Chapter 2

Second-Order Linear Equations

2.1 Second-Order Linear Equations with Constant Coefficients

1. Find the general solution of each of the following differential equations. See the table.

2. Find the solution of each of the following initial value problems:

(a) The associated polynomial $r^2 - 5r + 6$ has roots $r = 2, 3$. The general solution is $y = Ae^{3x} + Be^{2x}$. Initial conditions determine that $A = 0, B = 1/e$. Therefore, $\frac{1}{e}e^{3x}$.

(b) The associated polynomial $r^2 - 6r + 5$ has roots $r = 1, 5$. The general solution is $y = Ae^{x} + Be^{5x}$. Initial conditions determine that $A = 1, B = 2$. Therefore, $y = e^{x} + 2e^{5x}$.

(c) The associated polynomial $r^2 - 6r + 9$ has roots $r = 3, 3$. The general solution is $y = Ae^{3x} + Bxe^{3x}$. Initial conditions determine that $A = 3, B = -1$. Therefore, $y = 3e^{3x} - xe^{3x}$.

(d) The associated polynomial $r^2 + 4r + 5$ has roots $-2 \pm i$. The general solution is $y = Ae^{-2}\cos x + Be^{-2}\sin x$. Initial conditions determine that $A = e^2$ and $B = 0$. Therefore, $y = \cos x$.

(e) The associated polynomial $r^2 + 4r + 2$ has roots $-2 \pm \sqrt{2}$. The general solution is $y = Ae^{(-2+\sqrt{2})x} + Be^{(-2-\sqrt{2})x}$. Initial conditions determine that $A = 1, B = -2$. Therefore, $y = e^{(-2+\sqrt{2})x} - 2e^{(-2-\sqrt{2})x}$.

Table 2.1: The general solutions for Exercise 1.

	Assoc Poly	Roots	General Solution
(a)	r^2+r-6	$2,-3$	$Ae^{2x}+Be^{-3x}$
(b)	r^2+2r+1	$-1,-1$	$Ae^{-x}+Bxe^{-x}$
(c)	r^2+8	$\pm 2\sqrt{2}i$	$A\cos(2\sqrt{2}x)+B\sin(2\sqrt{2}x)$
(d)	$2r^2-4r+8$	$1\pm\sqrt{3}i$	$Ae^x\cos(\sqrt{3}x)+Be^x\sin(\sqrt{3}x)$
(e)	r^2-4r+4	$2,2$	$Ae^{2x}+Bxe^{2x}$
(f)	$r^2-9r+20$	$4,5$	$Ae^{4x}+Be^{5x}$
(g)	$2r^2+2r+3$	$-\frac{1}{2}\pm\frac{\sqrt{5}}{2}i$	$Ae^{-x/2}\cos(\sqrt{5}x/2)+Be^{-x/2}\sin(\sqrt{5}x/2)$
(h)	$4r^2-12r+9$	$3/2,3/2$	$Ae^{3x/2}+Bxe^{3x/2}$
(i)	r^2+r	$-1,0$	$Ae^{-x}+B$
(j)	$r^2-6r+25$	$3\pm 4i$	$e^{3x}(A\cos 4x+B\sin 4x)$
(k)	$4r^2+20r+25$	$\pm 5/2$	$Ae^{5x/2}+Be^{-5x/2}$
(l)	r^2+2r+3	$-1\pm\sqrt{2}i$	$Ae^{-x}\cos(\sqrt{2}x)+Be^{-x}\sin(\sqrt{2}x)$
(m)	r^2-4	± 2	$Ae^{2x}+Be^{-2x}$
(n)	$4r^2-8r+7$	$1\pm\frac{\sqrt{3}}{2}i$	$Ae^x\cos(\sqrt{3}x/2)+Be^x\sin(\sqrt{3}x/2)$
(o)	$2r^2+r-1$	$-1,1/2$	$Ae^{-x}+Be^{x/2}$
(p)	$16r^2-8r+1$	$1/4$	$Ae^{x/4}+Bxe^{x/4}$
(q)	r^2+4r+5	$-2\pm i$	$Ae^{-2x}\cos x+Be^{-2x}\sin x$
(r)	r^2+4r-5	$-5,1$	$Ae^{-5x}+Be^{x}$

(f) The associated polynomial $r^2 + 8r - 9$ has roots $r = -9, 1$. The general solution is $y = Ae^{-9x} + Be^x$. Initial conditions determine that $A = \frac{e^9}{5}, B = \frac{9}{5e}$. Therefore, $y = \frac{e^9}{5}e^{-9x} + \frac{9}{5e}e^x$.

3. The associated polynomial $r^2 + Pr + Q$ has roots $r = \frac{-P\pm\sqrt{P^2-4Q}}{2}$. Suppose that P and Q are both positive. Then $P^2 - 4Q \geq 0$ implies that the roots are real and negative so $y \to 0$ as $x \to \infty$ because both exponential terms in the solution have negative exponents. If $P^2-4Q < 0$, then the roots are complex with negative real part. Consequently, the solutions are of the form $y = Ae^{-Px/2}\cos\omega x + Be^{-Px/2}\sin\omega x$ and will oscillate towards 0 as $x \to \infty$. The other cases are handled similarly.

4. Take a derivative on both sides of $y'' + Py' + Qy = 0$ to obtain $y''' + Py'' + Qy' = 0$. As $y''' + Py'' + Qy' = (y')'' + P(y')' + Q(y')$, y' is a solution of the equation.

5. **Euler's equidimensional equation** Changing the independent variable using $x = e^z$ is equivalent to $z = \ln x$ so $y' = \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\dot{y}$ where the dot indicates differentiation with respect to the new independent variable, z. Similarly, $y'' = \frac{d}{dx}(\frac{1}{x}\dot{y}) = \frac{1}{x}\cdot\frac{1}{x}\ddot{y} - \frac{1}{x^2}\dot{y} = \frac{1}{x^2}(\ddot{y} - \dot{y})$. Making these substitutions into $x^2y'' + pxy' + qy = 0$ yields $x^2\cdot\frac{1}{x^2}(\ddot{y}-\dot{y})+px\cdot\frac{1}{x}\dot{y}+qy = 0$ which simplifies to $\ddot{y}+(p-1)\dot{y}+qy = 0$, an equation with constant coefficients. If $y = \phi(z)$ is the general solution to this equation, then $y = \phi(\ln x)$ will be the general solution to the Euler equidimensional equation. Note that the solution is only valid for $x > 0$.

(a) The z equation is $\ddot{y}+2\dot{y}+10y = 0$ with solution $y = Ae^{-z}\cos 3z + Be^{-x}\sin(3z)$. The x equation has the solution $y = Ax^{-1}\cos(3\ln x) + Bx^{-1}\sin(3\ln x)$.

(b) First divide the equation by 2. The z equation is then $\ddot{y}+4\dot{y}+4y = 0$ with solution $y = Ae^{-2z} + Bze^{-2z}$. The x equation has the solution $y = Ax^{-2} + Bx^{-2}\ln x$.

(c) The z equation is $\ddot{y}+\dot{y}-12y = 0$ with solution $y = Ae^{-4z}+Be^{3z}$. The x equation has the solution $y = Ax^{-4} + Bx^3$.

(d) Divide the equation by 4. The z equation is then $\ddot{y} - \dot{y} - \frac{3}{4}y = 0$ with solution $y = Ae^{-z/2}+Be^{3z/2}$. The x equation has the solution $y = Ax^{-1/2} + Bx^{3/2}$.

Table 2.2: The differential equations for Exercise 6.

	Roots	Assoc Poly	Differential equation
(a)	$1, -2$	$r^2 + r - 2$	$y'' + y' - 2y = 0$
(b)	$0, 2$	$r^2 - 2r$	$y'' - 2y' = 0$
(c)	$3, 5$	$r^2 - 8r + 15$	$y'' - 8y' + 15y = 0$
(d)	$1 \pm 3i$	$r^2 - 2r + 10$	$y'' - 2y' + 10y = 0$
(e)	$3, -1$	$r^2 - 2r - 3$	$y'' - 2y' - 3y = 0$
(f)	$-1, 4$	$r^2 + 5r + 4$	$y'' + 5y' + 4y = 0$
(g)	$2, -2$	$r^2 - 4$	$y'' - 4y = 0$
(h)	$-4 \pm i$	$r^2 + 8r + 17 = 0$	$y'' + 8y' + 17y = 0$

(e) The z equation is $\ddot{y} - 4\dot{y} + 4y = 0$ with solution $Ae^{2z} + Bze^{2z}$. The x equation has the solution $y = Ax^2 + Bx^2 \ln x$.

(f) The z equation is $\ddot{y} + \dot{y} - 6y = 0$ with solution $y = Ae^{-3z} + Be^{2z}$. The x equation has the solution $y = Ax^{-3} + Bx^2$.

(g) The z equation is $\ddot{y} + \dot{y} + 3y = 0$ with solution $y = Ae^{-z/2} \cos \frac{\sqrt{11}z}{2} + Be^{-z/2} \sin \frac{\sqrt{11}z}{2}$. The x equation has the solution

$$y = Ax^{-\frac{1}{2}} \cos \frac{\sqrt{11} \ln x}{2} + Bx^{-\frac{1}{2}} \sin \frac{\sqrt{11} \ln x}{2}.$$

(h) The z equation is $\ddot{y} - 2y = 0$ with solution $y = Ae^{-\sqrt{2}z} + Be^{\sqrt{2}z}$. The x equation has the solution $y = Ae^{-\sqrt{2}} + Bx^{\sqrt{2}}$.

(i) The z equation is $\ddot{y} - 16y = 0$ with solution $y = Ae^{-4z} + Be^{4z}$. The x equation as the solution $y = Ax^{-4} + Bx^4$.

6. Find the differential equation of each of the following general solution sets. See the table.

2.2 The Method of Undetermined Coefficients

1. Find the general solution of each of the following equations.

(a) The auxiliary roots are -5 and 2 so the homogeneous equation has the solution $y = Ae^{-5x} + Be^{2x}$. Try $y = \alpha e^{4x}$ as a particular solution. Substitute and simplify to obtain $18\alpha e^{4x} = 6e^{4x}$ which implies that $\alpha = 1/3$. The general solution is $y = Ae^{-5x} + Be^{2x} + \frac{1}{3}e^{4x}$.

(b) The auxiliary roots are $\pm 2i$ so the homogeneous equation has the solution $y = A\cos 2x + B\sin 2x$. Try $y = \alpha\cos x + \beta\sin x$ as a particular solution. Substitute and simplify to obtain $3\alpha\cos x + 3\beta\sin x = 3\sin x$ which implies that $\alpha = 0$ and $\beta = 1$. The general solution is $y = A\cos 2x + B\sin 2x + \sin x$.

(c) The auxiliary roots are $-5, -5$ so the homogeneous equation has the solution $y = Ae^{-5x} + Bxe^{-5x}$. Neither $y = \alpha e^{-5x}$ nor $y = \alpha xe^{-5x}$ can be a particular solution because they are solutions to the homogeneous equation. Try $y = \alpha x^2e^{-5x}$ instead. Substitute and simplify to obtain $2\alpha e^{-5x} = 14e^{-5x}$ which implies that $\alpha = 7$. The general solution is $y = Ae^{-5x} + Bxe^{-5x} + 7x^2e^{-5x}$.

(d) The auxiliary roots are $1 \pm 2i$ so the homogeneous equation has the solution $y = Ae^x\cos 2x + Be^x\sin x$. Try $y = \alpha x^2 + \beta x + \gamma$ as a particular solution. Substitute and simplify to obtain $5\alpha x^2 + (5\beta - 4\alpha)x + 2\alpha - 2\beta + 5\gamma = 25x^2 + 12$ which implies that $\alpha = 5, \beta = 4, \gamma = 2$. The general solution is $y = Ae^x\cos 2x + Be^x\sin x + 5x^2 + 4x + 2$.

(e) The auxiliary roots are $-2, 3$ so the homogeneous equation has the solution $y = Ae^{-2x} + Be^{3x}$. The function $y = \alpha e^{-2x}$ can not be a particular solution because it is a solution to the homogeneous equation. Try $y = \alpha xe^{-2x}$ instead. Substitute and simplify to obtain $-5\alpha xe^{-2x} = 20e^{-2x}$ which implies that $\alpha = -4$. The general solution is $y = Ae^{-2x} + Be^{3x} - 4xe^{-2x}$.

(f) The auxiliary roots are $1, 2$ so the homogeneous equation has the solution $y = Ae^x + Be^{2x}$. Try $y = \alpha\cos 2x + \beta\sin 2x$ as a particular solution. Substitute and simplify to obtain $(-6\alpha - 2\beta)\cos 2x + (-2\alpha + 6\beta)\sin 2x = 14\sin 2x - 18\cos 2x$ which implies that $\alpha = 3$ and $\beta = 2$. The general solution is $y = Ae^x + Be^{2x} + 3\cos 2x + 2\sin 2x$.

(g) The auxiliary roots are $\pm i$ so the homogeneous equation has the solution $y = A\cos x + B\sin x$. The function $y = \alpha\cos x + \beta\sin x$ can not be a particular solution because it is a solution to the homogeneous equation. Try $y = \alpha x\cos x + \beta x\sin x$ instead. Substitute and simplify to obtain $2\beta\cos x - 2\alpha\sin x = 2\cos x$ which implies that $\alpha = 0$ and $\beta = 1$. The general solution is $y = A\cos x + B\sin x + x\sin x$.

(h) The auxiliary roots are $0, 2$ so the homogeneous equation has the solution $y = A + Be^{2x}$. The function $y = \alpha x + \beta$ is not a particular solution because part of it is a solution to the homogeneous equation. Try $y = \alpha x^2 + \beta x$ instead. Substitute and simplify to obtain $-4\alpha x + 2\alpha - 2\beta = 12x - 10$ which implies that $\alpha = -3$ and $\beta = 2$. The general solution is $y = A + Be^{2x} - 3x^2 + 2x$.

(i) The auxiliary roots are $1, 1$ so the homogeneous equation has the solution $y = Ae^x + Bxe^x$. Neither $y = \alpha e^x$ not $y = \alpha xe^x$ is a particular solution because they are both solutions to the homogeneous equation. Try $y = \alpha x^2 e^x$ instead. Substitute and simplify to obtain $2\alpha e^x = 6e^x$ which implies that $\alpha = 3$. The general solution is $y = Ae^x + Bxe^x + 3x^2e^x$.

(j) The auxiliary roots are $1 \pm i$ so the homogeneous equation has the solution $y = Ae^x\cos x + Be^x\sin x$. This means that $y = \alpha e^x\cos x + \beta e^x\sin x$ can not be a particular solution. Try $y = \alpha xe^x\cos x + \beta xe^x\sin x$ instead. Substitute and simplify to obtain $2\beta e^x\cos x - 2\alpha e^x\sin x = e^x\sin x$ which implies that $\alpha = -1/2$ and $\beta = 0$. The general solution is $y = Ae^x\cos x + Be^x\sin x - \frac{1}{2}xe^x\cos x$.

(k) The auxiliary roots are $-1, 0$ so the homogeneous equation has the solution $y = Ae^{-x}+B$. This means that $y = \alpha x^4+\beta x^3+\gamma x^2+\delta x+\epsilon$ can not be a particular solution. Try $y = \alpha x^5+\beta x^4+\gamma x^3+\delta x^2+\epsilon x$ instead. Substitute and simplify to obtain $5\alpha x^4 + (20\alpha + 4\beta)x^3 + (12\beta + 3\gamma)x^2 + (6\gamma + 2\delta)x + 2\delta + \epsilon = 10x^4 + 2$ which implies that $\alpha = 2, \beta = -10, \gamma = 40, \delta = -120$ and $\epsilon = 242$. The general solution is $y = Ae^{-x} + B + 2x^5 - 10x^4 + 40x^3 - 120x^3 + 242x$.

2. Find the solution of the differential equation that satisfies the given initial conditions:

(a) The auxiliary roots are $\frac{3\pm\sqrt{5}}{2}$ so the homogeneous equation has the solution $y = Ae^{\frac{3+\sqrt{5}}{2}x} + Be^{\frac{3-\sqrt{5}}{2}x}$. Try $y = \alpha x + \beta$ as a particular solution. Substitute and simplify to obtain $-2\alpha x + \beta = x$ which implies that $\alpha = 1, \beta = 3$. The general solution is $y = x + 3 + Ae^{\frac{3+\sqrt{5}}{2}x} + Be^{\frac{3-\sqrt{5}}{2}x}$. Initial conditions determine that $A = \frac{2-\sqrt{5}}{\sqrt{5}}$ and $B = \frac{-2-\sqrt{5}}{\sqrt{5}}$. The solution that satisfies the given initial conditions is $y = x + 3 + \frac{2-\sqrt{5}}{\sqrt{5}}e^{\frac{3+\sqrt{5}}{2}x} + \frac{-2-\sqrt{5}}{e}^{\frac{3-\sqrt{5}}{2}x}$.

(b) The auxiliary roots are $-2 \pm \sqrt{2}i$ so the homogeneous equation has the solution $y = Ae^{-2x}\cos\sqrt{2}x + Be^{-2x}\sin\sqrt{2}x$. Try $y = \alpha\cos x + \beta\sin x$ as a particular solution. Substitute and simplify to obtain $\cos x(5\alpha+4\beta)+\sin x(5\beta-4\alpha) = \cos x$ which implies that $\alpha = \frac{5}{41}$ and $\beta = \frac{4}{41}$. The general solution is $y = Ae^{-2x}\cos\sqrt{2}x + Be^{-2x}\sin\sqrt{2}x + \frac{5}{41}\cos x + \frac{4}{41}\sin x$. Initial conditions determine that $A = -\frac{5}{41}$ and $B = \frac{88}{41\sqrt{2}}$. The solution that satisfies the given initial conditions is $y = -\frac{5}{41}e^{-2x}\cos\sqrt{2}x + \frac{88}{41\sqrt{2}}e^{-2x}\sin\sqrt{2}x + \frac{5}{41}\cos x + \frac{4}{41}\sin x$.

(c) The auxiliary roots are $\frac{-1\pm\sqrt{i}}{2}$ so the homogeneous equation has the solution $y = Ae^{-x/2}\cos\frac{\sqrt{3}}{2}x+Be^{-x/2}\sin\frac{\sqrt{3}}{2}x$. Try $y = \alpha\cos x+\beta\sin x$ as a particular solution. Substitute and simplify to obtain $\beta\cos x - \alpha\sin x = \sin x$ which implies that $\alpha = -1$ and $\beta = 0$. The general solution is $y = Ae^{-x/2}\cos\frac{\sqrt{3}}{2}x+Be^{-x/2}\sin\frac{\sqrt{3}}{2}x-\cos x$. Initial conditions determine that $A = (1+\cos 1)\cdot\sqrt{3}\cdot\cos\frac{\sqrt{3}}{2} - \frac{3+\cos 1+\sin 1}{\sqrt{3}}\cdot\sqrt{e}\cdot\sin\frac{\sqrt{3}}{2}$ and $B = \frac{3+\cos 1+\sin 1}{\sqrt{3}}\cdot\sqrt{3}\cdot\cos\frac{\sqrt{3}}{2} - \sqrt{e}\cdot(1+\cos 1)\cdot\sin\frac{\sqrt{3}}{2}$. The solution that satisfies the given initial conditions is $y = [(1+\cos 1)\cdot\sqrt{3}\cdot\cos\frac{\sqrt{3}}{2} - \frac{3+\cos 1+\sin 1}{\sqrt{3}}\cdot\sqrt{e}\cdot\sin\frac{\sqrt{3}}{2}]e^{-x/2}\cos\frac{\sqrt{3}}{2}x + [\frac{3+\cos 1+\sin 1}{\sqrt{3}}\cdot\sqrt{3}\cdot\cos\frac{\sqrt{3}}{2} - \sqrt{e}\cdot(1+\cos 1)\cdot\sin\frac{\sqrt{3}}{2}]e^{-x/2}\sin\frac{\sqrt{3}}{2}x - \cos x$.

(d) The auxiliary roots are $1, 2$ so the homogeneous equation has the solution $y = Ae^{x}+Be^{2x}$. Initial conditions determine that $A = -e$ and $B = e^2$. The solution that satisfies the given initial conditions is $y = -e\cdot e^x + e^2\cdot e^{2x} = -e^{x+1} + e^{2x+2}$.

(e) The auxiliary roots are $\frac{1\pm\sqrt{3}i}{2}$ so the homogeneous equation has the solution $y = Ae^{x/2}\cos\frac{\sqrt{3}}{2}x + Be^{x/2}\sin\frac{\sqrt{3}}{2}x$. Try $y = \alpha x^2 + \beta x + \gamma$ as a particular solution. Substitute and simplify to obtain $\alpha x^2 + (\beta - 2\alpha)x - \beta + 2\alpha = x^2$ which implies that $\alpha = 1$, $\beta = 2$ and $\gamma = 0$. The general solution is $y = Ae^{x/2}\cos\frac{\sqrt{3}}{2}x +$

$Be^{x/2}\sin\frac{\sqrt{3}}{2}x + x^2 + 2x$. Initial conditions determine that $A = 1$ and $B = -\frac{5}{\sqrt{3}}$. The solution that satisfies the given initial conditions is $y = e^{x/2}\cos\frac{\sqrt{3}}{2}x - \frac{5}{\sqrt{3}}e^{x/2}\sin\frac{\sqrt{3}}{2}x + x^2 + 2x$.

(f) The auxiliary roots are $-1, -1$ so the homogeneous equation has the solution $y = Ae^{-x} + Bxe^{-x}$. Try $y = \alpha x + \beta$ as a particular solution. Substitute and simplify to obtain $\alpha x + 2\alpha + \beta = 1$ which implies that $\alpha = 0$ and $\beta = 1$. The general solution is $y = Ae^{-x} + Bxe^{-x} + 1$. Initial conditions determine that $A = 0$ and $B = 0$. The solution that satisfies the given initial conditions is a constant $y = 1$.

3. The auxiliary roots are $-k, k$ so the homogeneous equation has the solution $y = Ae^{kx} + Be^{-kx}$. Try $y = \alpha\cos bx + \beta\sin bx$ as a particular solution. Substitute and simplify to obtain $(k^2 - b^2)\alpha\cos bx + (k^2 - b^2)\beta\sin bx = \sin bx$ which implies that $\alpha = 0$ and $\beta = \frac{1}{k^2-b^2}$. The general solution is $y = Ae^{kx} + Be^{-kx} + \frac{1}{k^2-b^2}\sin bx$.

4. Substitute $y = y_1 + y_2$ into the left side of the differential equation to get

$$(y_1+y_2)''+P(y_1+y_2)'+Q(y_1+y_2) = y_1''+Py_1'+Qy_1+y_2''+Py_2'+Qy_2 = R_1+R_2.$$

(a) The auxiliary roots are $\pm 2i$ so the homogeneous equation has the solution $y = A\cos 2x + B\sin 2x$.

A particular solution for $y'' + 4y = 4\cos 2x$ has the form $y_1 = \alpha x\cos 2x + \beta x\sin 2x$. Substitute to obtain $y_1 = x\sin 2x$.
A particular solution for $y'' + 4y = 6\cos x$ has the form $y_2 = \alpha\cos x + \beta\sin x$. Substitute to obtain $y_2 = 2\cos x$.
A particular solution for $y'' + 4y = 8x^2 - 4x$ has the form $y_3 = \alpha x^2 + \beta x + \gamma$. Substitute to obtain $y_3 = 2x^2 - x - 1$.

By the superposition principal $y = y_1+y_2+y_3$ is a particular solution to the given equation and $y = A\cos 2x + B\sin 2x + x\sin 2x + 2\cos x + 2x^2 - x + 1$ is a general solution.

(b) The auxiliary roots are $\pm 3i$ so the homogeneous equation has the solution $y = A\cos 3x + B\sin 3x$.

A particular solution for $y'' + 9y = 2\sin 3x$ has the form $y_1 = \alpha x\cos 3x + \beta x\sin 3x$. Substitute to obtain $y_1 = -\frac{1}{3}x\cos 3x$.

A particular solution for $y'' + 4y = 4\sin x$ has the form $y_2 = \alpha\cos x + \beta\sin x$. Substitute to obtain $y_2 = \frac{1}{2}\sin x$.
A particular solution for $y'' + 4y = -26e^{-2x}$ has the form $y_2 = \alpha e^{-2x}$. Substitute to obtain $y_4 = 3x^3 - 2x$.

By the superposition principal $y = y_1 + y_2 + y_3 + y_4$ is a particular solution to the given equation and $y = A\cos 3x + B\sin 3x - \frac{1}{3}x\cos 3x + \frac{1}{2}\sin x - 2e^{-2x} + 3x^2 - 2x$ is a general solution.

2.3 The Method of Variation of Parameters

1. Find a particular solution.

(a) The homogeneous solution is $y = A\sin 2x + B\cos 2x$ so the particular solution has the form $y = v_1\sin 2x + v_2\cos 2x$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'\sin 2x + v_2'\cos 2x &= 0 \\ 2v_1'\cos 2x - 2v_2'\sin 2x &= \tan 2x \end{aligned}$$

Therefore, $v_1' = \frac{1}{2}\sin x$ and $v_2' = -\frac{1}{2}\sin 2x\tan 2x$. Integrate to get $v_1 = -\frac{1}{4}\cos 2x$ and $v_2 = \frac{1}{4}\sin 2x - \frac{1}{4}\ln(\sec 2x + \tan 2x)$. Therefore, $y = -\frac{1}{4}\cos 2x\ln(\sec 2x + \tan 2x)$.

(b) The homogeneous solution is $y = Ae^{-x} + Bxe^{-x}$ so the particular solution has the form $y = v_1e^{-x} + v_2xe^{-x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{-x} + v_2'xe^{-x} &= 0 \\ -v_1'e^{-x} + v_2'(e^{-x} - xe^{-x}) &= e^{-x}\ln x \end{aligned}$$

Therefore, $v_1' = -x\ln x$ and $v_2' = \ln x$. Integrate to get $v_1 = -\frac{1}{2}x^2\ln x + \frac{1}{4}x^2$ and $v_2 = x\ln x - x$. Therefore, $y = \frac{1}{4}x^2e^{-x}(2\ln x - 3)$.

(c) The homogeneous solution is $y = Ae^{3x} + Be^{-x}$ so the particular solution has the form $y = v_1e^{3x} + v_2e^{-x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{3x} + v_2'e^{-x} &= 0 \\ 3v_1'e^{-x} - v_2'e^{-x} &= 64xe^{-x} \end{aligned}$$

Therefore, $v_1' = 16xe^{-4x}$ and $v_2' = -16x$. Integrate to get $v_1 = -(4x+1)e^{-4x}$ and $v_2 = -8x^2$. Therefore, $y = -e^{-x}(8x^2+4x+1)$. The last term can be dropped since $-e^{-x}$ is a solution to the homogeneous solution.

(d) The homogeneous solution is $y = Ae^{-x}\sin 2x + Be^{-x}\cos 2x$ so the particular solution has the form $y = v_1e^{-x}\sin 2x + v_2e^{-x}\cos 2x$ where v_1 and v_2 satisfy the system

$$\begin{aligned} &v_1'e^{-x}\sin 2x + v_2'e^{-x}\cos 2x = 0 \\ &v_1'(-e^{-x}\sin 2x + 2e^{-x}\cos 2x) + v_2'(e^{-x}\cos 2x + 2e^{-x}\sin 2x) \\ &\qquad = e^{-x}\sec 2x \end{aligned}$$

Therefore, $v_1' = \frac{1}{2}$ and $v_2' = -\frac{1}{2}\tan 2x$. Integrate to get $v_1 = \frac{1}{2}x$ and $v_2 = \frac{1}{4}\ln(\cos 2x)$. Therefore, $y = \frac{1}{2}xe^{-x}\sin 2x + \frac{1}{4}e^{-x}\cos 2x\ln(\cos 2x)$.

(e) The homogeneous solution is $y = Ae^{-x/2} + Be^{-x}$ so the particular solution has the form $y = v_1e^{-x/2} + v_2e^{-x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{-x/2} + v_2'e^{-x} &= 0 \\ -\frac{1}{2}v_1'e^{-x/2} - v_2'e^{-x} &= \frac{1}{2}e^{-3x} \end{aligned}$$

Therefore, $v_1' = e^{-5x/2}$ and $v_2' = -e^{-2x}$. Integrate to get $v_1 = -\frac{2}{5}e^{-5x/2}$ and $v_2 = \frac{1}{2}e^{-x}$. Therefore, $y = \frac{1}{10}e^{-3x}$.

(f) The homogeneous solution is $y = Ae^{x} + Be^{2x}$ so the particular solution has the form $y = v_1e^{x} + v_2e^{2x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{x} + v_2'e^{2x} &= 0 \\ v_1'e^{x} + 2v_2'e^{2x} &= (1+e^{-x})^{-1} \end{aligned}$$

Therefore, $v_1' = -\frac{e^{-x}}{1+e^{-x}}$ and $v_2' = \frac{e^{-2x}}{1+e^{-x}}$. Integrate to get $v_1 = \ln(1+e^{-x})$ and $v_2 = \ln(1+e^{-x}) - e^{-x}$. Consequently, the particular solution is $y = (e^{x}+e^{2x})\ln(1+e^{-x}) - e^{x}$.

2. the homogeneous solution is $y = A\cos x + B\sin x$ so the particular solution has the form $y = v_1\cos x + v_2\sin x$.

(a) The functions v_1 and v_2 satisfy the system

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0 \\ -v_1' \sin x + v_2' \cos x &= \sec x \end{aligned}$$

Therefore, $v_1' = -\tan x$ and $v_2' = 1$. Integrate to get $v_1 = \ln|\sec x|$ and $v_2 = x$. Therefore, $y = (A + \ln|\sec x|)\cos x + (B + x)\sin x$.

(b) The functions v_1 and v_2 satisfy the system

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0 \\ -v_1' \sin x + v_2' \cos x &= \cot^2 x \end{aligned}$$

Therefore, $v_1' = -\frac{\cos^2 x}{\sin x}$ and $v_2' = -\frac{\cos^3 x}{\sin^2 x}$. Integrate to get $v_1 = -\ln|\csc x - \cot x| - \cos x$ and $v_2 = \csc x + \sin x$. Therefore, $y = A\cos x + B\sin x + (-\ln|\csc x - \cot x| - \cos x)\cos x + (\csc x + \sin x)\sin x$.

(c) The functions v_1 and v_2 satisfy the system

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0 \\ -v_1' \sin x + v_2' \cos x &= \cot(2x) \end{aligned}$$

Therefore, $v_1' = -\frac{\cos 2x}{2\cos x}$ and $v_2' = -\frac{\cos 2x}{2\sin x}$. Integrate to get $v_1 = -\sin x + \frac{1}{2}\ln|\sec x + \tan x|$ and $v_2 = \frac{1}{2}\ln|\csc x - \cot x| + \cos x$. Therefore, $y = A\cos x + B\sin x + (-\sin x + \frac{1}{2}\ln|\sec x + \tan x|)\cos x + (\frac{1}{2}\ln|\csc x - \cot x| + \cos x)\sin x$.

(d) The functions v_1 and v_2 satisfy the system

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0 \\ -v_1' \sin x + v_2' \cos x &= x\cos x \end{aligned}$$

Therefore, $v_1' = -x\sin x\cos x$ and $v_2' = x\cos^2 x$. Integrate to get $v_1 = \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x$ and $v_2 = \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x$. Therefore, $y = A\cos x + B\sin x + (\frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x)\cos x + (\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x)\sin x$.

(e) The functions v_1 and v_2 satisfy the system

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0 \\ -v_1' \sin x + v_2' \cos x &= \tan x \end{aligned}$$

Therefore, $v_1' = -\frac{\sin^2 x}{\cos x}$ and $v_2' = \sin x$. Integrate to get $v_1 = -\ln|\sec x + \tan x| + \sin x$ and $v_2 = -\cos x$. Therefore, $y = A\cos x + B\sin x + (-\ln|\sec x + \tan x| + \sin x)\cos x + (-\cos x)\sin x$.

(f) The functions v_1 and v_2 satisfy the system

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0 \\ -v_1' \sin x + v_2' \cos x &= \sec x \tan x \end{aligned}$$

Therefore, $v_1' = -\frac{\sin^2 x}{\cos^2 x}$ and $v_2' = \tan x$. Integrate to get $v_1 = -\tan x + x$ and $v_2 = \ln|\sec x|$. Therefore, $y = A\cos x + B\sin x + (-\tan x + x)\cos x + (\ln|\sec x|)\sin x$.

(g) The functions v_1 and v_2 satisfy the system

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0 \\ -v_1' \sin x + v_2' \cos x &= \sec x \csc x \end{aligned}$$

Therefore, $v_1' = -\frac{\sin^2 x}{\cos^2 x}$ and $v_2' = \tan x$. Integrate to get $v_1 = -\tan x + x$ and $v_2 = \ln|\sec x|$. Therefore, $y = A\cos x + B\sin x + (-\tan x + x)\cos x + (\ln|\sec x|)\sin x$.

3. By Inspection The auxiliary polynomial is $r^2 - 2r + 1$ with roots $1, 1$ so the homogeneous solution is $y = Ae^x + Bxe^x$. Therefore, there is a particular solution of the form $y = \alpha x + \beta$. Substitute to find that $y = 2x + 4$ is a particular solution.

By Variation of Parameters The particular solution has the form $y = v_1 e^x + v_2 x e^x$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1' e^x + v_2' x e^x &= 0 \\ v_1' e^x + v_2'(xe^x + e^x) &= 2x \end{aligned}$$

Therefore, $v_1' = -2x^2 e^{-x}$ and $v_2' = 2xe^{-x}$. Integrate to get $v_1 = (2x^2 + 4x + 4)e^{-x}$ and $v_2 = -(2x + 2)e^{-x}$. Therefore, $y = 2x + 4$.

4. By undetermined coefficients THe auxiliary polynomial is $r^{-}r - 6$ with roots $-2, 3$ so the homogeneous solution is $y = Ae^{-2x} + Be^{3x}$. Therefore, there is a particular solution of the form $y = \alpha e^{-x}$. Substitute to find that $y = -\frac{1}{4}e^{-x}$.

By variation of parameters The particular solution has the form $y = v_1e^{-2x} + v_2e^{3x}$ where v_1 and v_2 satisfy the system

$$v_1'e^{-2x} + v_2'e^{3x} = 0$$
$$-2v_1'e^{-2x} + 3v_2'e^{3x} = e^{-x}$$

Therefore, $v_1' = -\frac{e^x}{5}$ and $v_2' = \frac{e^{-4x}}{5}$. Integrate to get $v_1 = -\frac{e^x}{5}$ and $v_2 = -\frac{1}{20}e^{-4x}$. Therefore, the particular solution is $y = -\frac{e^x}{5}e^{-2x} - \frac{1}{20}e^{-4x}e^{3x} = -\frac{1}{4}e^{-x}$.

5. The solution y_h to the homogeneous equation is given below. Use it to apply the variation of parameters technique to obtain the particular solution, labeled y_p. The general solution is $y = y_h + y_p$.

Warning Write the equation in the form

$$y'' + P(x)y' + Q(x)y = R(x)$$

before applying the variation of parameters algorithm.

(a) $y_h = Ax + B(x^2 + 1)$; $y_p = x^4/6 - x^2/2$.

(b) $y_h = Ax^{-1} + Be^x$; $y_p = -\frac{1}{3}x^2 - x - 1$.

(c) $y_h = Ax + Be^x$; $y_p = x^2 + x + 1$.

(d) $y_h = A(1 + x) + Be^x$; $y_p = \frac{1}{2}e^{2x}(x - 1)$.

(e) $y_h = Ax^2 + Bx$; $y_p = -xe^{-x} - (x^2 + x)\int \frac{e^{-x}}{x}dx$.
To obtain this solution formula you will have to apply integration by parts to $\int \frac{e^{-x}}{x^2}dx$. The remaining integral does not evaluate to an elementary function.

2.4 The Use of a Known Solution to Find Another

1. Find y_2 and the general solution, given y_1.

(a) Since $p(x) = 0$, $e^{-\int p(x)dx} = e^0 = 1$ and $y_2(x) = \sin(x)v(x)$ where $v(x) = \int \frac{1}{\sin^2 x}dx = \int \csc^2 x dx = -\cot x$. Therefore, $y_2(x) = -\cos x$. The general solution is $y = A\sin x + B\cos x$.

(b) Once more, $p(x) = 0$ and $e^{-\int p(x)dx} = e^0 = 1$. Therefore, $y_2(x) = e^x v(x)$ where $v(x) = \int \frac{1}{e^{2x}} dx = -\frac{1}{2}e^{-2x}$. Therefore, $y_2(x) = e^x(-\frac{1}{2}e^{-2x}) = -\frac{1}{2}e^{-x}$. The general solution is $y = Ae^x + Be^{-x}$.

2. Since $p(x) = \frac{3}{x}$, $e^{-\int p(x)dx} = x^{-3}$ and $y_2(x) = y_1(x)v(x) = v(x)$ where $v(x) = \int x^{-3}dx = -\frac{1}{2}x^{-2}$. The general solution is $y = A + Bx^{-2}$.

3. If $y = y_1 = x^2$, then $x^2y'' + xy' - 4y = x^2 \cdot 2 + x \cdot 2x - 4 \cdot x^2 = 0$. To find y_2 observe that $p(x) = 1/x$ and $e^{\int 1/x dx} = e^{-\ln x} = 1/x$. Therefore, $y_2 = x^2 \int \frac{1}{x^4} \cdot \frac{1}{4} = x^2 \cdot \frac{-4}{x^4} = -4/x^2$. The general solution is $y = Ax^2 + Bx^{-2}$.

4. Since $p(x) = \frac{-2x}{1-x^2}$, $e^{-\int p(x)dx} = \frac{1}{1-x^2}$. Therefore, $y_2 = x^2 \int \frac{1}{x^4} \cdot \frac{1}{1-x^2}dx = -\frac{1}{3x} - x - \log(1-x) + \frac{x^2}{2}\log(x+1)$. The general solution is $y = Ax^2 + B[-\frac{1}{3x} - x - \log(1-x) + \frac{x^2}{2}\log(x+1)]$.

5. If $y = y_1 = x^{-1/2}\sin x$, then

$$y' = x^{-1/2}\cos x - \frac{1}{2}x^{-3/2}\sin x$$
$$y'' = -x^{-1/2}\sin x - x^{-3/2}\cos x + \frac{3}{4}x^{-5/2}\sin x$$

Substitute carefully into (*).

To find y_2 observe that $p(x) = 1/x$ so $e^{-\int 1/x dx} = 1/x$. Therefore, $y_2 = x^{-1/2}\sin x \int \frac{1}{x^{-1}\sin^2 x} \cdot \frac{1}{x}dx = x^{-1/2}\sin x \int \csc^2 x dx = x^{-1/2}\sin x(-\cot x) = -x^{-1/2}\cos x$. Therefore, the general solution is $y = Ax^{-1/2}\sin x + Bx^{-1/2}\cos x$.

6. Find the general solution given y_1.

(a) Since $p(x) = \frac{x}{x-1}$, $e^{-\int p(x)dx} = e^{\int \frac{x}{x-1}dx} = e^{x-1+\ln(x-1)} = e^{x-1} \cdot (x-1)$. Therefore, $y_2 = x \cdot \int \frac{1}{x^2} \cdot e^{x-1} \cdot (x-1)dx = x \cdot \frac{e^{x-1}}{x} = e^{x-1}$. The general solution is $y = Ax + Be^{x-1}$.

(b) Since $p(x) = \frac{2x}{x^2} = \frac{2}{x}$, $e^{-\int p(x)dx} = e^{-2\ln x} = \frac{1}{x^2}$. Therefore, $y_2 = x\int \frac{1}{x^2} \cdot \frac{1}{x^2}dx = x \cdot (-\frac{1}{3x^3}) = -\frac{1}{3x^2}$. The general solution is $y = Ax + \frac{B}{x^2}$.w

(c) Since $p(x) = \frac{-x(x+2)}{x^2}$, $e^{-\int p(x)dx} = e^{x+\ln x^2} = x^2e^x$. Therefore, $y_2 = x\int \frac{1}{x^2} \cdot e^x \cdot x^2dx = xe^x$. The general solution is $y = Ax + Bxe^x$.

7. By inspection, $y = y_1 = x$ is one solution. Since $p(x) = -xf(x)$, the second solution has the form $y_2 = x \int \frac{1}{x^2} e^{\int xf(x)dx} dx$. The general solution has the form $y = Ax + Bx \int \frac{1}{x^2} e^{\int xf(x)dx} dx$.

8. Substitute $y_1 = y_1' = y_1'' = e^x$ to get $xe^x - (2x+1)e^x + (x+1)e^x = 0$. Since $p(x) = -\frac{2x+1}{x}$, $e^{-\int p(x)dx} = e^{2x|\ln x} = xe^{2x}$. Therefore, $y_2 = e^x \int \frac{1}{e^{2x}} \cdot e^{2x} \cdot xdx = \frac{1}{2}x^2 e^x$. The general solution is $y = Ae^x + Bx^2 e^x$.

9. If y_1 and y_2 are linearly dependent, then the function $v(x)$ is a constant and has a derivative that is identically 0. However, $v'(x) = \frac{1}{y_1^2} e^{-\int P(x)dx}$, which is never 0 (exponentials cannot vanish).

2.5 Vibrations and Oscillations

1. The amplitude $A = \frac{F_0}{\sqrt{(k-\omega^2 M)^2 + \omega^2 c^2}}$ attains its maximum at the ω value that minimizes the polynomial $\phi(\omega) = (k - \omega^2 M)^2 + \omega^2 c^2$. A simple calculation will show that $\phi'(\omega) = 0$ when $\omega = 0$ or $\omega = \pm\sqrt{\frac{k}{M} - \frac{c^2}{2M^2}}$. Thus if $\frac{k}{M} \leq \frac{c^2}{2M^2}$, i.e. $c \geq \sqrt{2kM}$, then there is no resonance frequency and as ω increases from 0, the amplitude A will steadily decrease to 0. On the other hand, if $0 < c < \sqrt{2kM}$, then A will increase as ω increases reaching its maximum value at the $\omega^* = \sqrt{\frac{k}{M} - \frac{c^2}{2M^2}}$ and decrease to 0 thereafter. The resonance frequency is $\frac{1}{2\pi}\sqrt{\frac{k}{M} - \frac{c^2}{2M^2}}$. This frequency is clearly less than the natural frequency $\frac{1}{2\pi}\sqrt{\frac{k}{M}}$.

3. Let b denote the density of the buoy (weight per unit volume) and ω the density of water. The volume of the buoy is $V = \frac{4}{3}\pi r^3$.

> The volume of a slice of the buoy from its center to a point y units from center is $\pi r^2 y - \frac{1}{3}y^3$ (exercise).

Since the buoy floats half-submerged, $b = \omega/2$. As it bobs up and down let y be the distance from its center to the surface of the water (up is positive). If $y > 0$, then the net force on the buoy is negative given by the difference between the upward buoyant force of the water:

$$w \cdot (\frac{V}{2} - \pi r^2 y + \frac{1}{3}\pi y^3),$$

and the downward weight of the buoy $b \cdot V$. Subtracting, the net force is $\omega(-\pi r^2 y + \frac{1}{3}\pi y^3)$. Newton's law ($ma = F$) applied to the sphere, at its center of mass, yields the following equation (g is the gravitational constant)

$$\frac{b \cdot V}{g} y'' = -\pi\omega r^2 y + \frac{1}{3}\omega\pi y^3.$$

This is a second-order non-linear differential equation. However, if the buoy is only "slightly" depressed, then the linearized version (ignore the y^3 term) provides an excellent model for the motion. The linearized equation simplifies to $y' + a^2 y = 0$ where $a = \sqrt{\frac{3g}{2r}}$. The period of the motion is $2\pi\sqrt{\frac{2r}{3g}}$ seconds.

4. Similar to the solution of Exercise 3.

5. Recall, Section 1.10 problem 4, that inside the Earth the force of gravity on an object is proportional to its distance from the center. Let x be the distance from the train to the center of a tunnel of length $2L$. Draw a picture to see that the distance from the train to the center of the Earth is $\sqrt{x^2 + R^2 - L^2}$ where R is the radius of the Earth. The magnitude of the force on the train, in the direction of the center of the Earth, is then $F_c = k\sqrt{x^2 + R^2 - L^2}$. The value of k can be found from this equation when the train is at the surface of the Earth: $mg = kR$, so $k = mg/R$.

The magnitude of the force on the train parallel to the tracks is the component of F_c in that direction: $F_c \cdot \cos\theta = F_c \cdot \frac{x}{\sqrt{x^2+R^2-L^2}} = kx$. When x is positive, the force is negative. Applying Newton's Second Law we have $mx'' = -kx = -\frac{mg}{R}x$. Thus $x'' + \frac{g}{R}x = 0$, and the period of motion is independent of L: $T = 2\pi\sqrt{\frac{R}{g}}$ seconds; this is approximately 90 minutes. The equation of motion for a particular L value is found from the initial conditions: $x(0) = L$ and $x'(0) = 0$. This yields $x(t) = L\cos\sqrt{\frac{g}{R}}t$. The greatest speed is $|x'(T/4)| = \sqrt{\frac{g}{R}}L \approx 4.43L$ miles per hour.

6. Adjusting the weight with a gravitational constant to be a mass, the differential equation becomes

$$4\frac{d^2x}{dt^2} + 64x = 32\sin 4t\,. \qquad (*)$$

Solving the homogeneous equation as usual, we find that $x(t)$ is either $\cos 4t$ or $\sin 4t$. We solve the inhomogeneous equation by undetermined coefficients. Since the obvious guess is already a solution of the homogeneous equation, we instead guess

$$x_p(t) = At\cos 4t + Bt\sin 4t\,.$$

Substituting this guess into $(*)$, and solving for the coefficients, we find that

$$x_p(t) = -\frac{8}{5}t\cos 4t\,.$$

Thus the general solution of $(*)$ is

$$x(t) = -\frac{8}{5}t\cos 4t + A\cos 4t + B\sin 4t\,.$$

The presence of the t factor guarantees that x will take arbitrarily large values.

2.6 Newton's Law of Gravitation and Kepler's Laws

1. Kepler's Third

(a) In astronomy the semi-major axis of the orbit is called the *mean distance* to the Sun because it is the average of the least and greatest values of r. Let a_u and T_u denote the semi-major axis and period of Uranus. These are known from Example 2.6.1. According to Kepler's Third Law, $\frac{T_u^2}{a_u^3} = \frac{T_m^2}{a_m^3}$ where a_m and T_m are Mercury's semi-major axis and period. Consequently, being careful with the units–see Example 2.6.1–we have

$$\begin{aligned} a_m = (\frac{T_m}{T_u})^{2/3} \cdot a_u &= (\frac{\frac{88}{365} \cdot (3.16 \times 10^7)}{26.516 \times 10^8})^{2/3} \cdot (2.87 \times 10^{14}) \\ &= 5.800 \times 10^{12} \text{ centimeters.} \end{aligned}$$

This is 5.800×10^{12} centimeters.

(b) When distance is measured in astronomical units and time in years, then $\frac{4\pi^2}{GM} = 1$ (verify). Therefore, in this system of units, $T^2 = a^3$. For example, the value of a_m calculated above can also be found (in astronomical units) using $a_m = T_m^{2/3} = (\frac{88}{365})^{2/3} = 0.3874$ au. Multiply by $93,000,000$ to obtain $a_m = 36,000,000$ miles. Regarding Saturn, $T_s = a_s^{3/2} = (9.54)^{3/2} = 29.5$ years.

3. According to Exercise 2, in the instant after the explosion, the motion of every particle that moves into an elliptical orbit about the Sun obeys the equation $v^2 = GM(\frac{2}{r} - \frac{1}{a})$. Consequently all of these particles move in an orbit with the same semi-major axis, a astronomical units, and (according to Kepler's Third Law) the same period, $T = a^{3/2}$ years. This means that T years later all of them will return to their original positions.

4. The formula derived in the text is

$$r = \frac{\|\mathbf{C}\|^2}{GM} \cdot \frac{1}{1 + \|\mathbf{K}\| \cos\theta} \,.$$

If we let $h = \|\mathbf{C}\|$ and $k = GM$ then we get the desired formula representing Kepler's first law.

Formula (22) in the text is

$$\mathcal{R} \times \mathcal{R}' = \mathbf{C} \,.$$

Taking lengths gives the formula in the exercise.

5. See Exercise 1, part (b).

(a) $T = 2^{3/2} = 2.83$ years.

(b) $T = 3^{3/2} = 5.20$ years.

(c) $T = 25^{3/2} = 125$ years.

2.7 Higher-Order Coupled Harmonic Oscillators

1-15. Find the general solution. See Table 2.3.

Table 2.3: Solutions for 1-15.

	Associated Polynomial	General Solution
1.	$r(r-1)(r-2)$	$y = A + Be^x + Ce^{2x}$
2.	$(r-1)(r^2-2r+2)$	$y = Ae^x + Be^x \cos x + Ce^x \sin x$
3.	$(r-1)(r^2+r+1)$	$y = Ae^x + e^{-x/2}(B\cos(\sqrt{3}x/2) + C\sin(\sqrt{3}x/2))$
4.	$(r+1)(r^2-r+1)$	$y = Ae^{-x} + Be^{x/2}\cos\frac{\sqrt{3}}{2}x + Ce^{x/2}\sin\frac{\sqrt{3}}{2}x$
5.	$(r+1)^3$	$y = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x}$
6.	$(r+1)^4$	$y = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x} + Dx^3e^{-x}$
7.	$(r^2-1)(r^2+1)$	$y = Ae^x + Be^{-x} + C\cos x + D\sin x$
8.	$(r^2+1)(r^2+4)$	$y = A\cos x + B\sin x + C\cos 2x + D\sin 2x$
9.	$(r-a)^2(r+a)^2$	$y = Ae^{ax} + Bxe^{ax} + Ce^{-ax} + Dxe^{-ax}$
10.	$(r^2+a^2)^2 = 0$	$y = A\cos ax + Bx\cos ax + C\sin ax + Dx\sin ax$
11.	$(r+1)^2(r^2+1)$	$y = Ae^{-x} + Bxe^{-x} + C\cos x + D\sin x$
12.	$(r-1)^2(r^2+4r+5)$	$y = Ae^x + Bxe^x + Ce^{-2x}\cos\frac{3}{2}x + De^{-2x}\sin\frac{3}{2}x$
13.	$(r-1)(r-2)(r-3)$	$y = Ae^x + Be^{2x} + Ce^{3x}$
14.	$(r-2)(r+1)^3$	$y = Ae^{2x} + Be^{-x} + Cxe^{-x} + Dx^2e^{-x}$
15.	$(r-6)(r-2)^2(r+2)^2$	$y = Ae^{6x} + Be^{2x} + Cxe^{2x} + De^{-2x} + Exe^{-2x}$

16. The associated polynomial is r^4 so the general solution to the homogeneous equation is $y_g = A + Bx + Cx^2 + Dx^3$. Try $y = \alpha\cos x + \beta\sin x + \gamma x^4$ as a particular solution. Substitute and simplify to get $\alpha\cos x + \beta\sin x + \gamma = \sin x + 24$ which implies that $\alpha = 0, \beta = 1, \gamma = 24$. Therefore, the general solution is $y = A + Bx + Cx^2 + Dx^3 + \sin x + 24x^4$.

17. The associated polynomial is $r(r-1)(r-2)$ so the general solution to the homogeneous equation is $y_g = A + Be^x + Ce^{2x}$. Based on the forcing function our fist choice for y_p is $y = A + Be^{3x}$. However, this will not work because $y = A$ is a solution to the homogeneous equation. Try $y = Ax + Be^{3x}$ instead. Substitute this into the forced equation to see that $A = 5$ and $B = 7$. The general solution is $y = A + Be^x + Ce^{2x} + 5x + 7e^{3x}$.

18. The associated polynomial is $r(r-1)(r+1)$ so the general solution to the homogeneous equation is $y_g = A + Be^{-x} + Ce^x$. Try $y_p = \alpha x$. Substitute and simply to get $\alpha = -1$. The general solution is $y = A + Be^{-x} + Ce^x - x$. Initial conditions determine that $A = 0$, $B = -1/2$, and $C = 9/2$. Therefore, $y = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x - x$.

19. The Euler Equidimensional Equation (order 3) Using $x = e^z$ is equivalent to $z = \ln x$ so $y' = \frac{1}{x}\dot{y}$ and $y'' = \frac{1}{x^2}(\ddot{y} - \dot{y})$. The dot indicates differentiation with respect to the new independent variable, z. See Section 2.1 problem 5. For the third derivative,

$$y''' = \frac{d}{dx}(\frac{1}{x^2}(\ddot{y} - \dot{y})) = \frac{1}{x^2}(\dddot{y} - \dot{y})\frac{1}{x} - \frac{2}{x^3}(\ddot{y} - \dot{y})$$
$$\frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y}).$$

Making these substitutions into $x^3y''' + a_2x^2y'' + a_1xy' + a_0y = 0$ yields $x^3 \cdot \frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y}) + a_2x^2 \cdot \frac{1}{x^2}(\ddot{y} - \dot{y}) + a_1x \cdot \frac{1}{x}\dot{y} + a_0y = 0$ which simplifies to

$$\dddot{y} + (a_2 - 3)\ddot{y} + (a_1 - a_2 + 2)\dot{y} + a_0y = 0,$$

an equation with constant coefficients. If $y = \phi(z)$ is the general solution to the equation, then $y = \phi(\ln x)$ will be the general solution to the Euler equidimensional equation. Note that this solution is only valid for $x > 0$.

(a) The z equation is $\dddot{y} - \dot{y} = 0$ with associated polynomial $r^3 - r = r(r^2 - 1)$. The solution is $y = A + Be^x + Ce^{-x}$ so the solution to the original equation is $y = A + Bx + Cx^{-1}$.

(b) The z equation is $\dddot{y} - 2\ddot{y} - \dot{y} + 2y = 0$ with associated polynomial $r^3 - 2r^2 - r + 2 = (r - 1)(r + 1)(r - 2)$. The solution is $y = Ae^z + Be^{-z} + Ce^{2z}$ so the solution to the original equation is $y = Ax + Bx^{-1} + Cx^2$.

(c) The z equation is $\dddot{y} - \ddot{y} + \dot{y} - y = 0$ with associated polynomial $r^3 - r^2 + r - 1 = (r - 1)(r^2 + 1)$. The solution is $y = Ae^z + B\cos z + C\sin z$ so the solution to the original equation is $y = Ax + B\cos(\ln x) + C\sin(\ln x)$.

20. Substituting $w = y'$ yields $x^3w''' + 8x^2w'' + 8xw' - 8 = 0$. By Problem 19, the z equation is $\dddot{w} + 5\ddot{w} + 2\dot{w} - 8w = 0$ with associated polynomial $(r - 1)(r + 2)(r + 4)$. The solution is $w = Ae^{-4z} + Be^{-2z} + Ce^z = Ax^{-4} + Bx^{-2} + Cx$. Integrate to get $y = -\frac{1}{3}Ax^{-3} - Bx^{-1} + \frac{C}{2}x^2$.

21. The equation is

$$m_1m_2\frac{d^4x_1}{dt^4} + (m_1(k_2 + k_3) + m_2(k_1 + k_3))\frac{d^2x_1}{dt^2} + (k_1k_2 + k_1k_3 + k_2k_3)x_1 = 0.$$