164, r_{11} = 0.618652, r_{12} = 0.618896$,
$r_{13} = 0.619018, r_{14} = 0.619079$,

$r_{15} = 0.619049.$

(g) $f(x) = x - 2^{-x}$
The computed sequence of iterations is:
$r_1 = 0.500000, r_2 = 0.750000, r_3 = 0.625000, r_4 = 0.687500,$
$r_5 = 0.656250, r_6 = 0.640625, r_7 = 0.648437, r_8 = 0.644531,$
$r_9 = 0.642578, r_{10} = 0.641601, r_{11} = 0.641113, r_{12} = 0.641357,$
$r_{13} = 0.641235, r_{14} = 0.641174,$
$r_{15} = 0.641204.$

(h) $f(x) = 2x + 3\cos x - e^x$
The computed sequence of iterations is:
$r_1 = 0.50000, r_2 = 0.75000, r_3 = 0.875000, r_4 = 0.937500,$
$r_5 = 0.968750, r_6 = 0.9843750, r_7 = 0.992187, r_8 = 0.996093,$
$r_9 = 0.998046, r_{10} = 0.999023, r_{11} = 0.999511, r_{12} = 0.999755,$
$r_{13} = 0.999877, r_{14} = 0.999938, r_{15} = 0.999969, r_{16} = 0.999984,$
$r_{17} = 0.999992, r_{18} = 0.999996.$

5. $f(x) = \ln(1-x) - e^x$
The function $f(x) = \ln(1-x)$ is monotone increasing on its domain $(-\infty, 1)$, while the exponential function $y = e^x$ is monotone increasing on $(-\infty, +\infty)$, $\Rightarrow$ the 2 curves intersect at a unique point which is the root of f.
$\bullet -1 < root < 0$, as $f(-1) \times f(0) < 0$
The first 4 iterates computed by the bisection method are displayed in the following table:

n	a_n	b_n	r_{n+1}	$f(r_{n+1})$
0	-1	0	-0.5	-
1	-1	-0.5	-0.75	+
2	-0.75	-0.5	-0.6250	-
3	-0.75	-0.6250	-0.6875	.

6. $f(x) = e^x - 3x$
The function $f(x) = e^x$ is monotone increasing on its domain $(-\infty, +\infty)$, and the function $y = 3x$ is monotone increasing on $(-\infty, +\infty)$, $\Rightarrow$ the 2 curves intersect at a unique point which is the root of f.
$\bullet 0 < root < 1.5$, as $f(0) \times f(1.5) < 0$
The first 4 iterates computed by the bisection method are:
$r_1 = 0.75$, $r_2 = 0.375$, $r_3 = 0.5625$, $r_4 = 0.65625$.

7. (a) Incorrect. For example for $n = 0$, $r > \frac{a_0+b_0}{2}$.
(b) Always correct since:

$$\forall n,\ r \in (a_n, b_n), \text{therefore, } b_n - r \leq b_n - a_n = 2^{-n}(b_0 - a_0).$$

(c) Incorrect on the basis that $r_{n+1} = \frac{a_n+b_n}{2}$ with r_n being either a_n or b_n but definitely not always b_n.

(d) Incorrect on the basis that $r_{n+1} = \frac{a_n+b_n}{2}$ with r_n being either a_n or b_n but definitely not always a_n.

Newton's and Secant methods

8. $f(x) = e^{-x} - cosx,\ 0.5 < root < \frac{\pi}{2} = 1.5707$ as $f(1) \times f(2) < 0$.
$r_{n+1} = r_n - \frac{(e^{-r_n} - cosr_n)}{(-e^{-r_n} + sinr_n)}$, with initial condition $r_0 = \frac{0.5+\frac{\pi}{2}}{2} = 1.036$.
The first 3 iterates by Newton's method are: $r_1 = 1.342$; $r_2 = 1.294$; $r_3 = 1.293$.

9. $f(x) = x^5 - x^3 - 3,\ 1 < root < 2$ as $f(1) \times f(2) < 0$.
$r_{n+1} = r_n - \frac{r_n^5 - r_n^3 - 3}{5r_n^4 - 3r_n^2}$, with initial condition $r_0 = \frac{1+2}{2} = 1.5$.
The first 3 iterates by Newton's method are: $r_1 = 1.4343$; $r_2 = 1.4263$; $r_3 = 1.4262$.

10. $f(x) = x^4 + 2x^3 - 7x^2 + 3,\ 0 < root1 < 1$ as $f(0) \times f(1) < 0$, and $2 < root2 < 4$ as $f(2) \times f(4) < 0$.
$\Longrightarrow f(x)$ is a function with 2 roots:
$r_{n+1} = \frac{r_n^4 + 2x^3 - 7x^2 + 3}{4x^3 + 6x^2 - 14x}$, with $r_0 = 1.5$ and $r_0' = \frac{2+4}{2} = 3$ being the initial conditions for computing $root1$ and $root2$ respectively.
By Newton's method approximating $root1$ and $root2$ up to 4 decimals are:
$root1 = 3.7907$; $root2 = 0.6180$.

11. $x = \ln(3) \Rightarrow e^x - 3 = 0$. Therefore $f(x) = e^x - 3$ is a function with unique root $root = \ln(3)$, with $1 < root < 2$ as $f(1) \times f(2) < 0$
$r_{n+1} = r_n - \frac{e^x - 3}{e^x}$, with initial condition $r_0 = \frac{1+2}{2} = 1.5$.
The iterates of Newton's method approximating $root$ up to 5 decimals are:
$r_1 = 1.16939$; $r_2 = 1.10105$; $r_3 = 1.10106$; $r_4 = 1.09862$; $r_5 = 1.1.09861$

12. $x = \pm\sqrt{e} \Rightarrow x^2 - e = 0$. Therefore $f(x) = x^2 - e$ is a function with 2 roots: $root1 = +\sqrt{e}$, with $1 < root1 < 2$ and $root2 = -\sqrt{e}$ with

$-2 < root2 < -1$.
$r_{n+1} = \frac{1}{2}(r_n + \frac{e}{r_n})$, with $r_0 = 1.5$, being the initial condition for computing $root1$.
The iterates of Newton's method approximating $root1$ up to 7 decimals are:
$r_1 = 1.656093943$; $r_2 = 1.648737682$; $r_3 = 1.648721271$; $r_4 = 1.648721271$.
By symmetry, the initial condition for computing $root2$ is $r_0 = -1.5$.
The iterates of Newton's method approximating $root2$ up to 7 decimals are:
$r_1 = -1.656093943$; $r_2 = -1.648737682$; $r_3 = -1.648721271$; $r_4 = -1.648721271$.

13. $f(x) = x - \frac{e}{x}$. The roots of f are $x = \pm\sqrt{e}$.
The negative root $-2 < root < -1$, as $f(-2) \times f(-1) < 0$.
$r_{n+1} = r_n - \frac{r_n - \frac{e}{r_n}}{e/r_n^2}$, with initial condition $r_0 = \frac{-1-2}{2} = -1.5$. The first 4 iterates by Newton's method are:
$r_1 = -1.7584068$; $r_2 = -1.5166588$; $r_3 = -1.7498964$; $r_4 = -1.5285389$.

14. (a) $f(x) = e^x - 3x,\ 0 < root < 1$ as $f(0) \times f(1) < 0$.
$r_{n+1} = r_n - \frac{e^{r_n} - 3r_n}{e^{r_n} - 3} = \frac{e^{r_n}(r_n - 1)}{e^{r_n} - 3}$ with initial condition $r_0 = \frac{0+1}{2} = 0.5$.
The iterates of Newton's method approximating $root$ up to 5 decimals, are:
$r_1 = 0.610059$; $r_2 = 0.618996$; $r_3 = 0.619061$; $r_4 = 0.619061$.
Besides the bisection method, Newton's method took 3 iterations to compute the root up to 5 decimals, while the theoretical number is $k = 4$.

(b) $f(x) = x - 2^{-x},,\ 0 < root < 1$ as $f(0) \times f(1) < 0$.
$r_{n+1} = r_n - \frac{r_n - 2^{-r_n}}{1 + 2^{-r_n}\ln(2)}$ with initial condition $r_0 = \frac{0+1}{2} = 0.5$.
The iterates of Newton's method approximating $root$ up to 5 decimals, are:
$r_1 = 0.500000$; $r_2 = 2.32523$; $r_3 = 1.59232$ $r_4 = 1.28881$, $r_5 = 1.24090$, $r_6 = 1.23971$, $r_7 = 1.23971$, .

15. (a) $f(x) = \frac{1}{x} - 3$ is a non polynomial function with $root = \frac{1}{3}$.
$r_{n+1} = r_n - \frac{\frac{1}{r_n} - 3}{\frac{-1}{r_n^2}} = r_n(2 - 3r_n)$. To approximate the positive r_{n+1}, with $r_n > 0$ requires that $0 < r_n < 2/3$.

(b) (i) - Choosing $r_0 = 0.5 < 2/3 \Rightarrow r_1 = 0.2500$; $r_2 = 0.3125$; $r_3 = 0.3320$; $r_4 = 0.3333$ this sequence obviously converges to the exact value of $root = 0.3333333...$
(ii) - Choosing $r_0 = 1 > 2/3 \Rightarrow r_1 = -1$; $r_2 = -5$; $r_3 = -85$; $r_4 = -21845$ that is a divergent sequence.

16. (a) $f(x) = x^2 - \frac{1}{R}$ is a polynomial function with $root = \frac{1}{\sqrt{R}}$.
$r_{n+1} = \frac{1}{2}(r_n + \frac{1}{Rr_n})$. No restrictions on the Initial condition, but the formula divides by the iterate.
(b) $f(x) = \frac{1}{x^2} - R$ is a non-polynomial function with $root = \frac{1}{\sqrt{R}}$.

$r_{n+1} = \frac{r_n}{2}(3 - Rr_n^2)$. There is a restriction on the Initial condition: $0 < r_n < \sqrt{(3/R)}$, but the formula does not divide by the iterate.

17. (a) $f(x) = \frac{1}{x^2} - 7$ is a non-polynomial function with negative $root = \frac{-1}{\sqrt{7}}$.
$r_{n+1} = \frac{r_n}{2}(3 - 7r_n^2)$. There is a restriction on the Initial condition: $-\sqrt{(3/7)} < r_n < 0$, but the formula does not divide by the iterate.
(b) A valid initial condition is $r_0 = 0.45$. The required iterates are : $r_1 = 0.356063$; $r_2 = -0.376098$; $r_3 = -0.377951$; $r_4 = -0.377964$.

18. $f(x) = x^2 - 2$.
$r_{n+1} = \frac{1}{2}(r_n + \frac{2}{r_n})$. A valid interval is $(1, 2)$ since $f(1) * f(2) < 0$.
The required iterates are:
$r_1 = 1.416665$; $r_2 = 1.414215$; $r_3 = 1.414210$; $r_4 = 1.414213562$; $r_5 = 1.414213562$.
Newton's Method took 5 iterates: $\sqrt{2} \simeq 1.414213$.

19. To compute $\sqrt{R}$, with $R > 0$.

(a) $r_{n+1} = \frac{1}{2}(r_n + \frac{R}{r_n})$. No restriction on initial condition.

(b) $r_{n+1} = \frac{1}{2}r_n(3 - \frac{r_n^2}{R})$. Restriction on initial condition: $0 < r_n < \sqrt{3R}$.

(c) $r_{n+1} = \frac{2Rr_n}{R+r_n^2}$. No restriction on initial condition, as for large values of x, $c(x) \approx x$.

(d) $r_{n+1} = \frac{r_n}{2}(3 - \frac{r_n^2}{R})$. Restriction on initial condition: $0 < r_n < \sqrt{3R}$.

(e) $r_{n+1} = 2R\frac{r_n}{R+r_n^2}$. No restriction on initial condition, as for large values of x, $c(x) \approx x$.

(f) $r_{n+1} = \frac{r_n}{2}(3 - \frac{r_n^2}{R})$. Restriction on initial condition: $0 < r_n < \sqrt{3R}$.

20. $f(x) = \sin(x)$ is a function with first positive $root = \pi \Rightarrow r_{n+1} = r_n - \frac{\sin(r_n)}{\cos(r_n)} = r_n - \tan(r_n)$. Locating this root in the interval $(3\pi/4, 7\pi/2)$, a valid initial condition is $r_0 = 3.9270$ and the first 3 iterates are: $r_1 = 2.9270$; $r_2 = 3.1449$; $r_3 = 3.1416$.

21. $f(x) = x^3 - 5x + 3$
(a) This function has 3 roots: $0 < root1 < 1$, as $f(0) \times f(1) < 0$, $1 < root2 < 2$, as $f(1) \times f(2) < 0$ and $-3 < root3 < -2$, as $f(-3) \times f(-2) < 0$.

(b) Using the bisection method:

n	a_n	b_n	r_{n+1}	$f(r_{n+1})$
0	1	2	1.5	-
1	1.5	2	1.75	-
2	1.75	2	1.875	+
3	1.75	1.875	1.8125	-
4	1.8125	1.875	1.8438	+
5	1.8125	1.8438	1.8281	-
6	1.8281	1.8438	1.8359	+
7	1.8281	1.8359	1.8320	.

Using Newton's method, the first iterates computing the root up to 3 decimals are:
$r_1 = 1.5000; r_2 = 2.1429; r_3 = 1.9007; r_4 = 1.8385; r_5 = 1.8343$.

22. (a) $f(x) = x^3 - a \Rightarrow root = a^{1/3}$.
By Newton's method: $r_{n+1} = r_n - \frac{r_n^3 - a}{3r_n^2} = \frac{1}{3}(2r_n + \frac{a}{r_n^2})$.

(b) Since Newton's method is quadratic: $|r_{n+1} - r| = \frac{1}{2}\frac{f"(r_n)}{f'(r_n)}|r - r_n|^2$
.
As $r < c_n < r_n$, then $|r_{n+1} - r| = \frac{1}{2}\frac{6|c_n|}{3r_n^2}|r - r_n|^2 \leq \frac{r_n}{r_n^2}|r - r_n|^2$
As $r_n > r > 1$, then $|r_{n+1} - r| \leq |r - r_n|^2$.

(c) • For n=1, part (b) above: $|r - r_1| \leq |r - r_0|^2$
• Assume $|r - r_n| \leq |r - r_0|^{2^n}$
• As from (b): $|r_{n+1} - r| \leq |r - r_n|^2$, and from assumption $|r - r_n|^2 \leq [|r - r_0|^{2^n}]^2 = [|r - r_0|^{2.2^n}] = |r - r_0|^{2^{n+1}}$, then $|r_{n+1} - r| \leq r - r_0|^{2^{n+1}}$.

(d) $|r - r_{n_0}| \leq |r - r_0|^{2^{n_0}} \leq (\frac{1}{2})^{2^{n_0}}$
If also: $(\frac{1}{2})^{2^{n_0}} \leq (\frac{1}{2})^{32} = (\frac{1}{2})^{2^5}$, then $|r - r_{n_0}| \leq (\frac{1}{2})^{32}$
Therefore: $(\frac{1}{2})^{2^{n_0}} \leq (\frac{1}{2})^{2^5}$ if and only if: $-2^{n_0}\ln 2 \leq -2^5 \ln 2$, i.e. $n_0 \geq 5$

23. $f(x) = p(x) = c_2x^2 + c_1x + c_0$
(a) Since in Newton's method:

$$|r_{n+1} - r| = \frac{1}{2}|\frac{f"(c_n)}{f'(r_n}|(r_n - r)^2 = \frac{1}{2}\frac{2|c_2|}{|p'(r_n)|}(r_n - r)^2| \leq \frac{|c_2|}{d}(r_n - r)^2$$

i.e., $|r_{n+1} - r| = C(r_n - r)^2$ with $C = \frac{|c_2|}{d}$.

(b) Multiplying the last inequality by C and letting $e_n = C|r - r_n|$ yields $e_{n+1} \leq e_n^2$.
For $n = 0$, $e_0 = C|r - r_0| < 1$ if and only if $|r - r_0| < \frac{1}{C} = \frac{d}{|c_2|}$.
Hence for such choice of r_0 $e_0 < 1$ implies $e_1 < e_0^2 < 1$ and by recurrence $e_n < 1$, i.e. the sequence $\{r_n\}$ belongs to the interval:

$$(r - \frac{1}{C}, r + \frac{1}{C}) \subseteq (a, b).$$

(c) If $e_0 = \frac{1}{2} < 1$ then $e_1 \leq e_0^2$, $e_2 \leq e_1^2 \leq e_0^4 = e_0^{2^2}$. By recurrence, assuming $e_n \leq e_0^{2^n}$, then $e_{n+1} \leq e_n^2 \leq (e_0^{2^n})^2 = e_0^{2^{n+1}}$.
Thus, $\frac{|r_n - r|}{|r_0 - r|} = \frac{e_n}{e_0} \leq e_0^{2^n - 1}$. Therefore, the smallest n_p for which $\frac{|r_{n_p} - r|}{|r_0 - r|} \leq 2^{-p}$ can be estimated using the inequalities:

$$e_0^{2^{n_p} - 1} \leq 2^{-p} < e_0^{2^{n_p - 1} - 1}.$$

For $e_0 = \frac{1}{2}$, this is equivalent to:

$$n_p - 1 < \frac{\ln(p + 1)}{\ln 2} \leq n_p,$$

implying that $n_p = \lceil \frac{\ln(p+1)}{\ln 2} \rceil$.

24. The function: $f(x) = x^4 - 4^3$ has a unique root: $2 < root = 4^{3/4} < 3$, as $f(2) \times f(3) < 0$.
The initial conditions are obtained by the bisection method applied twice on the interval $(2, 3)$ $\Rightarrow$: $r_0 = 2.5,\ r_1 = 2.75$.
The first 3 computed iterates by the Secant method are:

$$r_2 = 2.8439;\ r_3 = 2.8278;\ r_4 = 2.8284.$$

25. The function $f(x) = x^3 - 2x + 2$ has a unique negative root: $-2 < root < -1$, as $f(-1) \times f(-2) < 0$.
The initial conditions are obtained by the bisection method applied twice on the interval $(-2, -1)$. This gives:
$r_0 = -1.5000$, $r_1 = -1.7500$ .The first 3 computed iterates using the secant method are:
$r_2 = -1.7737$, $r_3 = -1.7692$, $r_4 = -1.7693$

26. The standard formula of the secant method is:

$$r_{n+1} = r_n - \frac{f(r_n)(r_n - r_{n-1})}{f(r_n) - f(r_{n-1})}, n \geq 2$$

By writing the 2 terms under a common denominator, one gets: $r_{n+1} = \frac{(r_{n-1})f(r_n)-(r_n)f(r_{n-1})}{f(r_n)-f(r_{n-1})}$ Computing the iterates based on this last formula leads to loss of precision problems, as the 2 terms of the numerator are almost equal for large values of n.
In the standard formula: a small quantity is subtracted from r_n to improve the approximation.

27. (a) The function $f(x) = e^x - 3x$ has a unique root : $0 < root < 1$, as $f(0) \times f(1) < 0$.
The initial conditions are obtained by the bisection method applied twice on the interval $(0, 1) \Rightarrow$: $r_0 = 0.5,\ r_1 = 0.75$.
The first computed iterates by the Secant method are:
$r_2 = 0.631975;\ r_3 = 0.617418;\ r_4 = 0.619078;\ r_5 = 0.619061$.
Therefore: 3 iterations are needed to compute *root* up to 5 decimals; the 4^{th} one confirms reaching the required precision.

(b) The function $f(x) = x - 2^{-x}$ has a unique root : $0 < root < 1$, as $f(0) \times f(1) < 0$.
The initial conditions are obtained by the bisection method applied twice on the interval $(0, 1) \Rightarrow$: $r_0 = 0.5,\ r_1 = 0.75$.
The first computed iterates by the Secant method are:
$r_2 = 0.642830;\ r_3 = 0.641166;\ r_4 = 0.641185;\ r_5 = 0.641185$.
Therefore: 3 iterations are needed to compute *root* up to 5 decimals; the 4^{th} one confirms reaching the required precision.

(c) The function $f(x) = -3x + 2\cos(x) - e^x$has a unique root : $0 < root < 1$, as $f(0) \times f(1) < 0$.
The initial conditions are obtained by the bisection method applied twice on the interval $(0, 1) \Rightarrow$: $r_0 = 0.5,\ r_1 = 0.25$.
The first computed iterates by the Secant method are:
$r_2 = 0.231462;\ r_3 = 0.229743;\ r_4 = 0.229731;\ r_5 = 0.229731$.
Therefore: 3 iterations are needed to compute *root* up to 5 decimals; the 4^{th} one confirms reaching the required precision.