Hamilton's Principle of Least Action

1. An arbitrary four-vector $A_\mu = (A_1, A_2, A_3, A_4)$ is defined in terms of its four components. For two reference frames in relative uniform motion with velocity v along the z-direction, the components of A_μ are related in the two frames by

$$\begin{aligned} A'_1 &= A_1 \\ A'_2 &= A_2 \\ A'_3 &= \gamma\,(A_3 + i\beta A_4) \\ A'_4 &= \gamma\,(-i\beta A_3 + A_4), \end{aligned}$$

where γ is given by (1.9) and $\beta = v/c$. This is known as a *Lorentz* transformation. Show that the inner product of any two four-vectors A_μ and B_μ satisfies

$$\sum_{\mu=1}^{4} A'_\mu\, B'_\mu = \sum_{\mu=1}^{4} A_\mu\, B_\mu.$$

An inner product of two four-vectors is thus said to be invariant under a Lorentz transformation.

Solution. The inner product of two four-vectors A_μ and B_μ is given by

$$\begin{aligned} \sum_{\mu=1}^{4} A'_\mu\, B'_\mu &= A_1\, B_1 + A_2\, B_2 + \gamma^2 (A_3 + i\beta A_4)\,(B_3 + i\beta B_4) \\ &\quad + \gamma^2(-i\beta A_3 + A_4)\,(-i\beta B_3 + B_4) \\ &= \sum_{\mu=1}^{4} A_\mu\, B_\mu \end{aligned}$$

2. Derive the Lorentz force law (1.15) from the Euler-Lagrange equations of motion (1.14).

Solution. Beginning with the Lagrangian L we have

$$L = -mc^2\sqrt{1 - v^2/c^2} + q\mathbf{v}\cdot\mathbf{A}(\mathbf{x}, t) - q\,\phi(\mathbf{x}, t)$$

$$\begin{aligned}
\frac{\partial L}{\partial x_i} &= -q\,\frac{\partial \phi}{\partial x_i} \\
\frac{\partial L}{\partial v_i} &= \gamma m v_i + q A_i \\
\frac{d}{dt}(\gamma m v_i + q A_i) &= -q\,\frac{\partial \phi}{\partial x_i} \\
\frac{d}{dt}(\gamma m v_i) &= -q\,\mathbf{v}\cdot\nabla A_i - q\,\frac{\partial A_i}{\partial t} - q\,\frac{\partial \phi}{\partial x_i} \\
\frac{d}{dt}(\gamma m \mathbf{v}) &= q\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}),
\end{aligned}$$

This represents the solution, where we have made use of the identities

$$\begin{aligned}
\frac{dA_i}{dt} &= \mathbf{v}\cdot\nabla A_i + \frac{\partial A_i}{\partial t} \\
\frac{\partial}{\partial x_i}(\mathbf{v}\cdot\mathbf{A}) - \mathbf{v}\cdot\nabla A_i &= [\,\mathbf{v}\times(\nabla\times\mathbf{A})\,]_i \\
\mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{B} &= \nabla\times\mathbf{A}
\end{aligned}$$

The Hamiltonian function

1. Show from the above analysis that

$$E^2 = p^2c^2 + m^2c^4,$$

where $p^2 \equiv \mathbf{p}\cdot\mathbf{p}$.

Solution. We make use of the identities

$$\begin{aligned}
E &= \gamma m c^2 \\
\mathbf{p} &= \gamma m \mathbf{v},
\end{aligned}$$

from which it follows:

$$E^2 = p^2c^2 + m^2c^4$$

2. Prove the identity

$$pc = \beta E,$$

where $\beta = v/c$.

Solution. We make use of

$$\begin{aligned}
\beta E &= \beta\gamma mc^2 \\
&= (\gamma mv)\,(\beta c^2/v) \\
&= pc
\end{aligned}$$

The Lagrange Invariant

1. Show that the functions V, W, X and Y all lead to the same Lagrange invariant.

Solution. By definition,

$$V_{ab}(\mathbf{x}_a, \mathbf{P}_b) = \mathbf{P}_b \cdot \mathbf{x}_b - W_{ab}(\mathbf{x}_a, \mathbf{x}_b).$$

It follows that the perturbation δV_{ab} is given by

$$\delta V_{ab}(\mathbf{x}_a, \mathbf{P}_b) = \delta\mathbf{P}_b \cdot \mathbf{x}_b + \mathbf{P}_a \cdot \delta\mathbf{x}_a,$$

where we have made use of the perturbation δW_{ab} derived in the text. A second, independent perturbation yields

$$d(\delta V_{ab}) = d(\delta\mathbf{P}_b) \cdot \mathbf{x}_b + \delta\mathbf{P}_b \cdot d\mathbf{x}_b + d\mathbf{P}_a \cdot \delta\mathbf{x}_a + \mathbf{P}_a \cdot d(\delta\mathbf{x}_a).$$

Interchanging the order of perturbations and subtracting, we obtain the Lagrange invariant as before,

$$d\mathbf{P}_a \cdot \delta\mathbf{x}_a - \delta\mathbf{P}_a \cdot d\mathbf{x}_a = d\mathbf{P}_b \cdot \delta\mathbf{x}_b - \delta\mathbf{P}_b \cdot d\mathbf{x}_b.$$

Similar logic applies to the functions X and Y.