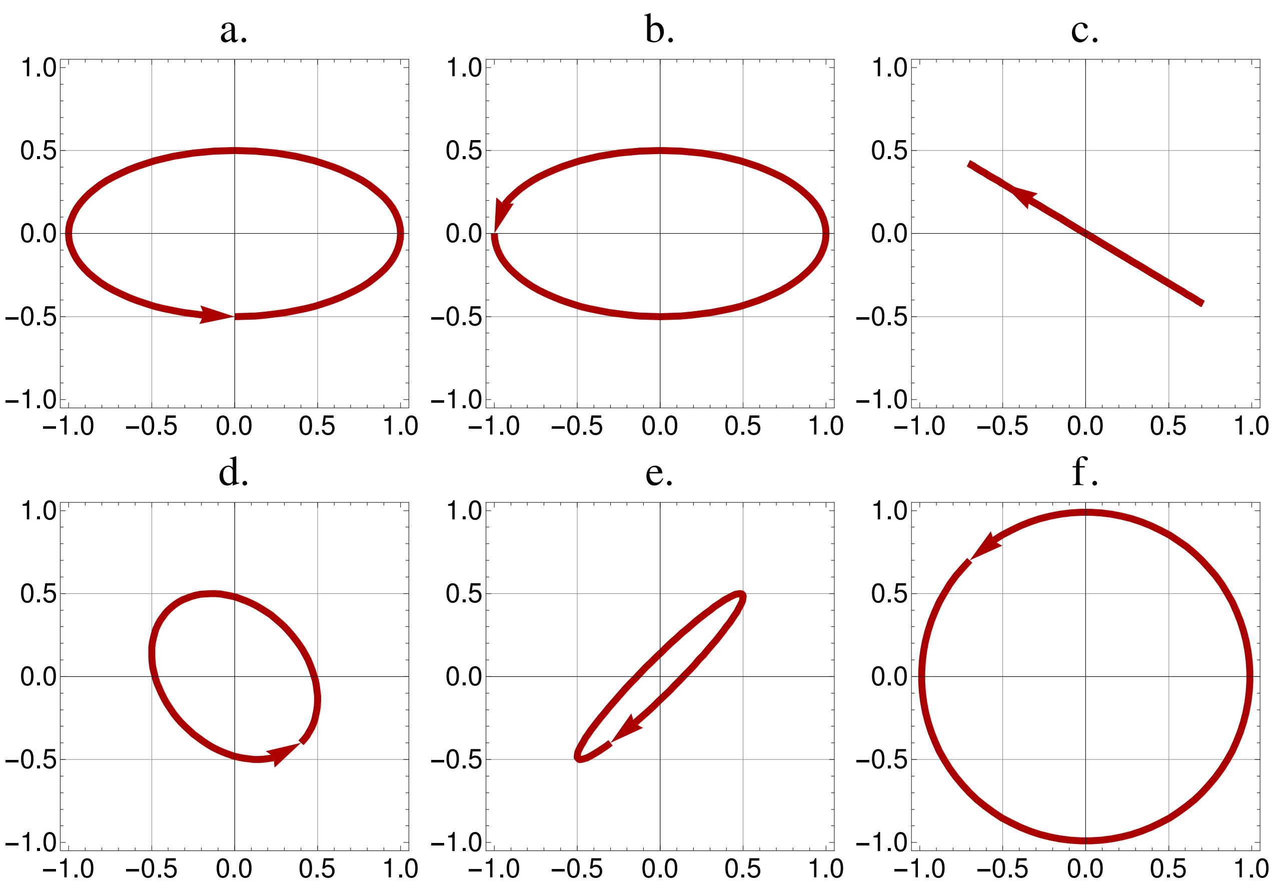
Polarized Light & Optical Systems

Chapter 2 Polarized Light

Problem Sets and Solutions

2.1 Polarization Ellipses

Estimate the Jones vectors for the following polarization ellipses. Find the phases to place the arrow correctly at t = 0.



Solution

$\mathbf{E}(t)=\mathrm{Re}\left[\begin{pmatrix} E_x\, e^{-i\phi x} \\ E_y\, e^{-i\phi y} \end{pmatrix} e^{-i\frac{2\pi}{T}t}\right]$, where

Jones vector: $\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}$ or $\begin{pmatrix} a_x + i\, b_x \\ a_y + i\, b_y \end{pmatrix}$, where $e^{-i\phi}$=cos(ϕ)-iSin(ϕ)

the phases are ϕx, ϕy

Solving the phase for complex number $a + i\, b$:

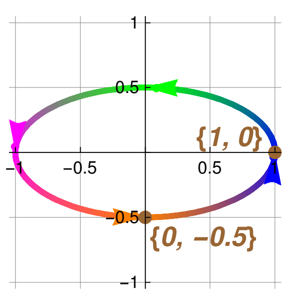
$$\phi = \begin{cases} \tan^{-1}(b/a) & \text{when } a > 0 \\ \tan^{-1}(b/a) + \pi & \text{when } a < 0,\ \ b \geq 0 \\ \tan^{-1}(b/a) - \pi & \text{when } a < 0,\ \ b < 0 \\ \pi/2 & \text{when } a = 0,\ b > 0 \\ -\pi/2 & \text{when } a = 0,\ b < 0 \\ \text{indeterminate} & \text{when } a = 0,\ b = 0 \end{cases}$$

$\mathbf{E}$(t=0)=Re$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]$ and $\mathbf{E}$(t=T/4)=Im$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]$

t	t=0	t=T/4	t=T/2	t=3T/4	t=T
$e^{-i\frac{2\pi}{T}t}$	1	-i	-1	i	1
E(t)	Re$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]$	Re$\left[\begin{pmatrix} E_x e^{-i(\phi x+\frac{\pi}{2})} \\ E_y e^{-i(\phi y+\frac{\pi}{2})} \end{pmatrix}\right]$	Re$\left[\begin{pmatrix} E_x e^{-i(\phi x+\pi)} \\ E_y e^{-i(\phi y+\pi)} \end{pmatrix}\right]$	Re$\left[\begin{pmatrix} E_x e^{-i(\phi x+\frac{3\pi}{2})} \\ E_y e^{-i(\phi y+\frac{3\pi}{2})} \end{pmatrix}\right]$	Re$\left[\begin{pmatrix} E_x e^{-i(\phi x+2\pi)} \\ E_y e^{-i(\phi x+2\pi)} \end{pmatrix}\right]$
E(t)	Re$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]$	Im$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]$	-Re$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]$	-Im$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]$	Re$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]$

The real and imaginary parts of the Jones vector are calculated at where the arrow starts (t=0) and where the arrow at after a quarter period (t=T/4).

a. t=0=T, $\mathbf{E}$=Re$\left[\begin{pmatrix} E_x e^{-i\phi x} \\ E_y e^{-i\phi y} \end{pmatrix}\right]=\begin{pmatrix} 0 \\ -0.5 \end{pmatrix}$ t=T/4, $\mathbf{E}$=Re $\left[\begin{pmatrix} E_x e^{-i(\phi x+\frac{\pi}{2})} \\ E_y e^{-i(\phi y+\frac{\pi}{2})} \end{pmatrix}\right]=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



t= 0, T/4, T/2, 3T/4, T

$\{\{ex \to 1,\ \phi x \to \frac{3\pi}{2}\}\}$

$\{\{ey \to \frac{1}{2},\ \phi y \to \pi\}\}$

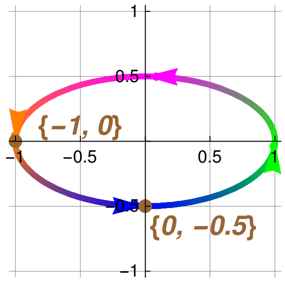
$E = \begin{pmatrix} i \\ -\frac{1}{2} \end{pmatrix}$

At t=0, $\phi x = \frac{3\pi}{2}$ and $\phi y = \pi$

$\phi = |\phi x - \phi y| = \frac{\pi}{2}$

b. When t=0, $\mathbf{E}$={-1,0}

When t=T/4, $\mathbf{E}$={0,-0.5}



t= 0, T/4, T/2, 3T/4, T

{{ex → 1, ϕx → π}}

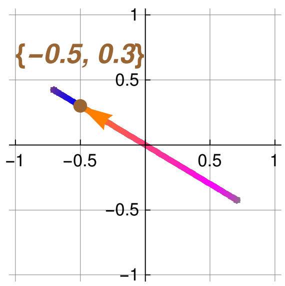
{{ey → $\frac{1}{2}$, ϕy → $\frac{\pi}{2}$}}

$E = \begin{pmatrix} -1 \\ -\frac{i}{2} \end{pmatrix}$

At t=0, ϕx=π and ϕy=$\frac{\pi}{2}$

ϕ=|ϕx−ϕy|=$\frac{\pi}{2}$

c. When t=0, **E**={-0.5,0.3}
When t=T/4, **E**={-0.5,0.3}



t= 0, T/4, T/2, 3T/4, T

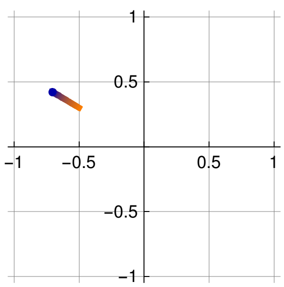
{{ex → 0.707, ϕx → 2.35619}}

{{ey → 0.424, ϕy → −0.785398}}

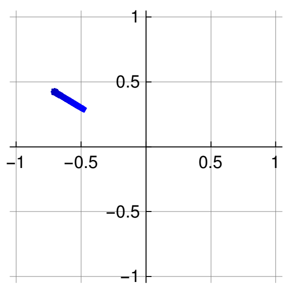
$E = \begin{pmatrix} -0.5 - 0.5\,i \\ 0.3 + 0.3\,i \end{pmatrix}$

At t=0, ϕx=2.35619 and ϕy=−0.785398

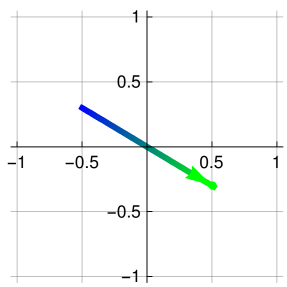
ϕ=|ϕx−ϕy|=3.14159



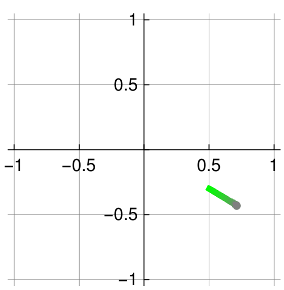
t from 0 to T/8



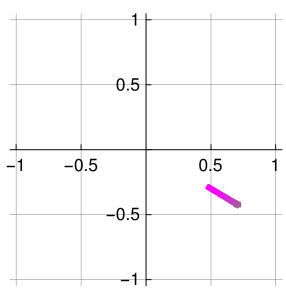
t from T/8 to T/4



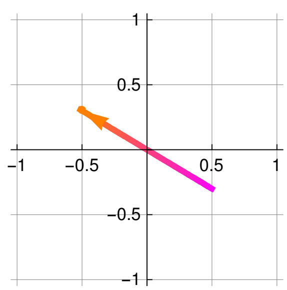
t from T/4 to T/2



t from T/2 to 5T/8

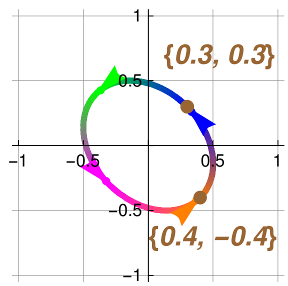


t from 5T/8 to 3T/4



t from 3T/4 to T

d. When t=0, **E**={0.4,-0.4}
When t=T/4, **E**={0.3,0.3}



{{ex → 0.5, ϕx → −0.643501}}

{{ey → 0.5, ϕy → −2.49809}}

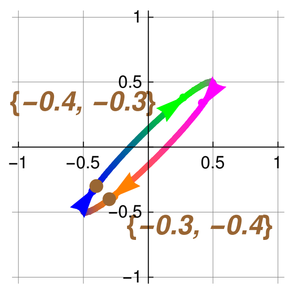
$E = \begin{pmatrix} 0.4 + 0.3\,i \\ -0.4 + 0.3\,i \end{pmatrix}$

At t=0, ϕx=−0.643501 and ϕy=−2.49809

ϕ=|ϕx−ϕy|=1.85459

t= ***0, T/4, T/2, 3T/4, T***

e. When t=0, **E**={-0.3,-0.4}
When t=T/4, **E**={-0.4,-0.3}



{{ex → 0.5, ϕx → 2.2143}}

{{ey → 0.5, ϕy → 2.49809}}

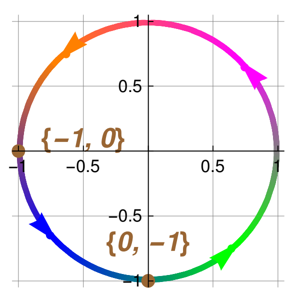
$E = \begin{pmatrix} -0.3 - 0.4\,i \\ -0.4 - 0.3\,i \end{pmatrix}$

At t=0, ϕx=2.2143 and ϕy=2.49809

ϕ=|ϕx−ϕy|=0.283794

t= ***0, T/4, T/2, 3T/4, T***

f. When t=T/8, **E**={-1,0}
When t=3T/8, **E**={0,-1}



{{ex → 1., ϕx → 2.35619}}

{{ey → 1., ϕy → 0.785398}}

$E = \begin{pmatrix} -0.707107 - 0.707107\,i \\ 0.707107 - 0.707107\,i \end{pmatrix}$

At t=0, ϕx=2.35619 and ϕy=0.785398

ϕ=|ϕx−ϕy|=1.5708

t= ***0, T/4, T/2, 3T/4, T***

2.2 Linear, circular, and elliptical Jones vectors

Which of the following Jones vectors are (a) linearly polarized, (b) circularly polarized, 90° or $\pi/2$ out of phase with equal amplitudes, (c) elliptically polarized, with arbitrary phase relationship.

a. (2, 2)

b. (i/2, 1)

c. (i, -i)

d. (1, -4)

e. $(2+2i, -2+2i)$

f. $(2+2i, -2-2i)$

g. $(0, 1+i)$

h. $(3, -6i)$

i. $(2+3i, -3+2i)$

j. $(2, -2i)$

Solution

Conditions for determination:
Linearly polarized: the x and y components are in phase or 180° out of phase.
Circularly polarized: the x and y components are 90° ($\pi/2$) out of phase with equal amplitudes, i. e. real and imaginary.
Elliptically polarized: the x and y components have arbitrary phases or different amplitudes

Therefore,
Linearly polarized: a, c, d, f, g
Circularly polarized: e, j
Elliptically polarized: b, h, i

2.3 Polarization Vector

Consider the plane wave

$$\mathbf{E}(r, t) = \mathrm{Re}\left\{e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\begin{pmatrix} a_x + i\, b_x \\ a_y + i\, b_y \\ a_z + i\, b_z \end{pmatrix}\right\}$$

Find the electric field and the Poynting vector at the following times and locations.

a. $t=0$, $\mathbf{r} = (0, 0, 0)$

b. $t=0$, $\mathbf{r} = \lambda^2\, \mathbf{k}/4\pi$,

c. $t = \pi / \omega$, $\mathbf{r} = (0, 0, 0)$

d. $t = 4\pi / \omega$, $\mathbf{r} = 8\,\lambda^2 \mathbf{k}/\pi$

Solution

In a, b, c, and d, first calculate $e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, then multiply it to $\begin{pmatrix} a_x + i\, b_x \\ a_y + i\, b_y \\ a_z + i\, b_z \end{pmatrix}$ and take the real part to get electric field.

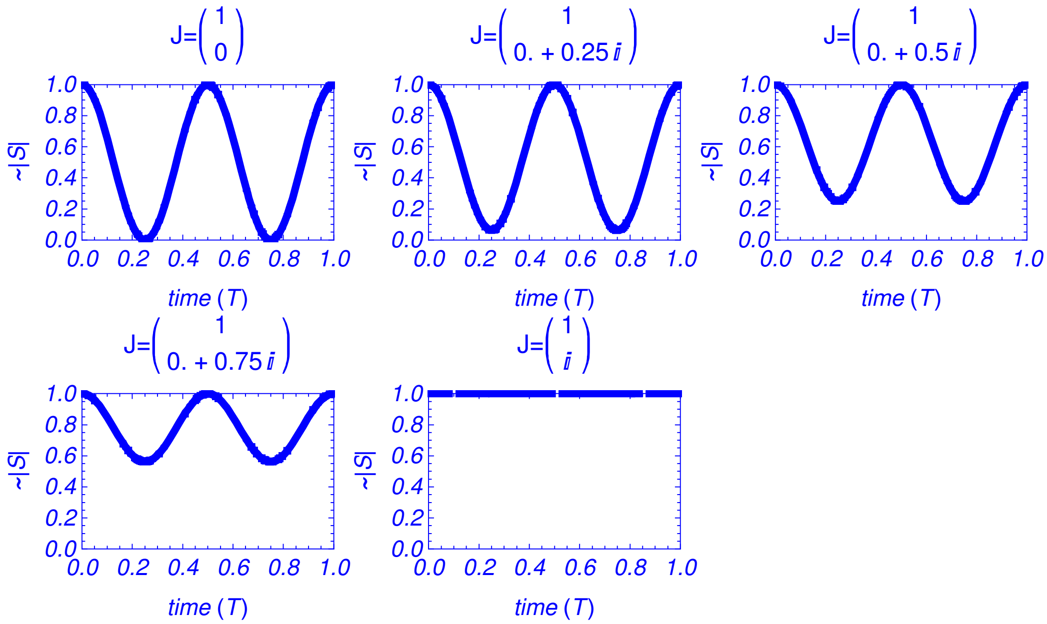
Poynting vector: $\mathbf{S}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$

Since $\mathbf{B}(\mathbf{r},t)=\mu_o\, \mathbf{H}(\mathbf{r},t)$, $\mathbf{S}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)\times\frac{1}{\mu_o}\mathbf{B}(\mathbf{r},t)=\epsilon_o\,(c^2)\, \mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)$.

Also, $\mathbf{B}(\mathbf{r},t)=\frac{1}{c}\hat{\boldsymbol{k}}\times\mathbf{E}(\mathbf{r},t)$, then

$$\begin{aligned}\mathbf{S}(\mathbf{r},t) &= \epsilon o\ c\ \mathbf{E}(\mathbf{r},t)\times\hat{\boldsymbol{k}}\times\mathbf{E}(\mathbf{r},t)\\ &= \epsilon o\ c\ \{\hat{\boldsymbol{k}}[\mathbf{E}(\mathbf{r},t)\cdot\ \mathbf{E}(\mathbf{r},t)]-\mathbf{E}(\mathbf{r},t)[\mathbf{E}(\mathbf{r},t)\cdot\ \hat{\boldsymbol{k}}]\}\\ &= \epsilon o\ c\ \hat{\boldsymbol{k}}[\mathbf{E}(\mathbf{r},t)\cdot\ \mathbf{E}(\mathbf{r},t)]\\ &= \epsilon o\ c\ \hat{\boldsymbol{k}}\left[\mathrm{Re}\left\{e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\begin{pmatrix}a_x+i\,b_x\\ a_y+i\,b_y\\ a_z+i\mathrm{b}_z\end{pmatrix}\right\}\cdot\mathrm{Re}\left\{e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\begin{pmatrix}a_x+i\,b_x\\ a_y+i\,b_y\\ a_z+i\mathrm{b}_z\end{pmatrix}\right\}\right]\end{aligned}$$

Its oscillates twice per period for linearly polarized light, and stays constant for circularly polarized light.



a.

$\mathbf{r}=\{0,\ 0,\ 0\},\ t=0$

$e^{-i\mathbf{k}\cdot\mathbf{r}}=e^{-i0}e^{-i\,k.\{0,0,0\}}=1$ $\quad e^{i\omega t}=e^{i0}=1$

$e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=1$

$\mathrm{E}=\mathrm{Re}\left\{\begin{pmatrix}a_x+i\,b_x\\ a_y+i\,b_y\\ a_z+i\,b_z\end{pmatrix}\right\}=\begin{pmatrix}a_x\\ a_y\\ a_z\end{pmatrix}$

$\mathrm{S}=c\ \epsilon o\ \left(a_x^2+a_y^2+a_z^2\right)\hat{\boldsymbol{k}}$

b.

$\mathbf{r}=\ \lambda^2\ \mathbf{k}/4\pi,\ \ t=0$

$e^{-i\mathbf{k}\cdot\mathbf{r}}=e^{-i\lambda^2\mathbf{k}\cdot\mathbf{k}/4\pi}=e^{-i(\lambda^2/4\pi)(4\pi^2/\lambda^2)}=e^{-i\pi}=-1$

$e^{i\omega t}=e^{i0}1$

$e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=-1$

$$E=\mathrm{Re}\left\{-\begin{pmatrix} a_x+i\,b_x \\ a_y+i\,b_y \\ a_z+i\,b_z \end{pmatrix}\right\}=\begin{pmatrix} -a_x \\ -a_y \\ -a_z \end{pmatrix}$$

$S=c\,\epsilon o\,(a_x^2+a_y^2+a_z^2)\,\hat{k}$

c.

$r=\ \{0,0,0\},\ \ t=\frac{\pi}{\omega}$

$e^{-i\mathbf{k}\cdot\mathbf{r}}=e^{-i0}=1$

$e^{i\omega t}=e^{i\omega(\pi/\omega)}=e^{i\pi}=-1$

$e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=-1$

$$E=\mathrm{Re}\left\{-\begin{pmatrix} a_x+i\,b_x \\ a_y+i\,b_y \\ a_z+i\,b_z \end{pmatrix}\right\}=\begin{pmatrix} -a_x \\ -a_y \\ -a_z \end{pmatrix}$$

$S=c\,\epsilon o\,(a_x^2+a_y^2+a_z^2)\,\hat{k}$

d.

$r=8\,\lambda^2\mathbf{k}/\pi,\ \ t=4\pi/\omega$

$e^{-i\mathbf{k}\cdot\mathbf{r}}=e^{-i(8\lambda^2/\pi)\mathbf{k}\cdot\mathbf{k}}=e^{-i(8\lambda^2/\pi)(4\pi/\lambda^2)}=e^{-i32\pi}=-1$

$e^{i\omega t}=e^{i\omega(4\pi/\omega)}=e^{i4\pi}=-1$

$e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=1$

$$E=\mathrm{Re}\left\{\begin{pmatrix} a_x+i\,b_x \\ a_y+i\,b_y \\ a_z+i\,b_z \end{pmatrix}\right\}=\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

$S=c\,\epsilon o\,(a_x^2+a_y^2+a_z^2)\,\hat{k}$

2.4 Orthogonal Polarization States

a. Find the equation for the normalized Jones vector **ff** orthogonal to **ee**

ee = $\{A_x\,e^{-i\,\phi_x},\ A_y\,e^{-i\,\phi_y}\}$

b. Verify the equation with right circularly polarized light.

c. Why is the phase of the orthogonal Jones vector a free parameter which can be chosen arbitrarily?

d. Given a propagation direction **k** and a polarization vector **F**

```
k = {kx, ky, kz}
```

$F = \{A_x e^{-i\phi_x}, A_y e^{-i\phi_y}, A_z e^{-i\phi_z}\}$

find the polarization vector **hh** orthogonal to **F**, normalized or unnormalized.

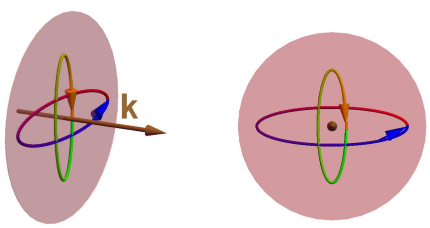
Solution

a. Find the equation for the normalized Jones vector **ff** orthogonal to **ee**

$ee = \{A_x e^{-i\phi_x}, A_y e^{-i\phi_y}\}$

$ff = e^{-i\xi}\{-A_y e^{i\phi_y}, A_x e^{i\phi_x}\}$

ee and ff are orthogonal, because **ee.ff***$=e^{-i\xi}(-A_x e^{-i\phi_x} A_y e^{-i\phi_y} + A_y e^{-i\phi_y} A_x e^{-i\phi_x})=0$



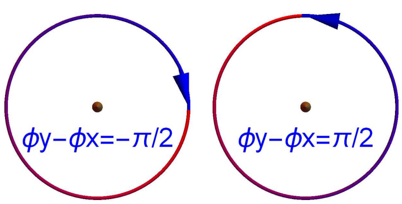
b. Verify the equation with right circularly polarized light.

Set **ee** $= \begin{pmatrix} 1 \\ -i \end{pmatrix}$ right circularly polarized in decreasing phase convention,

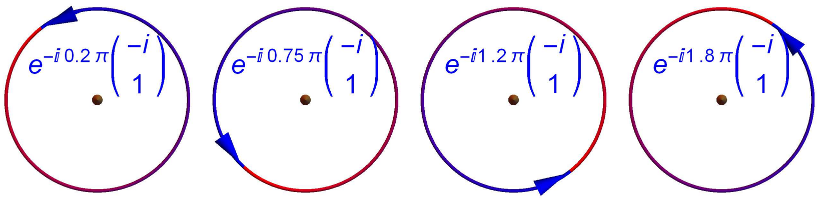
then $A_x=1$, $\phi_x=0$, $A_y=1$, $\phi_y=\pi/2$.

So **ff** $= e^{-i\xi}\{-1\, e^{i\pi/2}, 1\, e^{i0}\} = e^{-i\xi}\begin{pmatrix} -i \\ 1 \end{pmatrix} = e^{-i(\xi+\pi/2)}\begin{pmatrix} 1 \\ i \end{pmatrix}$ which is left circularly polarized.

RCP: **ee** $= \begin{pmatrix} 1 \\ -i \end{pmatrix}$ LCP: **ff** $= \begin{pmatrix} -i \\ 1 \end{pmatrix}$



$e^{-i\xi}\begin{pmatrix}-i\\1\end{pmatrix}$:



c. Why is the phase of the orthogonal Jones vector a free parameter which can be chosen arbitrarily?

Orthogonality does not depend on the phases.
ee.**ff***=0 regardless of the absolute phase: $e^{-i\xi}$ in **ff**.

d. Given a propagation direction **k** and a polarization vector **F**

$$k = \{kx, ky, kz\}$$
$$F = \{A_x e^{-i\phi_x}, A_y e^{-i\phi_y}, A_z e^{-i\phi_z}\}$$

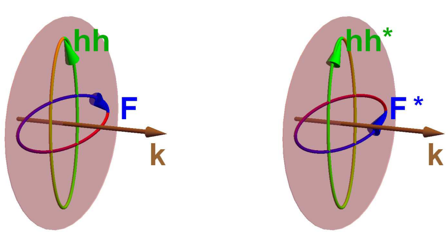
find the polarization vector **hh** orthogonal to **F**, normalized or unnormalized.

Polarization vector **hh** is orthogonal to propagation vector **k**, which means **hh**·**k**=0.
Polarization vector **F** is orthogonal to polarization vector **hh**, which means **F***·**hh**=0.
Therefore **hh** is **k**×**F***=

$$\begin{pmatrix} -e^{i\phi_y}\, kz\, A_y + e^{i\phi_z}\, ky\, A_z \\ e^{i\phi_x}\, kz\, A_x - e^{i\phi_z}\, kx\, A_z \\ -e^{i\phi_x}\, ky\, A_x + e^{i\phi_y}\, kx\, A_y \end{pmatrix}$$

This can be verified by {**hh**·**k**, **F***·**hh**}=

{0, 0}



For example, similar to part a and b, but now in 3D: say **k**={0,0,1} along *z*,
F={1,-i,0} right circularly polarized,

```
k={0, 0, 1} along z
```

```
F={1, -I, 0} right circularly polarized
```

```
hh={-I, 1, 0} is left circularly polarized
```

2.5 Rotate Jones vectors

Rotate the following Jones vectors 45° counterclockwise (+x toward +y)

a. v1=(1, i)
b. v2=(3, 3)
c. v3=(0, -2)
d. v4=(1+i, 1-i)

Solution

Rotation matrix for 45° counterclockwise $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

a. $\left\{\frac{1-i}{\sqrt{2}}, \frac{1+i}{\sqrt{2}}\right\}$
b. $\left\{0, 3\sqrt{2}\right\}$
c. $\left\{\sqrt{2}, -\sqrt{2}\right\}$
d. $\left\{i\sqrt{2}, \sqrt{2}\right\}$

2.6 Flux of Jones vectors

a. What is the normalized flux of the Jones vector E3 = $\begin{pmatrix} w + i\,x \\ y + i\,z \end{pmatrix}$?
b. What is the flux of E3 in W/m^2, (Watts per meter squared)?
c. What are the units of the Jones vector elements w+i x and y+i z?

Solution

a. The normalized flux is $E3^{\dagger} \cdot E3 = (w - i\,x, y - i\,z) \cdot \begin{pmatrix} w + i\,x \\ y + i\,z \end{pmatrix} = w^2 + x^2 + y^2 + z^2$.
b. The flux is $\frac{\epsilon_0 c}{2}(w^2 + x^2 + y^2 + z^2)$
c. Jones vector elements are expressed in volts/meter.

2.7 Circularly polarized basis

a. Find the matrix for a change of basis from the left and right circularly polarized basis states to the xy-basis.
b. Convert the Jones vectors $(E_L, E_R) = (e^{i\eta}, e^{-i\eta})/\sqrt{2}$ from the circular basis into the xy-basis and identify the type of polarization states.

Solution

To change the basis from Cartesian xy coordinates into LR, the rows of the change of basis matrix $R_{xy\to LR}$ are the conjugates of the states

$$R_{xy\to LR} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

To convert in the reverse direction, the change of basis matrix is $R_{LR\to xy}$ is the matrix inverse

$$R_{LR\to xy} = \left(R_{xy\to LR}\right)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

a. $\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \cdot \begin{pmatrix} E_L \\ E_R \end{pmatrix}$

b. $\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \cos\eta \\ \sin\eta \end{pmatrix}$ which is linearly polarized at angle η.

2.8 Basis conversion

Convert the six basis polarization states Table 2.2 from Jones vectors in the ordinary linear xy-basis into the LR circular basis

Solution

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix};$$

```
R.{1, 0}
```

$$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

```
R.{0, 1}
```

$$\left\{-\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}$$

$$\text{R.}\{1, 1\}\Big/\sqrt{2}$$

$$\left\{\frac{1}{2} - \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\right\}$$

$$\text{R.}\{1, -1\}\Big/\sqrt{2}$$

$$\left\{\frac{1}{2} + \frac{i}{2}, \frac{1}{2} - \frac{i}{2}\right\}$$

$$\text{R.}\{1, i\}\Big/\sqrt{2}$$

```
{1, 0}
```

$$\text{R.}\{1, -i\}\Big/\sqrt{2}$$

```
{0, 1}
```

2.10 Polarization vector

What are the two directions that light propagating with polarization vector

$$E=\begin{pmatrix} 4\,i \\ 6 \\ 4\,i \end{pmatrix}$$

might be propagating? This can be determined by taking the cross product between the E-field at two different times using the real field representation, not the exponential form.

Solution

$$\frac{\text{Re}[\text{polV}]\times\text{Im}[\text{polV}]}{|\text{Re}[\text{polV}]\times\text{Im}[\text{polV}]|}=\left\{\frac{1}{\sqrt{2}},\,0,\,-\frac{1}{\sqrt{2}}\right\}\text{ or}$$

$$\frac{\text{Im}[\text{polV}]\times\text{Re}[\text{polV}]}{|\text{Im}[\text{polV}]\times\text{Re}[\text{polV}]|}=\left\{-\frac{1}{\sqrt{2}},\,0,\,\frac{1}{\sqrt{2}}\right\}$$

2.11 Circularly Polarized Light

```
e2 = {6 i , 10, -8 i}
```

a. Show that the state e2 is circularly polarized.

b. Find the axis of light propagation. The direction along the axis is undetermined.

c. Which direction would the light be propagating to be left circularly polarized?

Solution

a. Electric Field for decreasing phase convention: $\mathbf{E}(t)=\text{Re}\left[\begin{pmatrix} Ex \\ Ey \\ Ez \end{pmatrix} e^{-i\frac{2\pi}{T}t}\right]$

$$\mathbf{E}(t) = \text{Re}\left[\begin{pmatrix} 6\,i \\ 10 \\ -8\,i \end{pmatrix} e^{-i\frac{2\pi}{T}t}\right]$$

$$= \text{Re}\left\{\begin{pmatrix} 6\,i \\ 10 \\ -8\,i \end{pmatrix}\left[\cos\left(\frac{2\pi}{T}t\right) - i\sin\left(\frac{2\pi}{T}t\right)\right]\right\}$$

$$= \begin{pmatrix} -6\sin\left(\frac{2\pi}{T}t\right) \\ 10\cos\left(\frac{2\pi}{T}t\right) \\ 8\sin\left(\frac{2\pi}{T}t\right) \end{pmatrix}$$

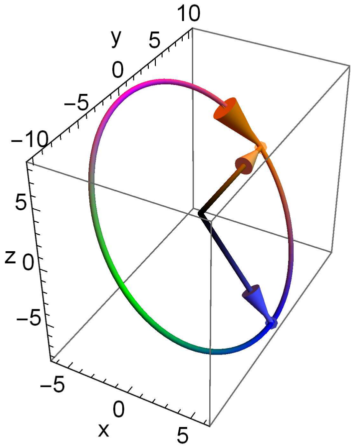
$$|\mathbf{E}(t)| = \sqrt{6^2\sin^2\left(\frac{2\pi}{T}t\right) + 10^2\cos^2\left(\frac{2\pi}{T}t\right) + 8^2\sin^2\left(\frac{2\pi}{T}t\right)}$$

$$= 10$$

|**E**| is constant at all time, so e2 is circularly polarized.

b. Consider the **E** vector at two arbitrary times

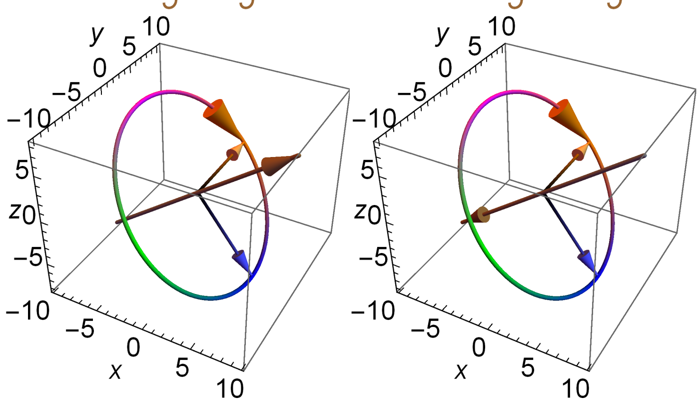
E(t=0) = {0,10,0}
E(t=T/4) = {6,0,-8}



The cross product of the two vectors gives the propagation axis, direction not specified:

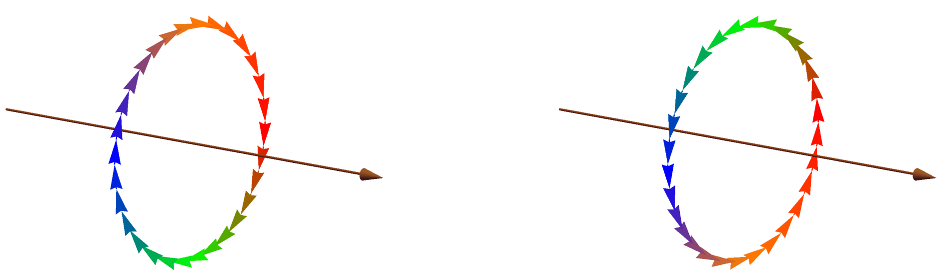
$\left\{-\frac{4}{5}, 0, -\frac{3}{5}\right\}$ or $\left\{\frac{4}{5}, 0, \frac{3}{5}\right\}$

$k=\left\{\frac{4}{5}, 0, \frac{3}{5}\right\}$ $\quad$ $k=\left\{-\frac{4}{5}, 0, -\frac{3}{5}\right\}$



c. Here right and left circular polarizations with the same propagation directions are plotting in time. As time progress, **E** evolves follwing the arrows:

Right Circularly Polarized through Time (*Left Hand Rule*) $\quad$ *Left Circularly Polarized through Time* (*Right Hand Rule*)



From part b, **k**=$\left\{-\frac{4}{5}, 0, -\frac{3}{5}\right\}$ follows the *Right hand rule*, and therefore is left circular polarized

Decreasing Phase Convention	Right Circularly Polarized	Left Circularly Polarized
In time: $\mathbf{E}(t)=\text{Re}[\begin{pmatrix}Ex\\Ey\\Ez\end{pmatrix} e^{-i\frac{2\pi}{T}t}]$	LHR	RHR
In space: $\mathbf{E}(\mathbf{r})=\text{Re}[\begin{pmatrix}Ex\\Ey\\Ez\end{pmatrix} e^{+i\vec{k}.\vec{r}}]$	RHR	LHR

The propagation direction: $\mathbf{E}(t=0)\times\mathbf{E}(t=T/4)=\text{Re}\left[\begin{pmatrix}Ex\\Ey\\Ez\end{pmatrix} e^{-i\,0}\right]\times\text{Re}\left[\begin{pmatrix}Ex\\Ey\\Ez\end{pmatrix} e^{-i\frac{\pi}{2}}\right]=\text{Re}\begin{pmatrix}Ex\\Ey\\Ez\end{pmatrix}\times\text{Im}\begin{pmatrix}Ex\\Ey\\Ez\end{pmatrix}$ obeys

Right hand rule,
and therefore corresponds to Left circularly polarization.

2.12 3-D Polarization Helices

The equation for a circularly polarized monochromatic plane wave electric field traces a helix for a given point in space or time. Graph this helix for right circularly polarized light with amplitude 1, $E(z, t) = \text{Re}\left[e^{i\left(\frac{2\pi}{\lambda}kz-\omega t-\phi\right)} \frac{A}{\sqrt{2}}(1, -i, 0)\right]$.

a. Generate a three dimensional view of the helix generated in space by the tip of a right circularly polarized vector field (time is fixed).

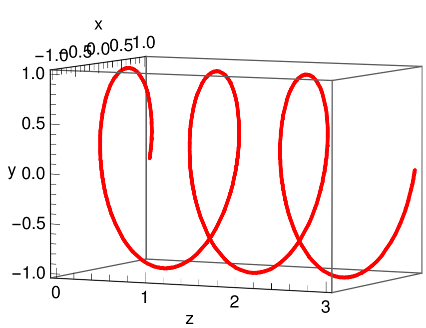
b. What is the sign of the helicity, left or right handed?

c. Generate a three dimensional view of the helix generated in x, y, and t by the tip of a right circularly polarized vector field.

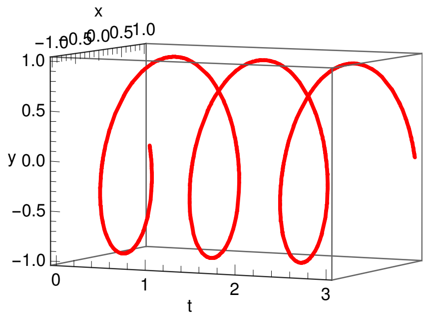
d. What is the sign of the helicity, left or right handed? Consider the plane wave

Solution

a. Space helix is right handed



c. Time helix is left handed



2.13

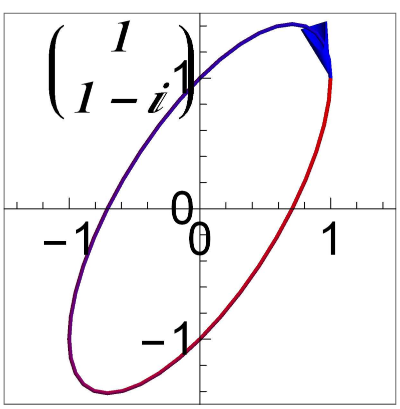
For each of the following Jones vectors

$$E1 = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix};\ E2 = \begin{pmatrix} -i \\ -i \end{pmatrix};\ E3 = \begin{pmatrix} -i \\ 1 \end{pmatrix};\ E4 = \frac{1}{4}\begin{pmatrix} 5 \\ 3\,i \end{pmatrix};\ E5 = \frac{1}{2}\begin{pmatrix} i \\ ! + i\sqrt{2} \end{pmatrix};\ E6 = \begin{pmatrix} -i \\ -\pi/3 \end{pmatrix};$$

(a) Plot the polarization ellipse and indicate the direction the electric field is rotating.
(b) Calculate the phase difference between the x- and y-components, $\delta(\phi) = \phi$k- ϕy, the orientation of the major ellipse, ψ,and the normalized flux, P. For circularly polarized light, the orientation may be undefined.
(c) Calculate the degree of circular polarization (DoCP) defined as |PL-PR|/(PL+PR).

Solution

1.



Orientation $\frac{1}{2}(\pi - \text{ArcTan}[2])$ = 58.2825 degrees

ϕx−ϕy $\frac{\pi}{4}$

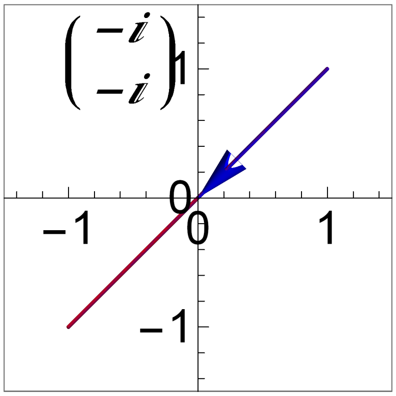
Flux 3

P_L $\frac{1}{2}$

P_R $\frac{5}{2}$

DoCP $\frac{2}{3}$

2.



Orientation $\frac{\pi}{4}$ = 45. degrees

ϕx−ϕy 0

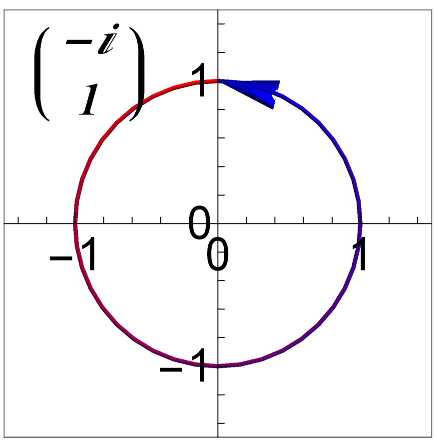
Flux 2

P_L 1

P_R 1

DoCP 0

3



Orientation ψ=0=0 degrees

ϕx−ϕy $-\frac{\pi}{2}$

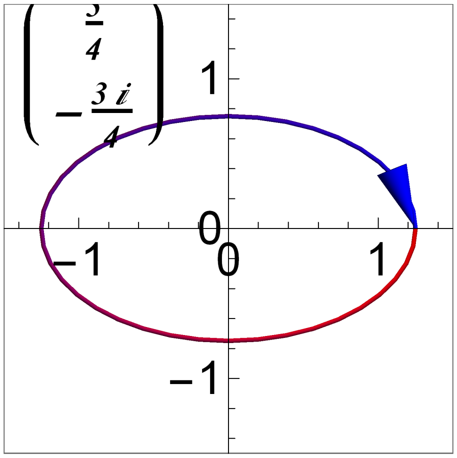
Flux 2

P_L 2

P_R 0

DoCP 1

4



Orientation θ = 0. degrees

ϕx−ϕy $\frac{\pi}{2}$

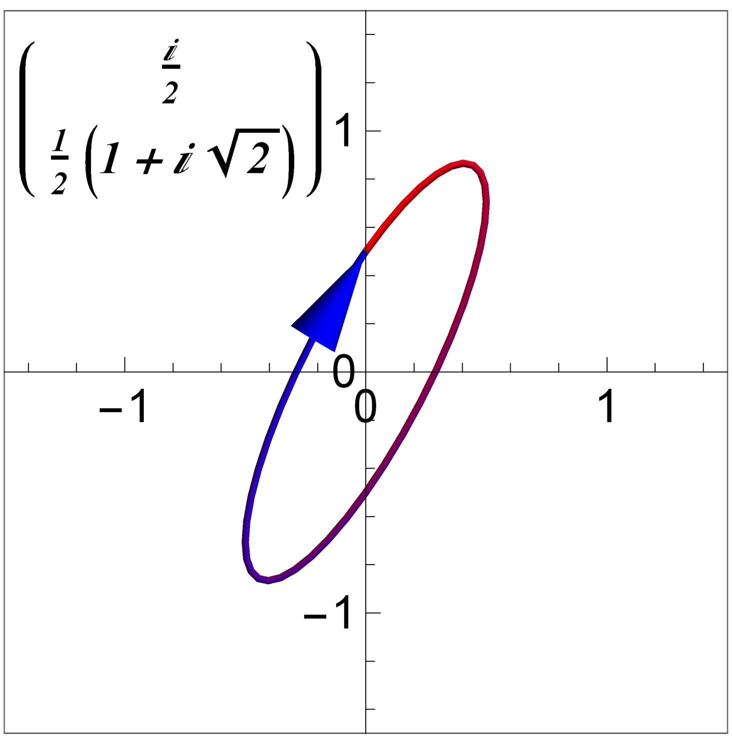
Flux $\frac{17}{8}$

P_L $\frac{1}{8}$

P_R 2

DoCP $\frac{15}{17}$

5



$\phi x - \phi y \quad \frac{\pi}{2} - \text{ArcTan}\left[\sqrt{2}\right] = 0.61548$

$\text{Orientation} \quad \frac{1}{2}\left(\pi - \text{ArcTan}\left[\sqrt{2}\right]\right) = 62.6322 \text{ degrees}$

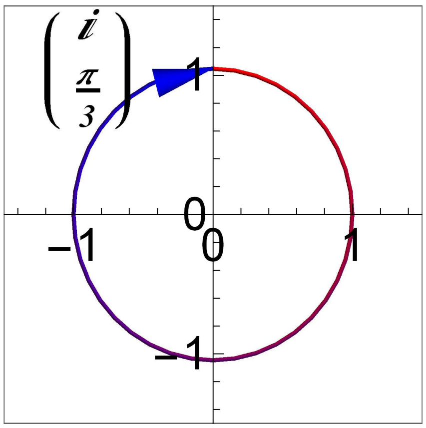
$\text{Flux} \quad 1 = 1.$

$P_L \quad \frac{1}{4} = 0.25$

$P_R \quad \frac{3}{4} = 0.75$

$\text{DoCP} \quad \frac{1}{2} = 0.5$

6



$\phi x - \phi y \quad \frac{\pi}{2} = 1.5708$

$\text{Orientation} \quad \frac{\pi}{2} = 90. \text{ degrees}$

$\text{Flux} \quad \frac{1}{9}\left(9 + \pi^2\right) = 2.09662$

$P_L \quad \frac{1}{18}(-3 + \pi)^2 = 0.0011138$

$P_R \quad \frac{1}{18}(3 + \pi)^2 = 2.09551$

$\text{DoCP} \quad \frac{6\pi}{9+\pi^2} = 0.998938$

2.14

a. Compute the normal vector ŵ to complete the right handed orthonormal basis set (**û**,**v̂**,**ŵ**) where

$$u = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}; \quad v = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}$$

b. Write the expression for a rotating unit vector **ŝ**(t) perpendicular to **û**, rotating clock-wise about the **û-**axis when looking into **û**, at an angular velocity of ω rad/sec, such

that $\hat{\mathbf{s}}(0) = \mathbf{v}$. This is an expression for the electric field at a point associated with right circularly polarized light propagating in the $\hat{\mathbf{u}}$ direction.
c. Calculate the polarization vector **E** for this wave

Solution

a. For a right-handed basis set, looking into u, by the right hand rule, moving from **v** to **w** the vector rotates clockwise.

Starting the vector at $\hat{\mathbf{v}}$ and rotating toward $\hat{\mathbf{w}}$ in time yields s(t)

To complete the orthonormal set,

$$\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

b.

$$s(t) = \left\{\frac{\mathrm{Cos}[2\pi t\omega]}{\sqrt{6}} - \frac{\mathrm{Sin}[2\pi t\omega]}{\sqrt{2}}, \frac{\mathrm{Cos}[2\pi t\omega]}{\sqrt{6}} + \frac{\mathrm{Sin}[2\pi t\omega]}{\sqrt{2}}, -\sqrt{\frac{2}{3}}\,\mathrm{Cos}[2\pi t\omega]\right\}$$

c. The polarization vector E will have its real part as $\hat{\mathbf{v}}$ and imaginary part as $\hat{\mathbf{w}}$

$$\mathbf{E} = \begin{pmatrix} -\frac{i}{\sqrt{2}} + \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Bonus problem sets

2.15 Polarization vectors and propagation vectors

Match each propagation vector **k** with the corresponding polarization vector **E**.

	k		**E**
a.	$\left\{\frac{1}{\sqrt{5}}, \sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}\right\}$	(1)	$\left\{-\frac{2}{3}, \frac{11}{3}, \frac{1}{3}\right\}$
b.	$\{0, 1, 0\}$	(2)	$\left\{\frac{2}{\sqrt{11}}, \frac{9}{\sqrt{11}}, -\frac{6}{\sqrt{11}}\right\}$
c.	$\left\{0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\}$	(3)	$\left\{\frac{1+i}{\sqrt{2}}, 0, \frac{1-i}{\sqrt{2}}\right\}$
d.	$\left\{\frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}\right\}$	(4)	$\left\{2\sqrt{\frac{10}{3}}, -2\sqrt{\frac{2}{15}}, 4\sqrt{\frac{2}{15}}\right\}$
e.	$\left\{\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right\}$	(5)	$\left\{2\sqrt{2}, -1+\frac{i}{\sqrt{2}}, -1-\frac{i}{\sqrt{2}}\right\}$

Solution

k and **E** pairs

a. and 5
b. and 4
c. and 2
d. and 3
e. and 1

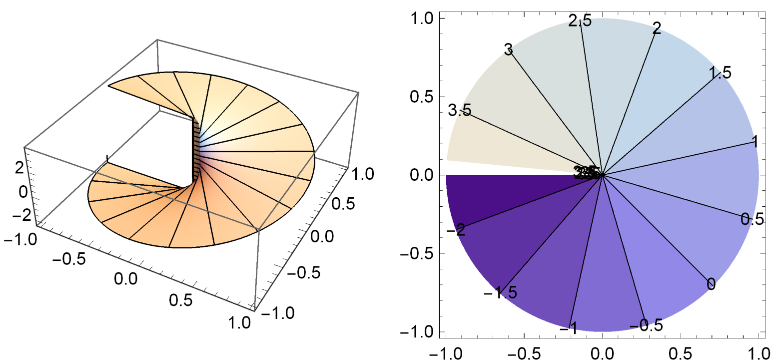
Fluxes:

(1)	14
(2)	11
(3)	2
(4)	16
(5)	11

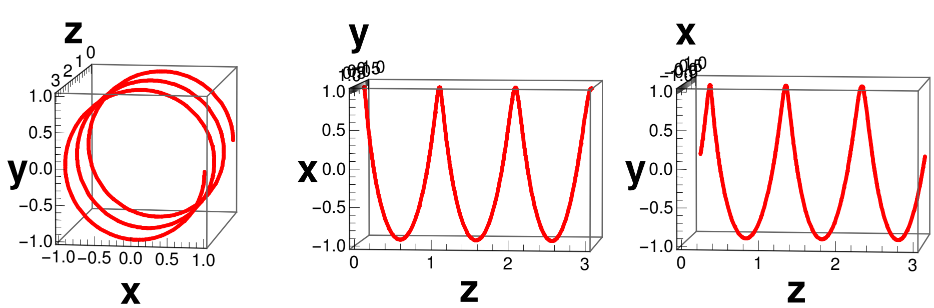
2.16 Identify the following functions

Write down an equation for the following functions.

a.



b.



Solution

a. $\tan^{-1}(y/x) + \frac{\pi}{4}$

b. $\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i 2 \pi z}$

2.17 Polarized Fluxes

a. Find the right and left circular polarized flux components of the following Jones vectors

```
e1 = {1, i};
e2 = {10 i, -10};
e3 = {2, -2};
e4 = {1 + i, 1 - i};
```

b. Find the matrix to transform from the usual cartesian basis for Jones vectors to the right $(1, -i)/\sqrt{2}$ and left circular $(1, i)/\sqrt{2}$ basis $\begin{pmatrix} E_R \\ E_L \end{pmatrix}$.

c. Transform the four Jones vectors into the circular basis.

Solution

a.

```
r = {1, -i}/√2; l = {1, i}/√2;
Conjugate[e1].r=0
{Abs[Conjugate[e1].r]^2,Abs[Conjugate[e1].l]^2}={0, 2}
{Abs[Conjugate[e2].r]^2,Abs[Conjugate[e2].l]^2}={0, 200}
{Abs[Conjugate[e3].r]^2,Abs[Conjugate[e3].l]^2}={4, 4}
{Abs[Conjugate[e4].r]^2,Abs[Conjugate[e4].l]^2}={4, 0}
```

b.

$$R=\{r,l\}=\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$R.e1=\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

$$R.e2=\begin{pmatrix} 10\, i \sqrt{2} \\ 0 \end{pmatrix}$$

$$R.e3=\begin{pmatrix} (1+i)\sqrt{2} \\ (1-i)\sqrt{2} \end{pmatrix}$$

$$R.e4=\begin{pmatrix} 0 \\ (1+i)\sqrt{2} \end{pmatrix}$$

2.18 Jones vectors

Consider the Jones vector

```
Ea = (17
      7);
```

$$Ea = \begin{pmatrix} 17 \\ 7 \end{pmatrix};$$

a. Is this state linear?

b. What is the orientation of the major axis?

c. What is the component of the flux (intensity) in the following states: **H, V, 45, 135**?

d. Convert E into the corresponding Stokes parameters?

e.. Rotate the stokes parameters 45 ° counterclockwise (x into y).

Solution

a. Yes, the phases of the x and y components are equal, both equal 0.

b. ArcTan[17,7] = ArcTan[7/17] = 22.3801°.

c.

```
PH=Abs[Ea.{1,0}]^2=           289
PV=Abs[Ea.{0,1}]^2=           49
P45=Abs[Ea.{1,1}/√2]^2=       288
P135=Abs[Ea.{1,-1}/√2]^2      50
```

d.

```
S={PH+PV,PH-PV,P45-P135,0}={338, 240, 238, 0}
```

e.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 338 \\ 240 \\ 238 \\ 0 \end{pmatrix} = \begin{pmatrix} 338 \\ -238 \\ 240 \\ 0 \end{pmatrix}$$

Supplement materials

Illustrate: How **E** field evolving as light propagates in time?

The oscillation of electric field can be plotted as a function of time:

$\mathbf{E}(t)=\mathrm{Re}\left[\begin{pmatrix} E_x\, e^{-i\phi x} \\ E_y\, e^{-i\phi y} \end{pmatrix} e^{-i\frac{2\pi}{T}t}\right]$. The following figures shows three examples of Jones vector **J** which is the $\begin{pmatrix} E_x\, e^{-i\phi x} \\ E_y\, e^{-i\phi y} \end{pmatrix}$ part of **E** for a propagation along *z* axis. At any particular time *t*, **E** is a vector pointing orthogonal from its propagation axis (shown in the second column). We often connect the tip of these arrows to represent the **E** field (shown in the third column). Looking at the transverse plane (*xy*-plane) into the propagation along *z* axis (shown in the 3rd column), we see the electric field oscillating in its local transverse plane. We can estimate **J** by estimating where the arrow tip is at as *t* increases (shown in the 4th column). The arrows in the 3rd and 4th columns start at time *t*=0 (red) and end at one period *t*=T (blue).