Chapter 2: Laser Beams

2.1

We begin with equation (2.34d)

$$\frac{\lambda}{\pi\omega_{04}^2} = \frac{\lambda}{\pi\omega_{01}^2}\frac{(f-p)}{(f-d)}$$

We also make use of equation (2.34c), i.e.

$$d = f + \frac{f^2(p-f)}{(p-f)^2 + z_{01}^2}$$

Therefore

$$\frac{1}{\omega_{04}^2} = \frac{1}{\omega_{01}^2}\frac{(f-p)}{(f-d)} = \frac{1}{\omega_{01}^2}\frac{(p-f)}{(d-f)} = \frac{1}{\omega_{01}^2}\frac{(p-f)}{f^2(p-f)}[(p-f)^2 + z_{01}^2] = \frac{1}{\omega_{01}^2}\frac{(p-f)^2 + z_{01}^2}{f^2}$$

or

$$\frac{1}{\omega_{04}^2} == \frac{1}{\omega_{01}^2}\frac{(p-f)^2 + z_{01}^2}{f^2} = \frac{1}{\omega_{01}^2}\left(1-\frac{p}{f}\right)^2 + \frac{1}{\omega_{01}^2}\frac{z_{01}^2}{f^2}$$

When p=f, then

$$\omega_{04}^2 = \omega_{01}^2\frac{f^2}{z_{01}^2} = \omega_{01}^2\frac{f^2}{(\pi\omega_{01}^2/\lambda)^2} = \left(\frac{f\lambda}{\pi\omega_{01}}\right)^2$$

This gives

$$\omega_{04} = \frac{f\lambda}{\pi\omega_{01}}$$

2.2

We use the following relations

$$\omega^2(z) = \omega_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right]$$

$$\omega^2(z+1) = \omega_0^2\left[1 + \left(\frac{z+1}{z_0}\right)^2\right] = \omega^2(z) + \omega_0^2\frac{2z+1}{z_0^2} = \omega^2(z) + \frac{(2z+1)\lambda^2}{\pi^2\omega_0^2}$$

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right] = z\left(\frac{z_0}{z}\right)^2\left[1 + \left(\frac{z}{z_0}\right)^2\right] = \frac{z_0^2}{z}\frac{\omega^2(z)}{\omega_0^2} = \frac{\pi^2\omega_0^2\,\omega^2(z)}{\lambda^2\,z}$$

$$z_0 = \frac{\pi\omega_0^2}{\lambda}$$

Given ω(z) = 1 x 10^{-3}m; ω(z+1) = 1.5 x 10^{-3}m and R = 2m. Substituting these values we have

$$2.25\,x10^{-6} - 1.0\,x\,10^{-6} = \frac{(2z+1)\lambda^2}{\pi^2\omega_0^2} \Longrightarrow 1.25\,x\,10^{-6}x\,\pi^2\omega_0^2 = (2z+1)\lambda^2$$

and

$$1.0\,x\,10^{-6}x\,\pi^2\omega_0^2 = 2z\lambda^2$$

Dividing, we obtain

$$1.25 = \frac{(2z+1)}{2z} \quad \Rightarrow \quad z = 2.0m$$

Substituting this value of z in the above equation, we obtain

$$1.0\ x\ 10^{-6} x\ \pi^2 \omega_0^2 = 2\ x\ 2\ \lambda^2 \ \Rightarrow \ \omega_0 = \frac{2\lambda}{10^{-3}\pi} = \frac{2\ x\ 632.8\ x 10^{-6}}{\pi} = 0.4mm$$

Given the value of ω_0, the distance z_0 is obtained as

$$z_0 = \frac{\pi\omega_0^2}{\lambda} = \frac{\pi}{\lambda}\left(\frac{2\lambda}{10^{-3}\pi}\right)^2 = \frac{4\lambda}{\pi 10^{-6}} = \frac{4\ x\ 632.8\ x 10^{-3}}{\pi} = 0.806m$$

Beam waist is 0.8mm and is located 0.806m to the left of the plane where the radius of curvature is 2m.

The radius of the spot at a distance of 1.806m is 1.5mm. The power received in an area of radius 2mm is

$$P = P_0\left[1 - e^{-2\frac{a^2}{\omega^2(z)}}\right] = 3mW\left(1 - e^{-8/2.25}\right) = 3(1 - 0.0286) = 2.91mW$$

2.3

The beam-waist radius at the focal plane is related to the beam-waist radius just before the lens through the equation (2.28b), i.e.,

$$\frac{\omega_{o3}^2}{\omega_{o1}^2} = \frac{1}{1 + \left(\frac{\pi\omega_{o1}^2}{\lambda f}\right)^2}$$

Given P_0 =5mW, ω_0 =1.0mm, λ = 632.8nm and f = 10cm. Substituting these values, we obtain

$$\omega_{03} = \frac{1.0\ x\ 10^{-3}}{[1 + (\pi\ x\ 10^{-6}/632.8\ x\ 10^{-9} x\ 0.1)^2]^{½}} = \frac{1.0\ x\ 10^{-3}}{[1 + (49.64)^2]^{½}} = \frac{1.0\ x\ 10^{-3}}{49.65} = 0.02mm$$

The beam waist at the focal plane is 0.04mm.

Power received within the focal spot is

$$P = P_0[1 - e^{-2}] = \ 4.3232mW$$

The intensity at the focal point is

$$I = \frac{4.3232\ x 10^{-3}}{\pi(0.02\ x\ 0.02\ x\ 10^{-6})} = 3.41MW/m^2$$

When there is a pinhole of 0.5mm diameter to restrict the size of the waist, the radius of the beam spot at the focal plane is given by

$$\omega = \frac{1.22\lambda\ f}{D} = \frac{1.22\ x\ 632.8\ x\ 10^{-9} x\ 0.1}{0.5\ x\ 10^{-3}} = 154.4\ x 10^{-6} = 0.154mm$$

This expression is written under the assumptions that (1) there is no diffraction at the pinhole and (2) the beam-waist is not of Gaussian profile, but of nearly constant amplitude. The beam-waist is now 0.154mm.

The pinhole allows reduced power through. The amount of power that is allowed by the pinhole is given by

$$P = P_0\left[1 - e^{-2\frac{a^2}{\omega^2(z)}}\right] = 5[1 - e^{-0.125}] = 0.588mW$$

The intensity at the focal spot is now

$$I = \frac{0.588\ x10^{-3}}{\pi(0.154\ x\ 0.154\ x\ 10^{-6})} = 7.88\ x\ 10^3 W/m^2$$

2.4

The radius of curvature of the beam is given by

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right] = z\left[1 + \left(\frac{\pi\omega_0^2}{z\lambda}\right)^2\right] \Rightarrow [R(z) - z]z = \left(\frac{\pi\omega_0^2}{\lambda}\right)^2$$

It is infinite at z=0 where the beam waist lies. It is also infinite at z=∞. In between these two extreme values, the beam takes different values of radius of curvature. For z values less than z_0, the beam propagation is governed by wave optics while for $z > z_0$, geometrical optics takes over. Substituting the values, we obtain

$$(2z - z)z = \left(\frac{\pi\ x\ 0.25\ x10^{-6}}{632.8\ x\ 10^{-9}}\right)^2 = (1.24)^2 = 1.54 \Rightarrow z = 1.24m$$

Alternately, it can be shown that R=2z leads to $z=z_0$. The value of z_0 is 1.24m.

2.5

Figure [2a] shows the stable resonator.

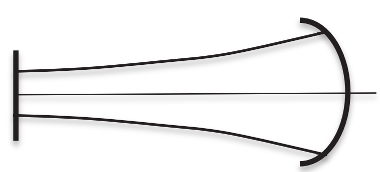


Figure [2a]

For the mode to be sustained, the curvatures of the wavefront should match with those of the mirrors. Therefore the beam waist would lie on the plane mirror and the radius of curvature of the beam must be R at the concave mirror. Therefore

$$R = d\left[1 + \left(\frac{z_0}{d}\right)^2\right] \Rightarrow \left(\frac{R}{d} - 1\right)^{1/2} = \frac{z_0}{d}$$

Write for $z_0 = \frac{\pi\omega_0^2}{\lambda}$, we obtain

$$\frac{z_0}{d} = \left(\frac{R}{d} - 1\right)^{1/2} \Rightarrow \frac{\pi\omega_0^2}{\lambda\, d} = \left(\frac{R}{d} - 1\right)^{1/2} \Rightarrow \omega_0 = \left(\frac{\lambda\, d}{\pi}\right)^{1/2}\left(\frac{R}{d} - 1\right)^{1/4}$$

2.6

(a) We use equation (233b), which is given by

$$\frac{\lambda}{\pi\omega_4^2} = \frac{z_{01}[(f-p)(f-d)+pf+d(f-p)]}{[pf+(f-p)d]^2+(f-d)^2 z_{01}^2} = \frac{z_{01}f^2}{[pf+(f-p)d]^2+(f-d)^2 z_{01}^2}$$

Given ω_{01} = 0.5mm; f = 10.0mm; p = 10.0cm, and d = 1.9m.

From this data, we obtain

$$z_{01} = \frac{\pi\omega_{01}^2}{\lambda} = \frac{\pi\ x\ 0.25\ x 10^{-6}}{632.8\ x\ 10^{-9}} = 1.24m$$

Hence

$$\frac{\lambda}{\pi\omega_4^2} = \frac{1.24\ x\ 10^{-4}}{(10^{-3}-0.009\ x\ 1.9)^2+(1.89\ x\ 1.24)^2} = 0.2246\ x\ 10^{-4}/m$$

$$\omega_4^2 = \frac{\lambda}{\pi\ x\ 0.2246\ x\ 10^{-4}} = \frac{632.8\ x\ 10^{-9}}{\pi\ x\ 0.2246\ x\ 10^{-4}} = 8.969\ x\ 10^{-3} \implies \omega_4 = 0.0947m$$

(b) The spot size just before the lens is

$$\omega_2 = \omega_{01} + \frac{\lambda\, p}{\pi\omega_{01}} = 0.5\ x\ 10^{-3} + \frac{0.6328\ X\ 10^{-6} x\ 0.1}{\pi\ x\ 0.5\ x\ 10^{-3}} = 0.54mm$$

The radius ω_f of the beam waist at the focal plane is

$$\omega_f = \frac{\lambda\, f}{\pi\omega_2} = \frac{632.8\ x\ 10^{-9} x\ 10^{-2}}{\pi\ x\ 0.54\ x\ 10^{-3}} = 3.7\ x\ 10^{-6}m$$

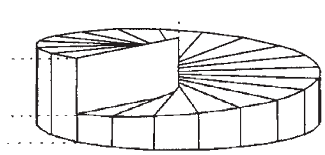
The intensity of the light wave at the focal plane is

$$I = \frac{5\ x\ 10^{-3}}{\pi(3.7\ x\ 10^{-6})^2} = 11.42\ x\ 10^7 = 114.2MW/m^2$$

(c) Using the relation (2.34a), it can be shown that the beam waist lies at a distance of 20.0 cm from the lens and its size is given by

$$\omega_c = \frac{f_c}{f}\omega_{01} = \frac{0.2}{0.01} 0.5\ x 10^{-3} = 10\ x 10^{-3} = 10^{-2}m = 1.0cm$$

2.7



The spiral phase plate (SPP) has the base thickness t_0 and continuously varying thickness as a function of azimuthal angle θ. At any azimuth, the thickness is constant radially. For a charge m, the step height is such that it introduces a path difference of mλ. Therefore the azimuthally varying thickness t_θ can be written as

$$t_\theta = \frac{m\,\lambda}{\mu-1}\frac{\theta}{2\pi}$$

Therefore the total thickness t(θ) can be expressed as

$$t(\theta) = t_0 + \frac{m\,\lambda}{\mu - 1}\frac{\theta}{2\pi}$$

In fact the constant thickness introduces a path difference of $(\mu-1)t_0$ and hence t_0 in the above expression should be $t_0/(\mu-1)$. Being a constant, it does not matter.

2.8

(a) Radius of the beam-waist at the focal plane is

$$\omega_f = \frac{\lambda f}{\pi\omega_0} = \frac{632.8\ x\ 10^{-9} x\ 0.2}{\pi\ x\ 10^{-3}} = 4.029\ x\ 10^{-5} m$$

The intensity at the focal plane is

$$I = \frac{P}{\pi\omega_f^2} = \frac{5\ x\ 10^{-3}}{\pi\left(4.029\ x\ 10^{-5}\right)^2} = 0.098\ x\ 10^7 W/m^2 \approx 1MW/m^2$$

(b) Radius of the spot at a plane 1.0cm from the focal plane

$$\omega(0.1) = \frac{1}{20}\ 1x\ 10^{-3} = 0.05mm$$

$$P = P_0\left[1 - e^{-2\frac{a^2}{\omega^2(z)}}\right] = 5\left[1 - e^{-2\frac{0.04}{0.0025}}\right] \cong 5mW$$

The aperture allows the entire beam through.

2.9

(a) The radius of curvature of the Gaussian beam is given by

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

Therefore

$$2z = z\left[1 + \left(\frac{z_0}{z}\right)^2\right] \Rightarrow z = \pm z_0$$

(b) The amplitude of a Gaussian wave is given by

$$U_o(x,y,z) = \frac{A}{\sqrt{1+\left(\frac{z}{z_o}\right)^2}} e^{-i\frac{k(x^2+y^2)}{2z\left(1+\frac{z_0^2}{z^2}\right)}} e^{-\frac{\pi}{\lambda}\frac{(x^2+y^2)}{z_0\left(1+\frac{z^2}{z_0^2}\right)}} e^{itan^{-1}\left(\frac{z}{z_o}\right)}$$

The amplitude at $z = z_0$ and at $r = 2\omega(z) = 2\omega(z_0)$ is given by

$$U_o(x,y,z_0) = \frac{A}{\sqrt{2}} e^{-\frac{\pi 4\omega^2(z_0)}{\lambda 2\pi\omega_0^2/\lambda}} = \frac{A}{\sqrt{2}} e^{-4}$$

The intensity is given as

$$I = \frac{I_0}{4}\ e^{-8} = 0.000084$$

The intensity is practically zero.

(c) The Gouy phase is given by $e^{itan^{-1}\left(\frac{z}{z_o}\right)}$, which is 45^0 at z = z_0.

(d) The phase at (r = 0, z = z_0) is

$$\phi(0; z_0) = -\frac{k(x^2 + y^2)}{4z_0} + tan^{-1}\left(\frac{z}{z_o}\right) = 45^0$$

The phase at [r = 2ω(z), z = z_0] is

$$\phi[2\omega(z_0); z_0] = -\frac{2\pi}{\lambda}\frac{4\left(\omega^2(z_0)\right)}{\frac{4\pi\omega_0^2}{\lambda}} + tan^{-1}\left(\frac{z}{z_o}\right) = -4 + 45^0$$

The phase difference $\phi(0; z_0) - \phi[2\omega(z_0); z_0] = 4\, rad$

(e)

The intensity at z = 0 plane is given by

$$I(0) = \frac{P}{\pi\omega_0^2} = \frac{5\ x\ 10^{-3}}{\pi\ x\ 10^{-6}}\frac{W}{m^2} = 1.6kW/m^2$$

The intensity at z = 5 z_0 plane is

$$I(z_0) = \frac{P}{\pi\omega^2(5z_0)} = \frac{P}{26\ x\ \pi\ x\ \omega_0^2} = \frac{1.6}{26} = 61.2W/m^2$$