

Chapter 2

Two

2.1. Solving differential equations. For the differential equation

$$\frac{dy}{dt} = 2y,$$

- (a) Show that $y(t) = ce^{2t}$, where c is any real valued constant, satisfies the differential equation by substituting into both sides of the equation. Is there any value of c that isn't a solution?
- (b) Find the one solution that corresponds to the initial condition $y(0) = 5$.

See Appendix A.1 if you need any revision of the meaning of differential equations.

Solution.

- (a) Substituting into the differential equation $y'(t) = 2y(t)$ we obtain

$$\text{LHS} = 2ce^{2t}, \quad \text{RHS} = 2 \times ce^{2t}$$

Since LHS=RHS, then $y(t) = c_1e^{2t}$ satisfies the differential equation so the set of solutions $y(t) = c_1e^{2t}$. This is true for all real values of c .

- (b) Setting $t = 0$ in $y(t) = ce^{2t}$ gives $y(0) = c$. Since $y(0) = 5$ this implies that $c = 5$. So $y(t) = 5e^{2t}$.

Answer given at back of textbook:

- (a) No. Is a solution for all c .
- (b) $y(t) = 5e^{2t}$.

2.2. Solving first-order DEs. For the following first-order differential equations, find the general solution, solving for the dependent variable. See Appendix A.3 if you need to revise how to solve first-order differential equations.

(a) $\frac{dy}{dt} = -3y$, (b) $\frac{dC}{dt} = 3C - 1$, (c) $\frac{dy}{dt} = 3yt^{-1}$.

Solution.

(a) By separating variables,

$$\frac{1}{y} dy = -3 dt, \quad \Rightarrow \quad \int \frac{1}{y} dy = \int -3 dt$$

and hence $\ln(y) = 3t + c_1$. Inverting the \ln function gives

$$y = e^{-3t+c_1} = e^{-3t} \times e^{c_1} = Ae^{-3t}.$$

where $A = e^{c_1}$.

Alternatively, by the constant coefficient method, let $y(t) = e^{mt}$ so $m + 3 = 0$ so general solution is $y(t) = Ae^{-3t}$.

(b) By separating variables

$$\int \frac{1}{3C-1} dy = \int dt$$

and integrating gives

$$\frac{1}{3} \ln(3C-1) = t + c_1$$

and rearranging gives

$$C(t) = Ae^{3t} + \frac{1}{3}.$$

Alternative method: the DE is a constant coefficient (non-homogeneous) DE

$$\frac{dC}{dt} - 3C = -1.$$

Let $C(t) = C_h + C_p$ where

$$\frac{dC_h}{dt} - 3C_h = 0, \quad \frac{dC_p}{dt} - 3C_p = -1,$$

and, substituting $C_h = e^{mt}$ into the DE for C_h gives $C_h(t) = c_1 e^{3t}$.

For a particular solution, the ‘forcing term’ is a constant function $f(t) = -1$, and not the same as the homogeneous solution, so we look for a particular solution of the form of a general constant solution, $y_p(t) = A$. Substituting this into the DE for C_p gives $0 - 3A = -1$ so $A = 1/3$ and $C_p = -1/3$. Hence $C_p(t) = -1/3$. The general solution is

$$C(t) = C_h(t) + C_p(t) = c_1 e^{3t} - \frac{1}{3}.$$

(c) Separating variables gives

$$\int \frac{1}{y} dy = 3 \int \frac{1}{t} dt$$

and integrating gives

$$\ln(y) = 3 \ln(t) + c_1$$

so

$$y = e^{3 \ln(t) + c_1} = e^{\ln(t^3 + 3c_1)} = At^3, \quad \text{where } A = e^{3c_1}.$$

2.3. Atmospheric pressure. The Earth's atmospheric pressure p is often modelled by assuming that dp/dh (the rate at which pressure p changes with altitude h above sea level) is proportional to p . Suppose that the pressure at sea level is 1,013 millibars and that the pressure at an altitude of 20 km is 50 millibars.

Answer the following questions and then check your calculations with Maple or MATLAB.

- (a) Use an exponential decay model

$$\frac{dp}{dh} = -kp$$

to describe the system, and then by solving the equation find an expression for p in terms of h . Determine k and the constant of integration from the initial conditions.

- (b) What is the atmospheric pressure at an altitude of 50 km?
(c) At what altitude is the pressure equal to 900 millibars?

Solution.

(a) Since the rate of change of pressure (p) with respect to altitude (h) is proportional to pressure, and pressure decreases with increasing altitude,

$$\frac{dp}{dh} = -kp,$$

where k is a constant. Separating the variables,

$$\frac{1}{p} \frac{dp}{dh} = -k,$$

and then integrating, with the condition that $p(0) = p_0$ (pressure at sea level),

$$\int \frac{1}{p} dp = \int -k dt \quad \Rightarrow \quad p(h) = p_0 e^{-kh} = 1013e^{-kh}.$$

Since $p(20) = 50$, solving $50 = 1013e^{-20k}$ gives $k = 0.1504$. So the solution is

$$p(h) = 1013e^{-0.1504h}. \quad (2.1)$$

- (b) Substitute $h = 50$ into the solution (2.1), which gives $p(50) \approx 0.55$ millibars.
(c) Substitute $p(h) = 900$ into the solution (2.1), which gives $h \approx 0.79$ kilometres.

Answer given at back of textbook:

- (a) $p(h) = 1013 e^{-0.1504h}$.
(b) Approximately 0.55 millibars.
(c) Approximately 0.79 kilometres.

2.4. The Rule of 72. Continuous compounding for invested money can be described by a simple exponential model, $M'(t) = 0.01rM(t)$, where $M(t)$ is the amount of money at time t and r is the percent interest compounding. Business managers commonly apply the Rule of 72, which says that the number of years it takes for a sum of money invested at $r\%$ interest to double, can be approximated by $72/r$. Show that this rule always overestimates the time required for the investment to double.

Solution.

$$t = \frac{100 \ell n(2)}{r} \approx \frac{69}{r} < 72r.$$

2.5. Dating a seashell. If an archaeologist uncovers a seashell which contains 60% of the ^{14}C of a living shell, how old do you estimate that shell, and thus that site, to be? (You may assume the half-life of ^{14}C to be 5,568 years.)

Solution.

Let $N(t)$ be the amount of ^{14}C per gram in the shell at time t . Let the current time, t_0 , occur at $t = 0$ and the time at which the shell is formed be T . For $t > T$, ^{14}C decays at

$$\frac{dN}{dt} = -kN, \quad N(0) = n_0,$$

where $k \approx 0.0001245$ per year, and so

$$N(T) = n_0 e^{-kT}.$$

This can be rearranged to give

$$T = -\frac{1}{k} \ell n\left(\frac{N(T)}{n_0}\right).$$

We know that

$$\frac{N(T)}{n_0} = \frac{N'(T)}{N'(0)} = \frac{N'(T)}{0.6N'(0)}.$$

Thus

$$T = -\frac{1}{k} \ell n\left(\frac{1}{0.6}\right) \approx -4,103 \text{ years}.$$

Approximately 4,103 years ago.

Answer given at back of textbook:

4,103 years.

2.6. Olduvai Gorge. (From Borelli and Coleman (1996).) Olduvai Gorge, in Kenya, cuts through volcanic flows, tuff (volcanic ash), and sedimentary deposits. It is the site of bones and artefacts of early hominids, considered by some to be precursors of man. In 1959, Mary and Louis Leakey uncovered a fossil hominid skull and primitive stone tools of obviously great age, older by far than any hominid remains found up to that time. Carbon-14 dating methods being inappropriate for a specimen of that age and nature, dating had to be based on the ages of the underlying and overlying volcanic strata.

The method used was that of potassium-argon decay. The potassium-argon clock is an accumulation clock, in contrast to the ^{14}C dating method. The potassium-argon method depends on measuring the accumulation of 'daughter' argon atoms, which are decay products of radioactive potassium atoms. Specifically, potassium-40 (^{40}K) decays to argon (^{40}Ar)

and to Calcium-40 (^{40}Ca) by the branching cascade illustrated below in Figure 2.1. Potassium decays to calcium by emitting a β particle (i.e. an electron). Some of the potassium atoms, however, decay to argon by capturing an extra-nuclear electron and emitting a γ particle.

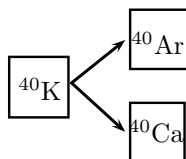


Figure 2.1: Compartment diagram for Question 2.6.

The rate equations for this decay process may be written in terms of $K(t)$, $A(t)$ and $C(t)$, the potassium, argon and calcium in the sample of rock:

$$\begin{aligned} K' &= -(k_1 + k_2)K, \\ A' &= k_1K, \\ C' &= k_2K, \end{aligned}$$

where

$$k_1 = 5.76 \times 10^{-11} \text{ year}^{-1}, \quad k_2 = 4.85 \times 10^{-10} \text{ year}^{-1}.$$

- Solve the system to find $K(t)$, $A(t)$ and $C(t)$ in terms of k_1 , k_2 , and $k_3 = k_1 + k_2$, using the initial conditions $K(0) = k_0$, $A(0) = C(0) = 0$.
- Show that $K(t) + A(t) + C(t) = k_0$ for all $t \geq 0$. Why would this be the case?
- Show that $K(t) \rightarrow 0$, $A(t) \rightarrow k_1 k_0 / k_3$ and $C(t) \rightarrow k_2 k_0 / k_3$ as $t \rightarrow \infty$.
- The age of the volcanic strata is the current value of the time variable t because the potassium-argon clock started when the volcanic material was laid down. This age is estimated by measuring the ratio of argon to potassium in a sample. Show that this ratio is

$$\frac{A}{K} = \frac{k_1}{k_3} (e^{k_3 t} - 1).$$

- Now show that the age of the sample in years is

$$\frac{1}{k_3} \ln \left[\left(\frac{k_3 A}{k_1 K} \right) + 1 \right].$$

- When the actual measurements were made at the University of California at Berkeley, the age of the volcanic material (and thus the age of the bones) was estimated to be 1.75 million years. What was the value of the measured ratio A/K ?

Solution.

(From Borelli and Coleman (1996).)

(a) The equation

$$K' = -(k_1 + k_2)K$$

is separable with solution

$$K(t) = C_1 e^{-k_3 t},$$

for an arbitrary constant C_1 . Since $K(0) = k_0$, $C_1 = k_0$ and so

$$K(t) = k_0 e^{-k_3 t},$$

Substituting for K in the differential equation for $A(t)$ gives

$$A' = k_1 k_0 e^{-k_3 t},$$

which can be solved to give

$$A(t) = -\frac{k_1 k_0}{k_3} e^{-k_3 t} + C_2,$$

where C_2 is an arbitrary constant. The initial condition $A(0) = 0$ implies that $C_2 = k_1 k_0 / k_3$ and so

$$A(t) = \frac{k_1 k_0}{k_3} (1 - e^{-k_3 t}).$$

Similarly, substitution for $K(t)$ into the differential equation for $C(t)$ leads to a linear first-order differential equation for C whose solution is

$$C(t) = -\frac{k_2 k_0}{k_3} e^{-k_3 t} + C_3,$$

for an arbitrary constant C_3 . The initial condition $A(0) = 0$ implies that $C_3 = k_2 k_0 / k_3$ and so

$$C(t) = \frac{k_2 k_0}{k_3} (1 - e^{-k_3 t}).$$

Summarising,

$$K(t) = k_0 e^{-k_3 t},$$

$$A(t) = \frac{k_1 k_0}{k_3} (1 - e^{-k_3 t}),$$

$$C(t) = \frac{k_2 k_0}{k_3} (1 - e^{-k_3 t}).$$

(b) Adding the three solutions together gives

$$\begin{aligned} K(t) + A(t) + C(t) &= k_0 e^{-k_3 t} + \frac{k_1 k_0}{k_3} (1 - e^{-k_3 t}) + \frac{k_2 k_0}{k_3} (1 - e^{-k_3 t}) \\ &= k_0, \end{aligned}$$

since $k_3 = k_1 + k_2$. So the sum of K , A and C is a constant at all times.

Alternative approach: adding the three differential equations

$$K'(t) + A'(t) + C'(t) = k_0 e^{-k_3 t} + \frac{k_1 k_0}{k_3} (1 - e^{-k_3 t}) + \frac{k_2 k_0}{k_3} (1 - e^{-k_3 t}).$$

This simplifies to

$$K'(t) + A'(t) + C'(t) = 0,$$

since $k_3 = k_1 + k_2$. Integrating with respect to t gives $K(t) + A(t) + C(t) = \text{constant}$. The sum of K , A and C represents the total number of atoms in the system. This number is always constant because the number of particles in the system must be conserved. (This result also implies that both potassium-40 and argon-40 are stable isotopes.)

(c) The limits as $t \rightarrow \infty$ are

$$\lim_{t \rightarrow \infty} K(t) = \lim_{t \rightarrow \infty} k_0 e^{-k_3 t} = 0,$$

$$\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} \frac{k_1 k_0}{k_3} (1 - e^{-k_3 t}) = \frac{k_1 k_0}{k_3},$$

and

$$\lim_{t \rightarrow \infty} C(t) = \lim_{t \rightarrow \infty} \frac{k_2 k_0}{k_3} (1 - e^{-k_3 t}) = \frac{k_2 k_0}{k_3}.$$

(d) From (a)

$$\begin{aligned} \frac{A}{K} &= \frac{\frac{k_1 k_0}{k_3} (1 - e^{-k_3 t})}{k_0 e^{-k_3 t}} \\ &= \frac{k_1}{k_3} (e^{k_3 t} - 1). \end{aligned}$$

(e) From (d)

$$e^{k_3 t} = \frac{k_3 A}{k_1 K} + 1$$

Solving for t ,

$$t = \frac{1}{k_3} \ln \left[\left(\frac{k_3 A}{k_1 K} \right) + 1 \right].$$

(f) From (a), $k_1 = 5.76 \times 10^{-11} \text{ year}^{-1}$ and $k_2 = 4.85 \times 10^{-10} \text{ year}^{-1}$. Since $k_3 = k_1 + k_2$, $k_3 = 5.426 \times 10^{-10} \text{ year}^{-1}$. If $t = 1.75 \times 10^6$ then the equation in (e) gives that

$$A/K \approx 1.008 \times 10^{-4}.$$

2.7. Storage time for radioactive chemicals.. (Adapted from Borelli and Coleman (1996).) In a biochemical laboratory radioactive phosphorus (^{32}P) was used as a tracer. (A tracer, through its radioactive emission, allows the course followed by a substance through a system to be tracked, which otherwise would not be visible.) ^{32}P decays exponentially with a half-life of 14.5 days and its quantity is measured in curies (Ci). (Although it is not necessary for the calculations, one curie is the quantity of a radioactive isotope undergoing 3.7×10^{10} disintegrations per second.) After the experiment the biochemists needed to dispose of the contents, but they had to store them until the radioactivity had decreased to the acceptably safe level of $1 \times 10^{-5}\text{Ci}$. The experiment required 8Ci of ^{32}P . Using a simple model of exponential decay, establish how long they had to store the contents of the experiment before it could be disposed of safely.

Solution.

Let $N(t)$ be the amount of ^{32}P present, in curies, and t be the time after the experiment. We assume the exponential decay model $N' = -kN$ with $N(0) = n_0$. The solution to this differential equation is $N(t) = n_0 e^{-kt}$. The half-life of ^{32}P is $\tau = 14.5$ days and so, from (9), $k = \frac{\ln 2}{\tau} \approx 0.0478$.

From the equation for N we then find that the acceptable level of 1×10^{-5} curies is reached after $t = (1/k) \ln(N(t)/n_0) = 1/0.0478 \ln(8/10^{-5})$ which is approximately 284 days. Let $N(t)$ be the amount of ^{32}P present, in curies, and t be the time after the experiment. We assume the exponential decay model $N' = -kN$ with $N(0) = n_0$. The solution to this differential equation is $N(t) = n_0 e^{-kt}$. The half-life of ^{32}P is $\tau = 14.5$ days and so, from (9), $k = \frac{\ln 2}{\tau} \approx 0.0478$.

From the equation for N we then find that the acceptable level of 1×10^{-5} curies is reached after approximately 284 days.

Answer given at back of textbook:

Approximately 284 days.

2.8. Lake Burley Griffin. Read the case study on Lake Burley Griffin. The average summer flow rate for the water into and out of the lake is $4 \times 10^6 \text{ m}^3/\text{month}$.

- (a) Using this summer flow, how long will it take to reduce the pollution level to 5% of its current level? How long would it take for the lake with pollution concentration of 10^7 parts/ m^3 , to drop below the safety threshold? (Assume in both cases that only fresh water enters the lake.)
- (b) Use Maple or MATLAB to replicate the results in the case study, for both constant and seasonal flow and constant and seasonal pollution concentrations entering the lake. Comment on the solutions.

Solution.

(a) The concentration is reduced to 5% of the original concentration after approximately 21 months. The safety threshold of 4×10^6 parts/m³ is reached after approximately 6 months.

The equation describing pollution in the lake is

$$c(t) = c_{in} - (c_{in} - c_0)e^{-Ft/V},$$

where $c(t)$ is the concentration of pollutant in the lake in parts/m³, c_{in} is the concentration of pollutant entering the lake and c_0 is the initial concentration of pollutant in the lake. The flow rate is $F = 4 \times 10^6$ m³/month and the volume of the lake is $V = 28 \times 10^6$ m³. Since only fresh water enters the lake, $c_{in} = 0$.

(b) See Figure 2.2.

2.9. Pollution with chemical activity. Consider the concentration, $C(t)$, of some pollutant chemical in a lake. Suppose that polluted water with concentration c_i flows into the lake with a flow rate of F and the well-stirred mixture leaves the lake at the same rate F .

In addition, suppose some chemical agent is present in the lake that breaks down the pollution at a rate r kg/day per kg of pollutant. Assuming that the volume of mixture in the lake remains constant and the chemical agent is not used up, formulate (but do not solve) a mathematical model as a single differential equation for the pollution concentration $C(t)$.

Solution.

We have conservation of the mass of the pollutant, which is $VC(t)$ where V is the constant volume of the lake. There is one input, due to the rate of mass of pollution flowing into the lake and two outputs, due to the rate of mass of pollutant flowing out of the lake and the rate of pollutant which is broken down by the chemical agent. This term is modelled as r multiplied by the mass of the pollutant, rVC .

Thus, conservation of mass leads to

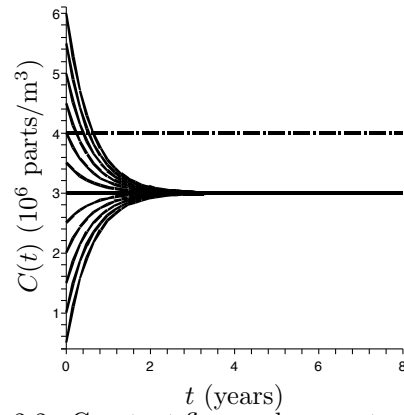
$$\frac{d(VC)}{dt} = Fc_i - FC(t) - rVC.$$

Since V is constant, $d(VC)/dt = V(dC/dt)$, and dividing through by V gives

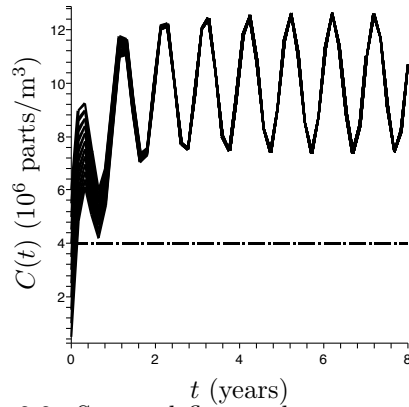
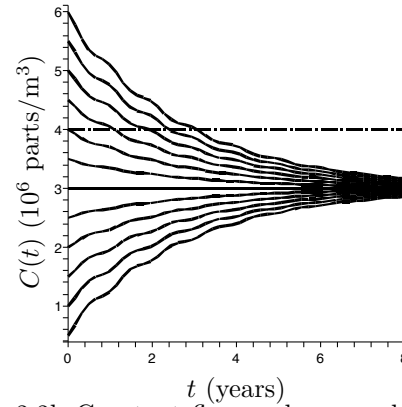
$$\frac{dC}{dt} = \frac{F}{V}c_i - \frac{F}{V}C - rC.$$

Answer given at back of textbook:

$$\frac{dC}{dt} = \frac{F}{V}c_i - \frac{F}{V}C - rC.$$



2.2a Constant flow and concentration. 2.2b Constant flow and seasonal concentration.



2.2c Seasonal flow and constant concentration. 2.2d Seasonal flow and concentration.

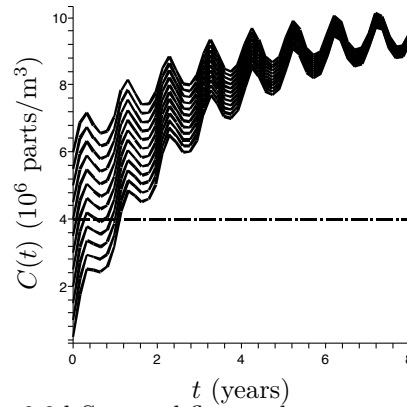


Figure 2.2: The effect of incorporating a seasonal inflowing pollutant concentration and flow rate on pollution levels in Lake Burley Griffin (Question 2.8). The constant concentration is $c_{in} = \times 10^6$ parts/ m^3 and the variable concentration is $c_{in} = 10^6(10 + 10 \cos(2\pi t))$ parts/ m^3 . The constant flow rate is $F = 4 \times 10^6$ m^3 /month and the variable flow rate is $F = 10^6(1 + 6 \sin(2\pi t))$ m^3 /year. In all plots $V = 28$ m^3 . The grey line is the pollution threshold level.

2.10. North American lake system. Consider the American system of two lakes: Lake Erie feeding into Lake Ontario. What is of interest is how the pollution concentrations change in the lakes over time. You may assume the volume in each lake to remain constant and that Lake Erie is the only source of pollution for Lake Ontario.

- Write a differential equation describing the concentration of pollution in each of the two lakes, using the variables V for volume, F for flow, $c(t)$ for concentration at time t and subscripts 1 for Lake Erie and 2 for Lake Ontario.
- Suppose that only unpolluted water flows into Lake Erie. How does this change the model proposed?
- Solve the system of equations to get expressions for the pollution concentrations $c_1(t)$ and $c_2(t)$.
- Set $T_1 = V_1/F_1$ and $T_2 = V_2/F_2$, and then $T_1 = kT_2$ for some constant k as V and F are constants in the model. Substitute this into the equation describing pollution levels in Lake Ontario to eliminate T_1 . Then show that, with the initial conditions $c_{1,0}$ and $c_{2,0}$, the solution to the differential equation for Lake Ontario is

$$c_2(t) = \frac{k}{k-1}c_{1,0} \left(e^{-t/(kT_2)} - e^{-t/T_2} \right) + c_{2,0}e^{t/T_2}.$$

(One way of finding the solution would be to use an integrating factor. See Appendix A.4.)

- Compare the effects of $c_1(0)$ and $c_2(0)$ on the solution $c_2(t)$ over time.

Solution.

(a)

$$\frac{dc_1}{dt} = \frac{F_1}{V_1}c_{1,in} - \frac{F_1}{V_1}c_1, \quad \frac{dc_2}{dt} = \frac{F_2}{V_2}c_1 - \frac{F_2}{V_2}c_2.$$

(b)

$$\frac{dc_1}{dt} = -\frac{F_1}{V_1}c_1, \quad \frac{dc_2}{dt} = \frac{F_2}{V_2}c_1 - \frac{F_2}{V_2}c_2.$$

(c) $c_1(t) = \alpha e^{-\frac{F_1 t}{V_1}}, \quad c_2(t) = \frac{F_2 \alpha}{V_2(\frac{F_2}{V_2} - \frac{F_1}{V_1})} e^{-\frac{F_1 t}{V_1}} + \beta e^{-\frac{F_2 t}{V_2}},$ where α and β are

arbitrary constants.

- As the initial concentration of pollution in Lake Erie increases, the peak concentration of pollution in Lake Ontario increases as well. It takes longer to reach this peak level and for the pollution concentration to decrease in Lake Ontario. Increasing the initial concentration of pollution in Lake Ontario increases the rate at which the pollution is initially removed.

2.11. Smoke in the bar. (Adapted from Fulford et al. (1997).) A public bar opens at 6 p.m. and is rapidly filled with clients of whom the majority are smokers. The bar is equipped with ventilators that exchange the smoke-air mixture with fresh air.

Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. The bar has a floor area of 20 m by 15 m, and a height of 4 m. It is estimated that smoke enters the room at a constant rate of $0.006 \text{ m}^3/\text{min}$, and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced.

The problem is to establish a good time to leave the bar, that is, sometime before the concentration of carbon monoxide reaches the lethal limit.

- Starting from a word equation or a compartmental diagram, formulate the differential equation for the changing concentration of carbon monoxide in the bar over time.
- By solving the equation above, establish at what time the lethal limit will be reached.

Solution.

- The compartment model is shown in Figure 2.3.

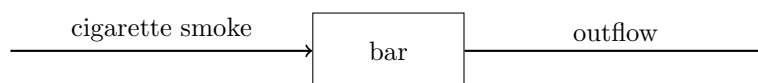


Figure 2.3: Compartment diagram for model of smoke concentration in a bar, Question 2.11.

The corresponding word equation is

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of amount of} \\ \text{carbon monoxide} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate carbon} \\ \text{monoxide produced} \\ \text{by smoke} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate carbon} \\ \text{monoxide removed} \\ \text{by ventilators} \end{array} \right\}.$$

The appropriate differential equation is:

$$C'(t) = \frac{F_{\text{in}}}{V} c_{\text{in}} - \frac{F_{\text{out}}}{V} C(t), \quad C(0) = 0$$

and its solution is

$$C(t) = \frac{F_{\text{in}} c_{\text{in}}}{F_{\text{out}}} (1 - e^{-(F_{\text{out}}/V)t}).$$

- Approximately 10 hours.

Answer given at back of textbook:

- Approximately 10 hours.

2.12. Detecting art forgeries. Based on methods used in the case study describing the detection of art forgeries (Section 2.3), comment on whether each of the paintings below is a possible forgery, based on the time it was painted:

- ‘Washing of Feet’, where the disintegration rate for ^{210}Po is 8.2 per minute per gram of white lead, and for ^{226}Ra is 0.26 per minute per gram of white lead.
- ‘Laughing Girl’, where the disintegration rate for ^{210}Po is 5.2 per minute per gram of white lead and for ^{226}Ra is 4 per minute per gram of white lead.

Solution.

(a) The original disintegration rate of Lead-210 is

$$\lambda n_0 = \lambda N e^{\lambda(t-t_0)} - R(e^{\lambda(t-t_0)} - 1).$$

Assuming ‘Washing of Feet’ is approximately 300 years old,

$$\lambda n_0 = 8.2 \times 2^{150/11} - 0.26(2^{150/11} - 1) > 101\,000.$$

This original rate of disintegration is too high. The painting is modern.

(b) With $\lambda N = 5.2$ and $R = 4$ and assuming the painting is 300 years old the equation gives

$$\lambda n_0 = 5.2 \times 2^{150/11} - 4(2^{150/11} - 1) \approx 15284.$$

The painting could be authentic.

2.13. Cold pills. *In Section 2.7, we developed the model*

$$\begin{aligned} \frac{dx}{dt} &= -k_1 x, & x(0) &= x_0, \\ \frac{dy}{dt} &= k_1 x - k_2 y, & y(0) &= 0, \end{aligned}$$

where $k_1, k_2 > 0$ determine the rate at which a drug, antihistamine or decongestant moves between two compartments in the body, the GI-tract and the bloodstream, when a patient takes a single pill. Here $x(t)$ is the level of the drug in the GI-tract and $y(t)$ is the level in the bloodstream at time t .

- (a) Find solution expressions for $x(t)$ and $y(t)$ that satisfy this pair of differential equations, when $k_1 \neq k_2$. Show that this solution is equivalent to that provided in the text.
- (b) The solution above is invalid at $k_1 = k_2$. Why is this, and what is the solution in this case?
- (c) For old and sick people, the clearance coefficient (that is, the rate at which the drug is removed from the bloodstream) is often much lower than that for young, healthy individuals. How does an increase or decrease in k_2 change the results of the model? Using **Maple** or **MATLAB** to generate the time-dependent plots, check your results.

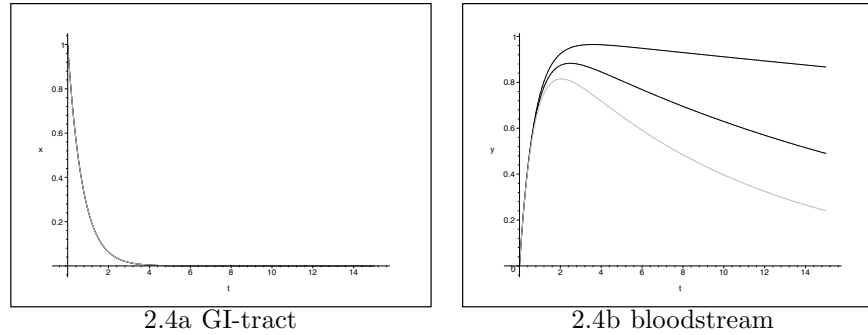


Figure 2.4: Amount of a single cold pill present in (a) the GI-tract and (b) the bloodstream with clearance coefficients of $k_2 = 0.1$ (grey line), $k_2 = 0.05$ (thick black line) and $k_2 = 0.01$ (thin black line). In all cases, $k_1 = 1.3860$, $x_0 = 1$ and $y_0 = 0$.

Solution.

(b) Cannot divide by $k_1 - k_2 = 0$.

We can replace k_2 by k_1 in the differential equation for y above to get

$$\frac{dy}{dt} = k_1 x_0 e^{-k_1 t} - k_1 y, \quad y(0) = 0.$$

The solution of this equation is $y(t) = k_1 x_0 t e^{-k_1 t}$.

(c) Lowering the clearance coefficient does not change the GI-tract drug concentration since the differential equation describing the amount in the tract is independent of k_2 . It does, however, increase the time taken for the drug to be removed from the blood stream. Accordingly, the peak concentration in the blood stream is higher. This is shown in Figure 2.4.

Answer given at back of textbook:

(b) Dividing by zero if $k_1 = k_2$.

$$y(t) = k_1 x_0 t e^{-k_1 t}.$$

(c) Hint: DE for the GI tract doesn't contain k_2 .

2.14. Cold pills. In Section 2.7, we also developed a model to describe the levels of antihistamine and decongestant in a patient taking a course of cold pills:

$$\begin{aligned} \frac{dx}{dt} &= I - k_1 x, & x(0) &= 0, \\ \frac{dy}{dt} &= k_1 x - k_2 y, & y(0) &= 0. \end{aligned}$$

Here k_1 and k_2 describe rates at which the drugs move between the two sequential compartments (the GI-tract and the bloodstream) and I denotes the amount of drug released into the GI-tract in each time step. The levels of the drug in the GI-tract and bloodstream are x and y , respectively. By solving the equations sequentially show that the solution is

$$x(t) = \frac{I}{k_1} (1 - e^{-k_1 t}), \quad y(t) = \frac{I}{k_2} \left[1 - \frac{1}{k_2 - k_1} (k_2 e^{-k_1 t} - k_1 e^{-k_2 t}) \right].$$

Hint: Substitute the solution for x into the differential equation for y . The differential equation for $x(t)$,

$$\frac{dx}{dt} = I - k_1x,$$

can be solved using an integrating factor to give

$$x(t) = \frac{I}{k_1} + C_1e^{-k_1t},$$

for some arbitrary constant C_1 . Applying the initial condition we find that the equation for $x(t)$ is

$$x(t) = \frac{I}{k_1}(1 - e^{-k_1t}).$$

Substituting for $x(t)$ in the differential equation for $y(t)$ gives

$$\frac{dy}{dt} = I(1 - e^{-k_1t}) - k_2y.$$

We again use the integrating factor technique and find that

$$y(t) = \frac{I}{k_2} - \frac{I}{k_1 - k_2}e^{-k_1t} + C_2e^{-k_2t},$$

where C_2 is an arbitrary constant. With $y(0) = 0$ we find that

$$C_2 = \frac{I}{k_2 - k_1} - \frac{I}{k_2} = \frac{Ik_1}{k_2(k_2 - k_1)}$$

and so

$$y(t) = \frac{I}{k_2} \left[1 - \frac{1}{k_2 - k_1}(k_2e^{-k_1t} - k_1e^{-k_2t}) \right].$$

2.15. Antibiotics. (Adapted from Borelli and Coleman (1996).) *Tetracycline* is an antibiotic prescribed for a range of problems, from acne to acute infections. A course is taken orally and the drug moves from the GI-tract through the bloodstream, from which it is removed by the kidneys and excreted in the urine.

- Write word equations to describe the movement of a drug through the body, using three compartments: the GI-tract, the bloodstream and the urinary tract. Note that the urinary tract can be considered as an absorbing compartment, that is, the drug enters but is not removed from the urinary tract.
- From the word equations develop the differential equation system that describes this process, defining all variables and parameters as required.
- The constants of proportionality associated with the rates at which *tetracycline* (measured in milligrams) diffuses from the GI-tract into the bloodstream, and then is removed, are 0.72 hour^{-1} and 0.15 hour^{-1} , respectively (Borelli and Coleman (1996)). Suppose, initially, the amount of *tetracycline* in the GI-tract is 0.0001 milligrams, while there is none in the bloodstream or urinary tract.

Use **Maple** or **MATLAB**(with symbolic toolbox) to solve this system analytically, and thus establish how the levels of tetracycline change with time in each of the compartments. In the case of a single dose, establish the maximum level reached by the drug in the bloodstream and how long it takes to reach this level with the initial conditions as given above.

- (d) Suppose that, initially, the body is free from the drug and then the patient takes a course of antibiotics: 1 unit per hour. Use **Maple** or **MATLAB** to examine the level of tetracycline (expressed as units) in each of the compartments over a 24-hour period. Use the constants as given above.

Solution.

(a)

$$\begin{aligned}\left\{ \begin{array}{l} \text{rate of change} \\ \text{of drug in} \\ \text{GI tract} \end{array} \right\} &= - \left\{ \begin{array}{l} \text{rate drug} \\ \text{leaves GI tract} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{rate of change} \\ \text{of drug} \\ \text{in blood} \end{array} \right\} &= \left\{ \begin{array}{l} \text{rate drug} \\ \text{enters blood} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate drug} \\ \text{leaves blood} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{rate of change} \\ \text{of drug in} \\ \text{urinary tract} \end{array} \right\} &= \left\{ \begin{array}{l} \text{rate drug enters} \\ \text{urinary tract} \end{array} \right\}\end{aligned}$$

(b) Let $x(t)$ be the amount of the drug in the GI-tract, $y(t)$ be the amount of drug in the blood stream and $z(t)$ be the amount of drug in the urinary tract at time t . Then

$$\begin{aligned}x(t) &= -k_1x, & x(0) &= x_0, \\ y(t) &= k_1x - k_2y, & y(0) &= 0, \\ z(t) &= k_2y, & z(0) &= 0,\end{aligned}$$

where $k_1, k_2 > 0$ represent the rates at which the drug is removed from or enters different parts of the system.

(c)

$$\begin{aligned}x(t) &= 0.0001e^{-0.72t}, \\ y(t) &= 0.000126(e^{-.15t} - e^{-.72t}), \\ z(t) &= 0.000026e^{-.72t} - 0.000126e^{-.15t} + 0.0001.\end{aligned}$$

The maximum level in the bloodstream is approximately 6.6×10^{-5} milligrams and that this maximum is reached after about 2.8 hours.

(d)

$$\begin{aligned}x(t) &= I - k_1x, & x(0) &= 0, \\ y(t) &= k_1x - k_2y, & y(0) &= 0, \\ z(t) &= k_2y, & z(0) &= 0,\end{aligned}$$

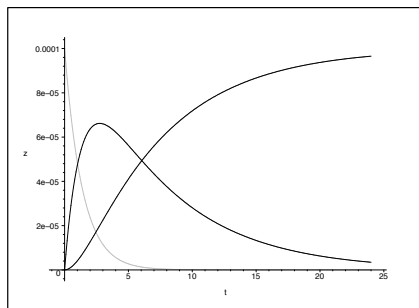


Figure 2.5: Amount of a single dose of tetracycline present in the GI-tract (grey line), the bloodstream (thick black line) and the urinary tract (thin black line) with $k_1 = 0.72$ and $k_2 = 0.15$, $x_0 = 0.0001$, $y_0 = 0$ and $z_0 = 0$.

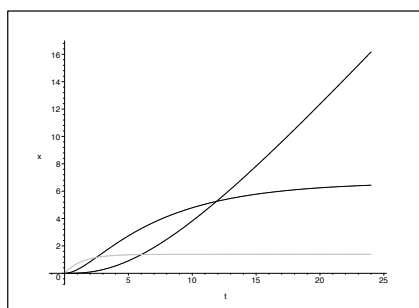


Figure 2.6: Amount of a course of tetracycline present in the GI-tract (grey line), the bloodstream (thick black line) and the urinary tract (thin black line) with $I = 1$, $k_1 = 0.72$ and $k_2 = 0.15$, $x_0 = 0$, $y_0 = 0$ and $z_0 = 0$.

where I represents the rate of ingestion of the drug.

The amount of tetracycline in the GI-tract levels out at just over 1 milligram after approximately 5 hours. The amount in the bloodstream rises in the first 24 hours, although this too appears to tend towards a constant value after 24 hours. The urinary tract is considered to be an absorbing compartment and so it is not surprising that the amount of tetracycline continues to rise.

Answer given at back of textbook:

$$\begin{aligned} (d) \quad & x(t) = I - k_1 x, \quad x(0) = 0, \\ & y(t) = k_1 x - k_2 y, \quad y(0) = 0, \\ & z(t) = k_2 y, \quad z(0) = 0. \end{aligned}$$

2.16. Alcohol consumption. Use the model from the case study on alcohol consumption (Dull, dizzy or dead, Section 2.8), to establish, for the case of drinking on an empty stomach, the following:

- Use Maple or MATLAB to generate graphs to investigate the effects of alcohol on a woman of 55 kg, over a period of time.
- Compare these results with those for a man of the same weight.
- Assuming the legal limit to be 0.05 BAL (the Australian limit), establish roughly how much alcohol the man and woman above can consume each hour and remain below this limit.

(d) Repeat (a)–(c) for the case of drinking together with a meal.

Solution.

(a) The BAL of a 55kg woman over time in both the bloodstream and the GI-tract are examined in Figure 2.7

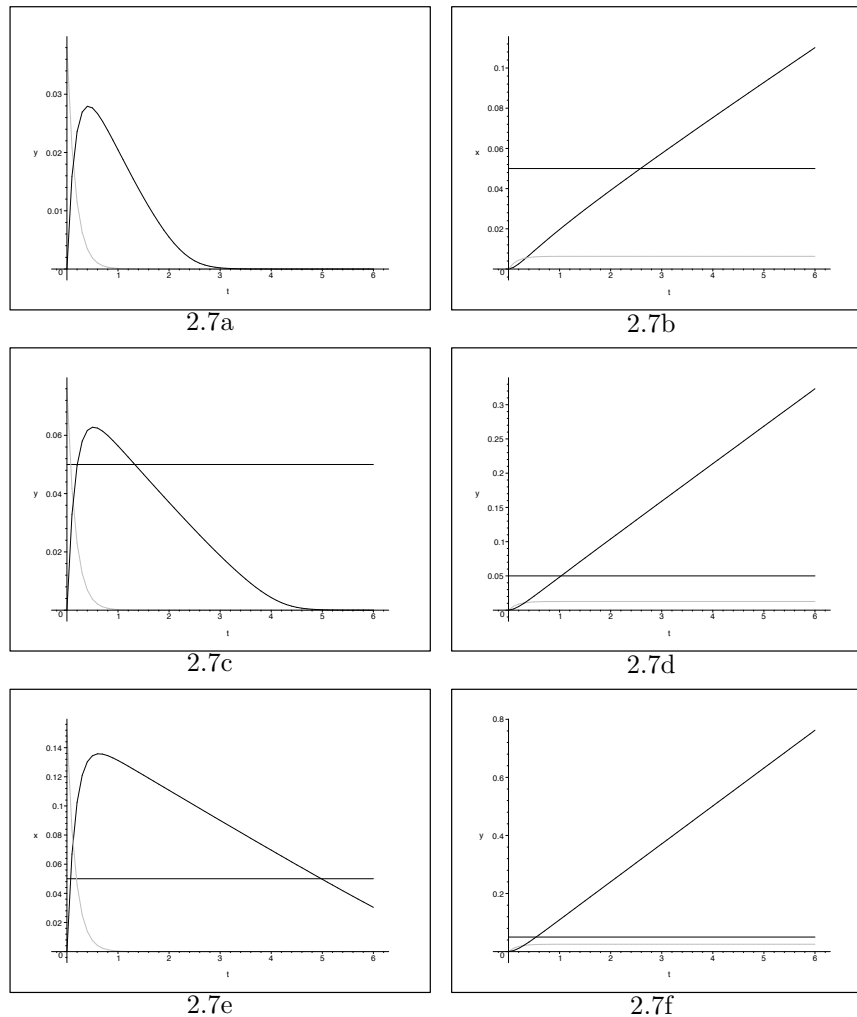


Figure 2.7: The BAL of a 55kg woman drinking on an empty stomach. In (a) and (b) 1 drink per hour are consumed, in (c) and (d) 2 drinks per hour are consumed and in (e) and (f) 4 drinks per hour are consumed. The left-hand diagrams are for a single drinking bout and the right-hand graphs for a continuous binge. The black line represents the BAL in the bloodstream and the grey line the level in the GI-tract. The legal limit BAL of 0.05 is also indicated.

(b) The BAL of a man is slightly lower than that of a woman, regardless of the number of standard drinks consumed, but tends to rise and fall in a qualitatively similar manner.

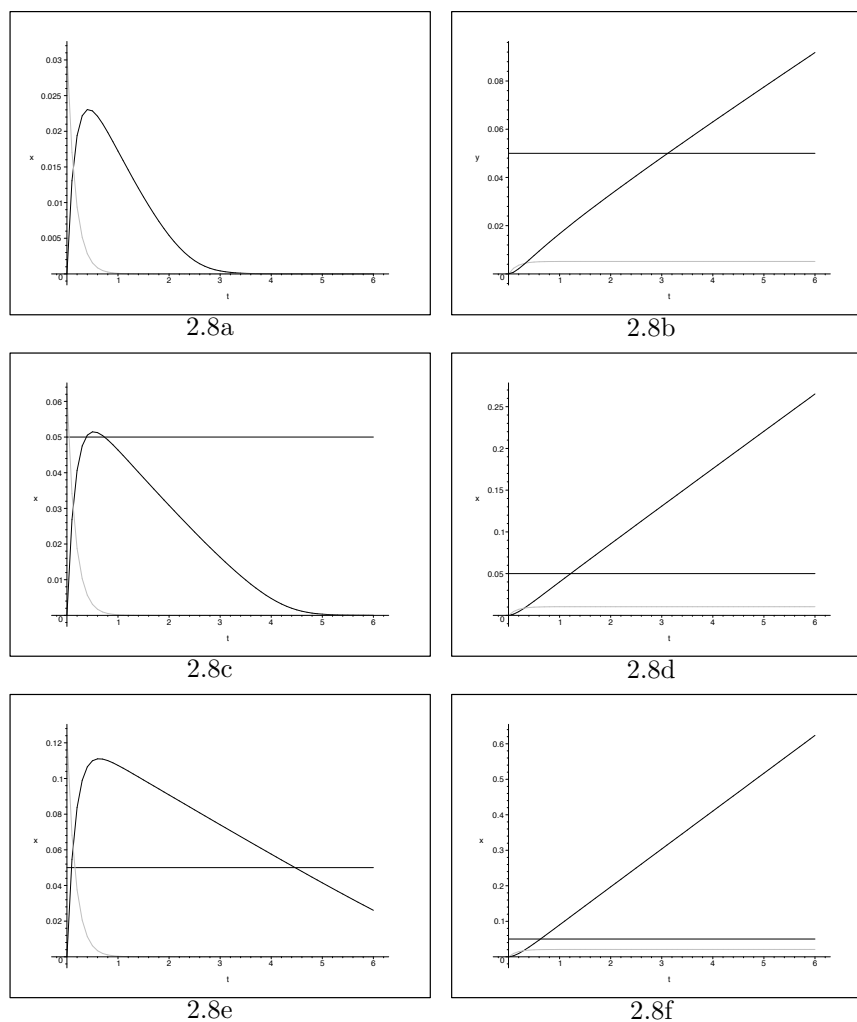


Figure 2.8: The BAL of a 55kg man drinking on an empty stomach. In (a) and (b) 1 drink per hour are consumed, in (c) and (d) 2 drinks per hour are consumed and in (e) and (f) 4 drinks per hour are consumed. The left-hand diagrams are for a single drinking bout and the right-hand graphs for a continuous binge. The black line represents the BAL in the bloodstream and the grey line the level in the GI-tract. The legal limit BAL of 0.05 is also indicated.

(c) To remain strictly under the legal limit men and woman can only consume one drink at the start of the first hour. If the man consumes two drinks his BAL just exceeds the legal limit of 0.05 in the second half of the first hour but then falls below this threshold.

(d) The BAL of a woman is again higher than that of a man of the same weight and who consumes the same amount of alcohol. The BAL of both women and men is lower if alcohol is consumed after a substantial meal than on an empty stomach, as expected. After a meal, women and men can consume three standard drinks at once and remain under the limit.

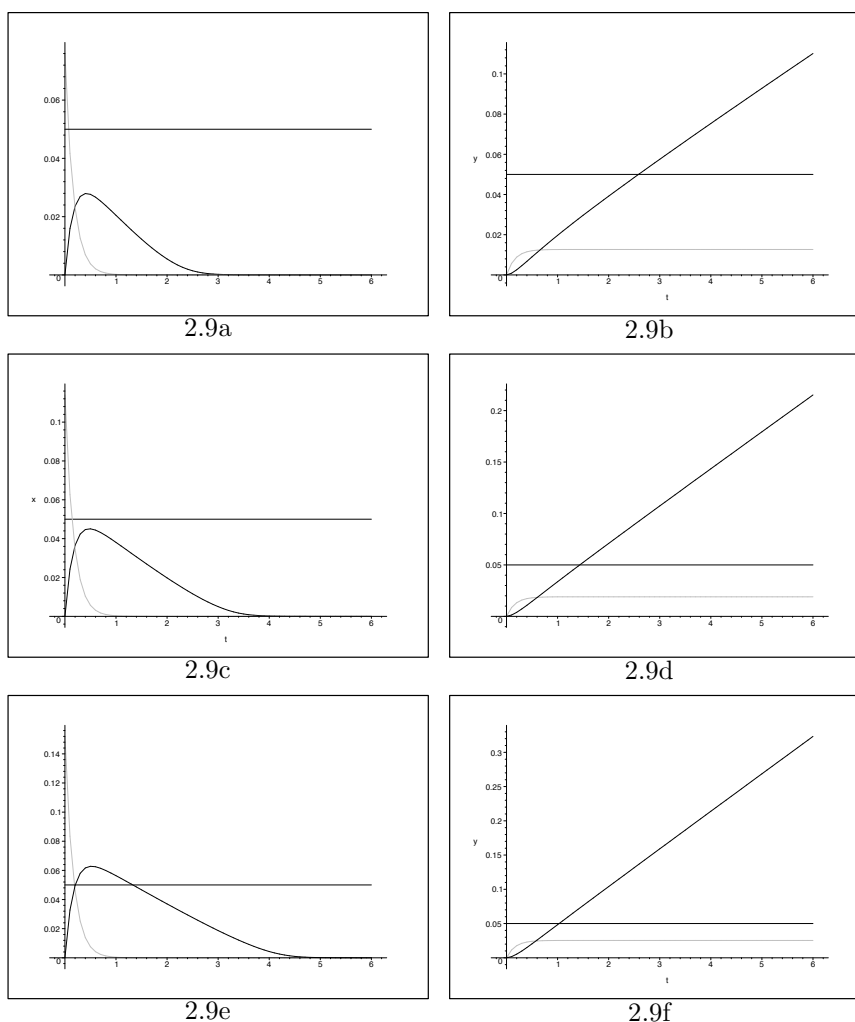


Figure 2.9: The BAL of a 55kg woman drinking after a substantial meal. In (a) and (b) 2 drinks per hour are consumed, in (c) and (d) 3 drinks per hour are consumed and in (e) and (f) 4 drinks per hour are consumed. The left-hand diagrams are for a single drinking bout and the right-hand graphs for a continuous binge. The black line represents the BAL in the bloodstream and the grey line the level in the GI-tract. The legal limit BAL of 0.05 is also indicated.

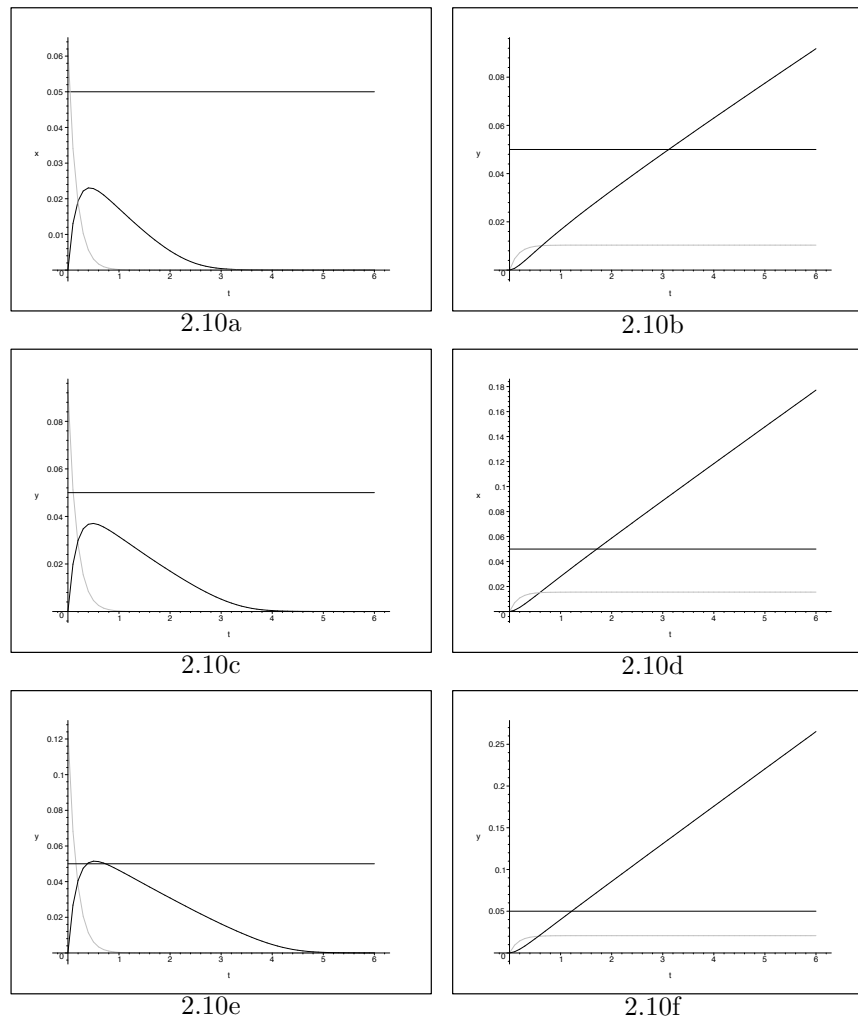


Figure 2.10: The BAL of a 55kg man drinking after a substantial meal. In (a) and (b) 2 drinks per hour are consumed, in (c) and (d) 3 drinks per hour are consumed and in (e) and (f) 4 drinks per hour are consumed. The left-hand diagrams are for a single drinking bout and the right-hand graphs for a continuous binge. The black line represents the BAL in the bloodstream and the grey line the level in the GI-tract. The legal limit BAL of 0.05 is also indicated.

2.17. Alcohol consumption. Alcohol is unusual in that it is removed (that is, metabolised through the liver) from the bloodstream by a constant amount each time period, independent of the amount in the bloodstream. This removal can be modelled by a Michaelis–Menten type function $y' = -k_3y/(y + M)$, where $y(t)$ is the ‘amount’ (BAL) of alcohol in the bloodstream at time t , k_3 is a positive constant and M a small positive constant.

- If y is large compared with M , then show that $y' \simeq -k_3$. Solve for y in this case.
- Alternatively, as y decreases and becomes small compared with M , show that then $y' \simeq -k_3y/M$. Solve for y in this case.
- Now sketch the solution function for $y' = -k_3y/(y + M)$ assuming that, initially, y is much greater than M . Indicate clearly how the graph changes in character when y is small compared with M , compared with when y is large compared with M . Show how the solution behaves as $t \rightarrow \infty$.
- When and why would this function be more suitable than simply using $y' = -k_3$ to model the removal rate?

Solution.

- $y(t) = -k_3t + y_0$, $y(0) = y_0$.
- $y(t) = y_0e^{-(k_3/M)t}$, $y(0) = y_0$.
- See Figure 2.11

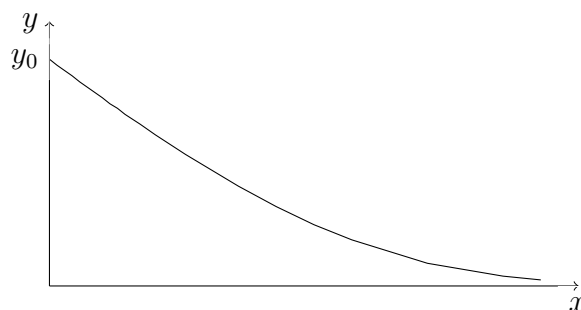


Figure 2.11: Sketch of the solution of the differential equation in Question 2.17.

- If the observed rate of removal frequently reaches a saturation value.

Answer given at back of textbook:

- $y(t) = -k_3t + y_0$, $y(0) = y_0$.
- $y(t) = y_0e^{-(k_3/M)t}$, $y(0) = y_0$.

2.18. Solving differential equations. Consider the differential equations

$$t \frac{dx}{dt} = x, \quad x(t_0) = x_0,$$

and

$$y^2 \frac{dx}{dy} + xy = 2y^2 + 1, \quad x(y_0) = x_0.$$

Put each equation into normal form and then use the integrating factor technique to find the solutions. Establish whether these solutions are unique, and which part of each solution is a response to the initial data and which part a response to the input or forcing.

Solution.

The normal form of

$$t \frac{dx}{dt} = x$$

is

$$\frac{dx}{dt} - \frac{x}{t} = 0$$

and its solution is

$$x(t) = \frac{x_0}{t_0} t.$$

A unique solution to the IVP exists on an interval containing t_0 provided $t_0 \neq 0$. The response to the initial data is $\frac{x_0}{t_0} t$. There is no response to the input.

The normal form of

$$y^2 \frac{dx}{dy} + xy = 2y^2 + 1$$

is

$$\frac{dx}{dy} + \frac{x}{y} = 2 + \frac{1}{y^2}.$$

Its solution is

$$x(y) = y + \frac{1}{y} (\ln y + x_0 y_0 - y_0^2 - \ln y_0).$$

A unique solution to the IVP exists on an interval containing y_0 provided $y_0 \neq 0$. The response to the initial data is $\frac{y_0 x_0}{y}$ and the response to the input is $y + \frac{1}{y} (\ln y - y_0^2 - \ln y_0)$.

2.19. Formulating DEs for alcohol case study. Read over the case study in Section 2.8. Consider two compartments, one for the GI-tract and one for the blood. Let $C_1(t)$ be the concentration of alcohol in the GI tract and $C_2(t)$ be the concentration in the blood, with both concentrations measured in BAL (g per 100 ml). Also let F_1 be the flow rate of fluid from the GI-tract and let F_2 be the flow rate of fluid from the blood to the tissues. Finally, we let i_0 be the rate of ingestion of alcohol (in g/hr). Use conservation of mass of alcohol to deduce the equations in the form

$$\begin{aligned} \frac{dC_1}{dt} &= I - k_1 C_1, \\ \frac{dC_2}{dt} &= k_2 C_1 - k_4 C_2 \end{aligned}$$

and determine I , k_1 , k_2 and k_3 all in terms of i_0 , F_1 , F_2 and V_g , the volume of the fluid in the GI-tract, V_b the volume of fluid in the blood, and α , where α is the proportion of the alcohol leaving the GI-tract goes into the bloodstream.

Note: In the case study we let the rate constant k_4 depend on the blood alcohol concentration

$$k_4 = \frac{k_3}{M + C_2},$$

where k_3 and M are positive constants, k_3 with the same units as k_1 and k_2 , namely hours^{-1} and M with the same units as C_2 , namely BAL.

Solution.

Conservation of mass requires

$$\begin{aligned}\frac{d}{dt}(V_b C_1) &= I - F_1 \times \frac{C_1 V_b}{V_g}, \\ \frac{d(V_b C_2)}{dt} &= \alpha F_1 \times \frac{C_1 V_b}{V_g} - F_2 \times C_2,\end{aligned}$$

So

$$I = \frac{i_0}{V_b}, \quad k_1 = \frac{F_1}{V_g}, \quad k_2 = \frac{\alpha F_1}{V_g}, \quad k_4 = \frac{F_2}{V_b}.$$

Since C_1 is the concentration in the GI-tract measured in terms of the volume in the blood, the mass of alcohol is therefore $V_b C_1$ and since F_1 measures flow rate as volume of fluid in the GI-tract we need to multiply by the fraction $(C_1 V_b)/V_g$ as the appropriate concentration in the first equation.

Answer given at back of textbook:

$$\begin{aligned}\frac{d}{dt}(V_b C_1) &= I - F_1 \times \frac{C_1 V_b}{V_g}, \\ \frac{d}{dt}(V_b C_2) &= \alpha F_1 \times \frac{C_1 V_b}{V_g} - F_2 \times C_2.\end{aligned}$$

2.20. Economic growth. Read over the case study on a model of economic growth in Section 2.12. In this model the Cobb–Douglas function was used to model production. An alternative model is the Harrod–Domar model of fixed proportions, $Y = \min\{K/a, L/b\}$ is the minimum of the two values, with a units of capital and b units of labour required to produce a unit of output. The expression for Y describes the ‘bottlenecks’ for the system, that is, whether it is limitations in capital or labour that determine the outcome for production.

(a) For the case $r/a < 1/b$, show that

$$\frac{dr}{dt} = \left(\frac{s}{a} - n\right) r$$

and solve this to obtain

$$r(t) = r_0 e^{(s/a - n)t}.$$

where $r(0) = r_0$.

(b) Consider the case when $n > s/a$ and $r_0 > a/b$. Provide an interpretation of what this scenario means in terms of capital and the demand for labour.

Solution.

(a) Note that, from the definition of parameters a and b , $Y \equiv F\left(\frac{K}{L_0 e^{nt}}, 1\right)$. From the case study, the ratio of capital to labour, $r(t)$ satisfies

$$\left(\frac{dr}{dt} + nr\right) L_0 e^{nt} = sF(K, L_0 e^{nt})$$

so

$$\frac{dr}{dt} = s \min\left\{\frac{r}{a}, \frac{1}{b}\right\} - nr,$$

For $r/a < 1/b$ we have $r < a/b$ and the relevant equation is

$$\frac{dr}{dt} = \left(\frac{s}{a} - n\right) r.$$

The solution, satisfying an initial condition $r(t) = r_0$, is

$$r(t) = r_0 e^{(s/a - n)t}.$$

(b) For $s/a - n < 0$ then r is always decreasing. With $r_0 > a/b$, then r decreases towards $s/nb < a/b$. When $r(t)$ reaches a/b it decreases further.

To interpret this scenario, note that at $r = a/b$ labour supply and capital stock are in balance (from the definition of a and b). Thereafter, the capital-labour ratio decreases with labour becoming redundant, and this redundancy growing. Thus the conditions above predict a *possible* scenario for growing unemployment.

2.21. Return to scale property. Show that the Cobb–Douglas function, from Section 2.12,

$$Y = F(K, L) = K^a L^{1-a}$$

has the return to scale property.

Solution.

From the definition

$$\begin{aligned} F(aK, aL) &= (aL)^\alpha (aK)^{1-\alpha} \\ &= aK^\alpha L^{1-\alpha} \\ &= aF(K, L). \end{aligned}$$

2.22. Stability of equilibrium solution. Consider Figure 2.16(b), in Section 2.12. Establish the stability of each of the equilibrium points, $r_e^{(1)}$, $r_e^{(2)}$ and $r_e^{(3)}$, from the underlying equation.

Table 2.1: Stability of the steady states of $r(t)$ for Question 2.22.

r	Value of $sf(r)$	Sign of dr/dt
$r < r_e^{(1)}$	$sf(r) > nr$	+
$r_e^{(1)} < r < r_e^{(2)}$	$sf(r) < nr$	-
$r_e^{(2)} < r < r_e^{(3)}$	$sf(r) > nr$	+
$r > r_e^{(3)}$	$sf(r) < nr$	-

Solution.

The stability of the steady states can be determined by examining the sign of dr/dt for different values of r , as shown in Table 2.1. If $r < r_e^{(1)}$, dr/dt is positive and r increases towards $r_e^{(1)}$. On the other hand, if $r_e^{(1)} < r < r_e^{(2)}$, r decreases towards $r_e^{(1)}$. Thus $r_e^{(1)}$ is a stable steady state.

By similar arguments, we see that $r_e^{(2)}$ is unstable and $r_e^{(3)}$ is stable.

2.23. Equilibria. Each of the following differential equations has only one equilibrium solution. Find that equilibrium solution and determine if it is stable or unstable?

(a) $\frac{dy}{dt} = y - 1$.

(b) $\frac{dC}{dt} = \frac{F}{V}c_i - \frac{F}{V}C$, where F, V, c_i are positive constants.

Solution.

(a) Set $dy/dt = 0$ so $y - 1 = 0$ hence $y = 1$. So the equilibrium solution is $y_e = 1$. Now $F'(y) = \frac{d}{dy}(y - 1) = 1$, so $F'(y_e) = 1 > 0$, so $y_e = 1$ is unstable.

(b) $C_e = c_i$ is the only equilibrium. Now $F'(C) = -(F/V)$ so $F'(c_i) = -F/V < 0$ for all positive F and V . So $C_e = c_i$ is stable for all parameter values.

Answer given at back of textbook:

(a) $y_e = 1$ which is unstable.

(b) $C_e = c_i$. $F'(C) = -(F/V) < 0$.

$C_e = c_i$ is **stable** for all values of parameters.