Electomagnetic Waves and Materials
Week 2

Instructor:

What will be covered today

- Uniform plane waves
- Good conductor approximation
- Skin effect
- Boundary condition between PEC and dielectric medium
- AC resistance of round wires
- Bounded transmission line

Uniform plane waves

What is a sourceless medium:

$\rho_v = 0$ and

$\mathbf{J}_{source} = 0$

One-dimensional solution for a simple lossy medium with parameters $\varepsilon_r\ \mu_r\ \sigma$ in Cartesian coordinates :

$$E_z = H_z = 0 \qquad \text{. . . . . . (2.1)}$$

$$\eta \mathbf{H} = \hat{z} \times \mathbf{E} \qquad \text{. . . (2.2)}$$

$$\mathbf{E} = \mathbf{E}_t \begin{Bmatrix} e^{+jkz} \\ e^{-jkz} \end{Bmatrix} \qquad \text{. . . . . . (2.3)}$$

where

$$k^2 = \omega^2 \mu\varepsilon - j\omega\mu\sigma \qquad \text{. . . . . . (2.4)}$$

is complex.

Uniform plane waves

The characteristic impedance is also complex

$$\eta = \left(\frac{j\omega\mu}{\sigma + j\omega\varepsilon} \right)^{1/2} \qquad \text{. . . . . . (2.5)}$$

Define: $k = \beta - j\alpha$

where,

α is the attenuation constant (Np/m)

β is the phase constant (rad/m),

which can be obtained by solving the equation (2.4),

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{ \frac{1}{2}\left[\sqrt{1+T^2} - 1 \right] \right\}^{1/2} \qquad \text{. . . (2.6)}$$

$$\beta = \omega\sqrt{\mu\varepsilon}\left\{ \frac{1}{2}\left[\sqrt{1+T^2} + 1 \right] \right\}^{1/2} \qquad \text{. . . . . . (2.7)}$$

where loss tangent *T* is defined as

$$T = \frac{\sigma}{\omega\varepsilon} \qquad \text{. . . . . . (2.8)}$$

Uniform plane waves

For a low-loss dielectric, the loss tangent *T << 1* and

$$\alpha \approx \frac{\sigma}{2}\sqrt{\mu/\varepsilon}, \quad T << 1 \qquad \text{. . . . . . (2.9)}$$

$$\beta \approx \omega\sqrt{\mu\varepsilon}, \qquad T << 1 \qquad \text{. . . . . . (2.10)}$$

$$\eta \approx \sqrt{\frac{\mu}{\varepsilon}} \angle \tan^{-1}\frac{\sigma}{2\omega\varepsilon}, \quad T << 1 \qquad \text{. . . . . . (2.11)}$$

Good conductor approximation

Definition for good conductor

$$T >> 1$$

We can get

$$\alpha \approx \beta \approx \sqrt{\pi f \mu \sigma} = \frac{1}{\delta}, \quad T >> 1 \qquad \ldots\ldots (2.12)$$

$$\eta \approx \frac{\sqrt{2}}{\sigma\delta} \angle 45^{\circ}, \qquad T >> 1 \qquad \ldots\ldots (2.13)$$

δ is called the skin depth. The characteristic impedance Z_s or surface impedance

$$\eta = Z_s = R_s + jX_s = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} \qquad \ldots\ldots (2.14)$$

Skin effect

When waves go through the good conductor, they will be attenuated exponentially as

$$E(z,t) = E_0 e^{-z/\delta} \cos(\omega t - z/\delta) \qquad \ldots\ldots (2.15)$$

$$J(z,t) = J_0 e^{-z/\delta} \cos(\omega t - z/\delta) \qquad \ldots\ldots (2.16)$$

where

$$J_0 = \sigma E_0 \qquad \ldots\ldots (2.17)$$

For DC, $f = 0$, current density is uniform inside the good conductor

for the time-harmonic AC case ($f \neq 0$), we define skin depth

$$\delta = 1/\sqrt{\pi f \mu\sigma}$$

- The amplitude of the current density drops to e^{-1} (38.8%) for a distance of δ.
- It will drop to zero (practically) for a distance of 4δ.
- At high frequencies the current is practically confined to the skin of the conductor and the phenomenon is hence described as **skin effect**.

B.C. between PEC and dielectric medium

For a perfect conductor, $\sigma = \infty \Rightarrow \delta = 0$, we can get ***E = 0*** Inside the conductor

If medium 1 is a perfect conductor and medium 2 is a dielectric,
we can get the boundary conditions as

$$\hat{n}_{12} \cdot \mathbf{D}_2 = \rho_s \qquad \ldots\ldots (2.19)$$

$$\hat{n}_{12} \cdot \mathbf{B}_2 = 0 \qquad \ldots\ldots (2.20)$$

$$\hat{n}_{12} \times \mathbf{E}_2 = 0 \qquad \ldots\ldots (2.21)$$

$$\hat{n}_{12} \times \mathbf{H}_2 = \mathbf{K} \qquad \ldots\ldots (2.22)$$

$\hat{n}$ is the normal unit vector on the boundary.

B.C. between PEC and dielectric medium

Since ***E = 0*** Inside the conductor, we get: (2.24) and (2.26) see problem P2.2

$$E_t = 0 \qquad \text{.......(2.13)}$$

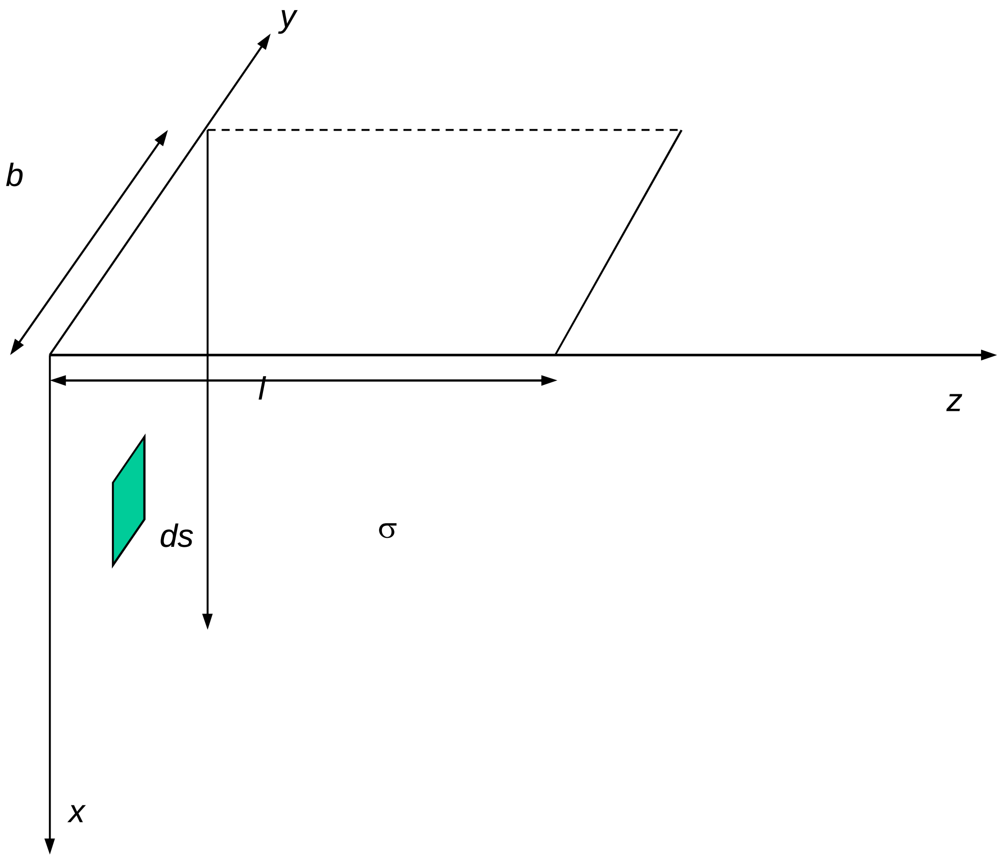
$$\frac{\partial E_n}{\partial n} = 0 \qquad \text{.......(2.24)}$$

$$H_n = 0 \qquad \text{.......(2.25)}$$

$$\frac{\partial H_t}{\partial n} = 0 \qquad \text{.......(2.26)}$$

n is the normal direction and t is the tangential direction.

AC resistance



For an infinitely deep good conductor of conductivity σ defined by the half-space $0 < x < \infty$, shown in the figure, E and H can be expressed as in next slide:

AC resistance

E and ***H*** inside half-space conductor

$$\tilde{\mathbf{E}}(x) = \hat{z} E_0 e^{-x/\delta - jx/\delta} \qquad \text{. . . . . . (2.27)}$$

$$\tilde{\mathbf{H}}(x) = -\hat{y} E_0 \frac{\sigma\delta}{\sqrt{2}} e^{-x/\delta} e^{-jx/\delta} e^{-j45^{\circ}} \qquad \text{. . . . . . (2.28)}$$

Real parts of the fields are

$$\mathbf{E}(x,t) = \hat{z} E_0 e^{-x/\delta} \cos(\omega t - \frac{x}{\delta}) \qquad \text{. . . . . . (2.29)}$$

$$\mathbf{H}(x,t) = \hat{y} E_0 \frac{\sigma\delta}{\sqrt{2}} e^{-x/\delta} \cos(\omega t - \frac{x}{\delta} - 45^{0}) \qquad \text{. . . . . . (2.30)}$$

The time averaged power density

$$\langle \mathbf{S} \rangle = \hat{x} \frac{1}{2} E_0^2 \frac{\sigma\delta}{\sqrt{2}} e^{-2x/\delta} \cos 45^{\circ} = \hat{x} \frac{1}{4} E_0^2 \sigma\delta e^{-2x/\delta} \qquad \text{. . . . . . (2.31)}$$

AC resistance

Total power entering the conductor of width *b* and length *l* is given by

$$P = \langle \mathbf{S} \rangle_{x=0} bl = \frac{1}{4} E_0^2 \sigma \delta bl \qquad \ldots\ldots (2.32)$$

Total phasor current entering the conductor of width *b*

$$\tilde{I} = \iint \tilde{\mathbf{J}} \cdot d\mathbf{s} = \iint \sigma \tilde{\mathbf{E}} \cdot d\mathbf{s} = \int_0^b \int_0^\infty \sigma E_0 e^{-(1+j)x/\delta} dx\, dy \qquad \ldots\ldots (2.33)$$

After integrating, substituting the limits and transferring to time domain, we get

$$I(t) = \frac{\sigma E_0 b \delta}{\sqrt{2}} \cos(\omega t - 45^{\circ}) \qquad \ldots\ldots (2.35)$$

Define an AC equivalent resistance which consumes the same power as

$$\tilde{I}_{RMS}^2\ R_{AC} = Power\ consumed = P \qquad \ldots\ldots (2.36)$$

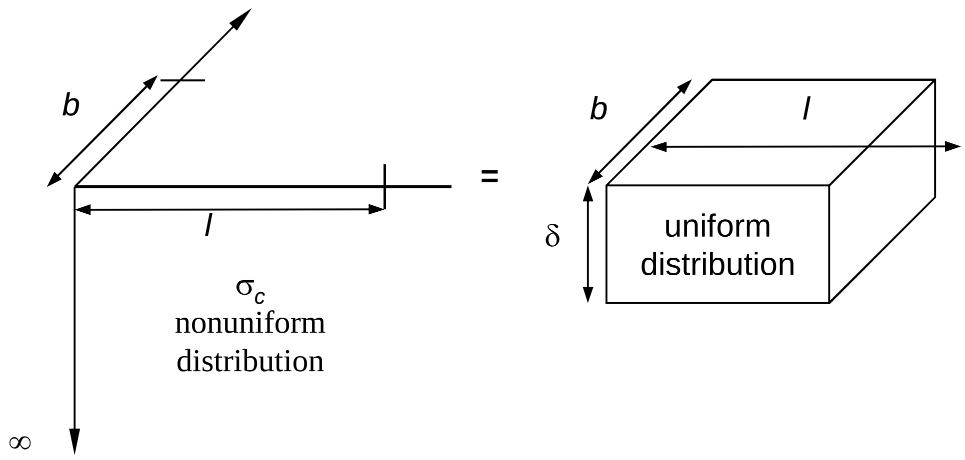
where $\tilde{I}_{RMS} = \frac{\sigma E_0 b \delta}{2}$ is the AC RMS equivalent current.

AC resistance

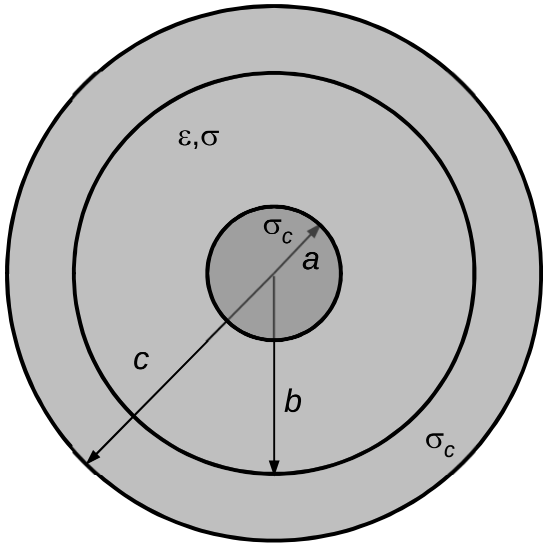
Finally, we get the AC equivalent resistance

$$R_{AC} = \frac{l}{\sigma b \delta} \qquad \text{. . . . . . (2.38)}$$

When AC waves go through an infinitely deep good conductor of conductivity σ defined by the half-space $0 < x < \infty$, shown in the figure 2.1, the equivalent DC conductor has δ depth.



AC resistance of round wires



The AC resistance of a round wire of radius *a* and length *l*.

1. $\delta >> a$, we get

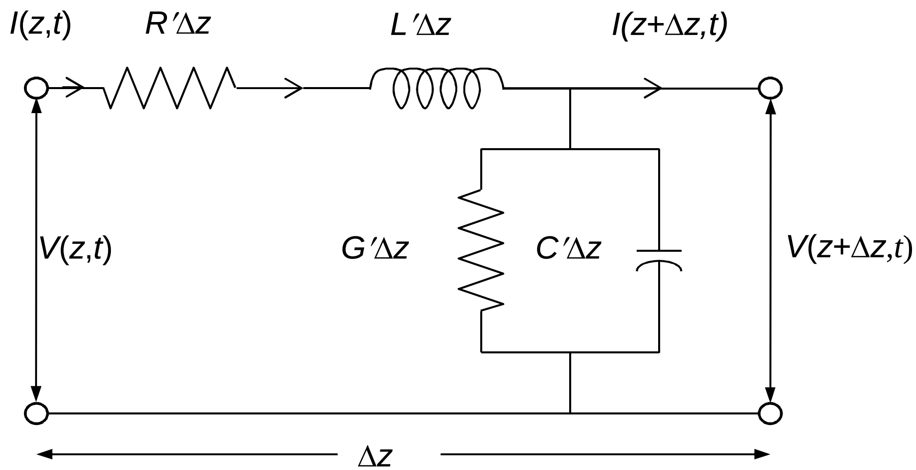
$$R = \frac{l}{\sigma s} = \frac{l}{\sigma \pi a^2}, \delta >> a \qquad \text{. . . . . . . (2.39)}$$

2. $\delta << a$, we get

$$R = \frac{l}{2\pi\sigma a\delta}, \delta << a \qquad \text{. . . . . . (2.42)}$$

Transmission lines

A transmission line is modeled by distributed circuit theory,



where,

R': The resistance due to imperfect conductors

L': The inductance due to magnetic flux generated by the currents in the conductors

G': The conductance due to an imperfect dielectric

C': The capacitance due to the surface charges on the conductors

Transmission lines

The parameters are computed as though the fields are static. For a coaxial cable shown (slide 14), the parameters are

$$G' = \frac{2\pi\sigma}{\ln(b/a)} \quad \text{......(2.42)}$$

$$L' = \frac{\mu}{2\pi}\ln(b/a) \quad \text{......(2.43)}$$

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)} \quad \text{......(2.44)}$$

For high frequencies, $\delta << a$

$$R' = \left[\frac{1}{2\pi\sigma_c\delta a} + \frac{1}{2\pi\sigma_c\delta b}\right] \quad \text{......(2.45)}$$

Transmission lines

L' including the correction due to the internal inductance (inductance due to the magnetic flux in the conductors) is given

$$L' = \frac{\mu}{2\pi}\ln(b/a) + L'' \qquad \text{. . . . . . (2.46)}$$

where $\omega L'' = R'$

For a two-port network shown in Figure 2.4, the following equations can be obtained

$$V(z,t) - V(z+\Delta z,t) = R'\Delta z I(z,t) + L'\Delta z\frac{\partial I(z,t)}{\partial t} \qquad \text{. . . . . . (2.48)}$$

When Δz -> *0*

$$-\frac{\partial V(z,t)}{\partial z} = R'I(z,t) + L'\frac{\partial I(z,t)}{\partial t} \qquad \text{. . . . . . (2.49)}$$

Similarly

$$-\frac{\partial I(z,t)}{\partial z} = G'V(z,.t) + C'\frac{\partial V(z,t)}{\partial t} \qquad \text{. . . . . . (2.51)}$$

Transmission lines

For a the lossless transmission line

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2 V}{\partial t^2} = 0, \qquad R' = G' = 0 \qquad \text{.......(2.52)}$$

where $\mathrm{v} = \dfrac{1}{\sqrt{L'C'}}$

If the source is harmonic

$$\frac{\partial^2 \tilde{V}}{\partial z^2} + \beta^2 \tilde{V} = 0 \qquad \text{.......(2.54)}$$

where $\beta = \dfrac{\omega}{\mathrm{v}}$

Solution

$$\tilde{V} = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{+j\beta z} \qquad \text{.......(2.57)}$$

Transmission lines

The relation between the voltages and the current

$$\tilde{V}^{+} = Z_0 \tilde{I}^{+} \qquad \text{. . . . . . (2.58)}$$

$$\tilde{V}^{-} = -Z_0 \tilde{I}^{-} \qquad \text{. . . . . . (2.59)}$$

where $Z_0 = \sqrt{L'/C'}$

For a lossy transmission line, we get

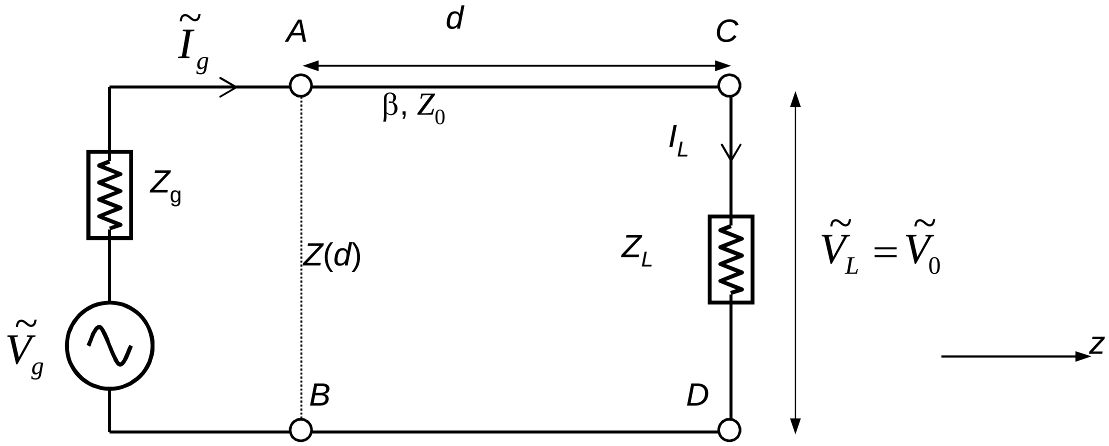
$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \qquad \text{. . . . . . (2.62)}$$

Parameters Comparison between TEM wave and transmission

TEM wave	Transmission wave
E	*V*
H	*I*
ε	C'
μ	L'
η	Z_0
$\beta = \omega\sqrt{\mu\varepsilon}$	$\beta = \omega\sqrt{L'C'}$

Bounded Transmission lines

For a bounded transmission line,



where,

d: transmission line of length

V_g: *Voltage* source

Z_g: internal impedance

Z_L: External load

Bounded Transmission lines

The voltages and the current

$$\tilde{V}(d) = \tilde{V}_0^+ e^{j\beta d} + \tilde{V}_0^- e^{-j\beta d} \qquad \text{......} (2.63)$$

$$\tilde{I}(d) = \frac{1}{Z_0}\left[\tilde{V}_0^+ e^{j\beta d} - \tilde{V}_0^- e^{-j\beta d}\right] \qquad \text{......} (2.64)$$

where

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = Z_0\left[\frac{1+\Gamma_0 e^{-j2\beta d}}{1-\Gamma_0 e^{-j2\beta d}}\right] \qquad \text{......} (2.65)$$

Defined Γ_0 as the reflection coefficient at the load

$$\Gamma_0 = \frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{......} (2.66)$$

By substituting (2.66) in (2.65), the input impedance can be obtained

$$Z(d) = Z_0 \frac{Z_L \cos\beta d + jZ_0 \sin\beta d}{Z_0 \cos\beta d + jZ_L \sin\beta d} \qquad \text{......} (2.67)$$

Bounded Transmission lines: see Apendices 2B and 2C for Smith Chart etc.

From 2.67, we get input impedance for special length

$$Z(\lambda/4) = Z_0^2/Z_L \qquad \text{. . . . . . (2.68)}$$

$$Z(\lambda/2) = Z_L \qquad \text{. . . . . . (2.69)}$$

$$Z(d) = Z_L, \ \frac{d}{\lambda} << 1 \qquad \text{. . . . . . (2.70)}$$

For a matched line, $Z_L = Z_0$, $\Gamma_0 = 0$

$$Z(d) = Z_L. \qquad \text{. . . . . . (2.71)}$$

Two special cases

$$Z(d) = jZ_0 \tan\beta d, \qquad (Z_L = 0) \qquad \text{. . . . . . (2.72)}$$

$$Z(d) = -jZ_0 \cot\beta d, \qquad (Z_L = \infty) \qquad \text{. . . . . . (2.73)}$$

Question (for answer and additional information read Appendix 2D

A lossfree nonuniform transmission line has

$L' = L'(z)$
$C' = C'(z)$
where L' and C' are per meter values of the series inductance and parallel capacitance of the transmission line.

Determine the partial differential equation for the instantaneous voltage $V(z,t)$.
For an exponential transmission line
$L'(z) = L_0 \exp(qz)$
$C'(z) = C_0 \exp(-qz)$,

assuming $V(z,t) = V_0 \exp[j(wt - kz)]$,
determine the relation between w and k.