Chapter 2

Solution to exercise 11:

- For the Bernoulli distribution $p(x) = p^x(1-p)^{1-x}$, with $p \in [0,1]$ and $x \in \{0,1\}$, we have

$$\varphi(s) \quad = \quad \mathbb{E}(s^X) = p\ s + (1-p)\ s^0 = ps + (1-p).$$

- For the Binomial distribution $p(x) = \begin{pmatrix} n \\ x \end{pmatrix} p^x(1-p)^{n-x}$, with $p \in [0,1]$ and $x \in \{0,\ldots,n\}$ for some $n \in \mathbb{N}$, we have

$$\begin{aligned}
\varphi(s) &= \sum_{0 \le x \le n} \begin{pmatrix} n \\ x \end{pmatrix} s^x\ p^x(1-p)^{n-x} \\
&= \sum_{0 \le x \le n} \begin{pmatrix} n \\ x \end{pmatrix} (sp)^x(1-p)^{n-x} = ((1-p)+ps)^n\ .
\end{aligned}$$

- For the Poisson distribution $p(x) = e^{-\lambda}\lambda^x/x!$, with $\lambda > 0$ and $x \in \mathbb{N}$, we have

$$\begin{aligned}
\varphi(s) &= e^{-\lambda} \sum_{x \ge 0} s^x\ \lambda^x/x! \\
&= e^{-\lambda} \sum_{x \ge 0} (s\lambda)^x/x! = e^{-\lambda+s\lambda} = e^{-\lambda(1-s)}.
\end{aligned}$$

- For the Geometric distribution $p(x) = (1-p)^{x-1}p$, with $\lambda > 0$ and $x \in \mathbb{N} - \{0\}$, we have

$$\begin{aligned}
\varphi(s) &= p \sum_{x \ge 1} s^x\ (1-p)^{x-1} \\
&= ps \sum_{x \ge 1} s^{x-1}\ (1-p)^{x-1} = ps \sum_{x \ge 0} (s(1-p))^x = ps/\left(1 - s(1-p)\right).
\end{aligned}$$

This ends the proof of the exercise. ∎

Solution to exercise 12:
By construction, we have

$$\begin{aligned}
\mathbb{E}(N_{n+1}) &= \mathbb{E}\left(\mathbb{E}\left(\sum_{1 \le i \le N_n} X_n^i \mid N_n\right)\right) \\
&= \mathbb{E}\left(\sum_{1 \le i \le N_n} \mathbb{E}\left(X_n^i \mid N_n\right)\right) = \mathbb{E}(N_n)\ m.
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}(N_{n+1}) &= \mathbb{E}(N_{n+1}^2) - (\mathbb{E}(N_{n+1}))^2 \\
&= \mathbb{E}\left(\mathbb{E}\left(\left(\sum_{1\le i\le N_n} X_n^i\right)^2 \mid N_n\right)\right) - (\mathbb{E}(N_{n+1}))^2 .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathbb{E}\left(\left(\sum_{1\le i\le N_n} X_n^i\right)^2 \mid N_n\right) &= N_n\mathbb{E}(X^2) + N_n(N_n-1)\,(\mathbb{E}(X))^2 \\
&= N_n\ \mathrm{Var}(X) + N_n^2 m^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\mathrm{Var}(N_{n+1}) &= \mathbb{E}(N_n)\ \mathrm{Var}(X) + \mathbb{E}(N_n^2)m^2 - (\mathbb{E}(N_{n+1}))^2 \\
&= \mathbb{E}(N_n)\ \mathrm{Var}(X) + \mathrm{Var}(N_n)m^2 + \left[(\mathbb{E}(N_n)\ m)^2 - (\mathbb{E}(N_{n+1}))^2\right] \\
&= m^2\ \mathrm{Var}(N_n) + \mathbb{E}(N_n)\ \mathrm{Var}(X).
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\mathrm{Var}(N_{n+1}) &= m^2\ \left[m^2\ \mathrm{Var}(N_{n-1}) + \mathbb{E}(N_{n-1})\ \mathrm{Var}(X)\right] + \mathbb{E}(N_n)\ \mathrm{Var}(X) \\
&= m^4\ \mathrm{Var}(N_{n-1}) + \left[m^2\mathbb{E}(N_{n-1}) + \mathbb{E}(N_n)\right]\ \mathrm{Var}(X) \\
&= m^6\ \mathrm{Var}(N_{n-2}) + \left[m^4\mathbb{E}(N_{n-2}) + m^2\mathbb{E}(N_{n-1}) + \mathbb{E}(N_n)\right]\ \mathrm{Var}(X) \\
&= \ldots \\
&= m^{2(n+1)}\ \mathrm{Var}(N_0) + \mathrm{Var}(X)\ \sum_{0\le k\le n} m^{2k}\mathbb{E}(N_{n-k}),
\end{aligned}
$$

so that

$$
\mathrm{Var}(N_{n+1}) = m^{2(n+1)}\ \mathrm{Var}(N_0) + \mathrm{Var}(X)\ m^n\ (\mathbb{E}(N_0))^n \sum_{0\le k\le n} (m/\mathbb{E}(N_0))^k.
$$

When $N_0 = 1$ we have $\mathrm{Var}(N_0) = 0$ and $\mathbb{E}(N_0) = 1$. In this case, we have

$$
\mathrm{Var}(N_n) = \mathrm{Var}(X)\ m^{n-1} \sum_{0\le k<n} m^k = \begin{cases} n\ \mathrm{Var}(X) & \text{when} \quad m = 1 \\ \mathrm{Var}(X)\ m^{n-1}\ \frac{m^n-1}{m-1} & \text{when} \quad m \neq 1. \end{cases}
$$

This ends the proof of the exercise. ■

Solution to exercise 13:
We have

$$
\begin{aligned}
\varphi_n(s) &:= \mathbb{E}\left(\mathbb{E}\left(s^{N_n} \mid N_{n-1}\right)\right) = \mathbb{E}\left(\prod_{1\le i\le N_{n-1}} \mathbb{E}\left(s^{X_n^i} \mid N_{n-1}\right)\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(s^X\right)^{N_{n-1}}\right) = \mathbb{E}\left(\varphi_1(s)^{N_{n-1}}\right) = \varphi_{n-1}\left(\varphi_1(s)\right).
\end{aligned}
$$

Recalling that $0^0 = 1$, this implies that

$$\varphi_n(0) = \mathbb{E}(0^{N_n}) = 1 \times \mathbb{P}(N_n = 0) = \mathbb{P}(N_n = 0).$$

For the Bernoulli offspring distribution $p(x) = p^x q^{1-x}$, with $q := (1-p), p \in [0,1]$ and $x \in \{0,1\}$, we have

$$\begin{array}{rcl} \varphi_1(s) & = & q + ps \\ \varphi_2(s) & = & \varphi_1(q+ps) = q + p(q+ps) = q(1+p) + p^2 s \\ \varphi_3(s) & = & \varphi_2(q+ps) = q(1+p) + p^2(q+ps) = q(1+p+p^2) + p^3 s \\ \ldots & = & \ldots \\ \varphi_n(s) & = & q(1+p+p^2+\ldots+p^{n-1}) + p^n s = \dfrac{q}{1-p}\,(1-p^n) + p^n s = (1-p^n) + p^n s. \end{array}$$

The last assertion follows from the fact that $\varphi_n(s) = (1-p^n)+p^n s$ is the moment generating function of a Bernoulli random variable N_n with parameter p^n; that is, we have that

$$\mathbb{P}(N_n = 1) = p^n \quad \text{and} \quad \mathbb{P}(N_n = 0) = 1 - p^n.$$

This ends the proof of the exercise. ■

Solution to exercise 14:
We set $g_0^i = \left(g_0^i(j)\right)_{j \in S}$. In this notation, we have

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\widehat{\xi}_0^i) \mid g_0^i,\ \xi_0\right) = \sum_{1\leq i\leq N_0} g_0^i\left(\xi_0^i\right) f(\xi_0^i).$$

This implies that

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\widehat{\xi}_0^i) \mid \xi_0\right) = \sum_{1\leq i\leq N_0} G\left(\xi_0^i\right)\ f(\xi_0^i)$$

and therefore

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\widehat{\xi}_0^i)\right) = \sum_{1\leq i\leq N_0} \eta_0(Gf) = N_0\eta_0(Gf). \tag{30.18}$$

In the same vein, we have

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\xi_1^i) \mid N_1,\ \widehat{\xi}_0\right) = \sum_{1\leq i\leq N_1} \mathbb{E}\left(f(\xi_1^i) \mid \widehat{\xi}_0^i\right) = \sum_{1\leq i\leq N_1} M(f)(\widehat{\xi}_0^i).$$

Using (30.18), we readily deduce that

$$\mathbb{E}\left(\sum_{1\leq i\leq N_1} f(\xi_1^i)\right) = \mathbb{E}\left(\sum_{1\leq i\leq N_1} M(f)(\widehat{\xi}_0^i)\right) = N_0\ \eta_0(GM(f)). \tag{30.19}$$

In much the same way, if we set $g_1^i = \left(g_1^i(j)\right)_{j\in S}$ then we have

$$\mathbb{E}\left(\sum_{1\leq i\leq N_2} f(\widehat{\xi}_1^i) \mid g_1^i,\ \xi_1\right) = \sum_{1\leq i\leq N_1} g_1^i\left(\xi_1^i\right) f(\xi_1^i).$$

This implies that

$$\mathbb{E}\left(\sum_{1\le i\le N_2} f(\widehat{\xi}_1^i)\mid \xi_1\right)=\sum_{1\le i\le N_1} G\left(\xi_1^i\right)\ f(\xi_1^i).$$

Using (30.18), we readily deduce that

$$\mathbb{E}\left(\sum_{1\le i\le N_2} f(\widehat{\xi}_1^i)\right)=\mathbb{E}\left(\sum_{1\le i\le N_1} G\left(\xi_1^i\right)\ f(\xi_1^i)\right)=N_0\ \eta_0(Q(Gf)). \tag{30.20}$$

Arguing as above we have

$$\mathbb{E}\left(\sum_{1\le i\le N_2} f(\xi_2^i)\mid N_2,\ \widehat{\xi}_1\right)=\sum_{1\le i\le N_2}\mathbb{E}\left(f(\xi_2^i)\mid \widehat{\xi}_1^i\right)=\sum_{1\le i\le N_2} M(f)(\widehat{\xi}_1^i).$$

Using (30.20) we deduce that

$$\mathbb{E}\left(\sum_{1\le i\le N_2} f(\xi_2^i)\right)=\mathbb{E}\left(\sum_{1\le i\le N_2} M(f)(\widehat{\xi}_1^i)\right)=N_0\ \eta_0(Q(GM(f)))=N_0\ \eta_0(Q^2(f)). \tag{30.21}$$

The last assertion is proved using induction. This ends the proof of the exercise. ■

Solution to exercise 15:
By construction, we have

$$\mathbb{P}\left(X_{n+1}=i\mid X_1,\ldots,X_n\right)=\frac{n}{n+\alpha}\ \frac{1}{n}\sum_{1\le p\le n} 1_{X_p}(i)+\frac{\alpha}{n+\alpha}\ \mu(i). \tag{30.22}$$

The number of different tables occupied by the first n customers is defined by

$$T_n:=\sum_{1\le p\le n}\epsilon_p$$

where ϵ_n stands for a sequence of independent Bernoulli random variables with distribution

$$\mathbb{P}\left(\epsilon_n=1\right)=1-\mathbb{P}\left(\epsilon_n=0\right)=\frac{\alpha}{\alpha+(n-1)}.$$

This implies that

$$\sum_{1\le p<n}\int_p^{p+1}\frac{dt}{1+(t/\alpha)}\le\mathbb{E}\left(T_n\right)=\sum_{0\le p<n}\frac{\alpha}{\alpha+p}\le\sum_{1\le p<n}\int_{p-1}^{p}\frac{dy}{1+(t/\alpha)}.$$

We conclude that

$$\int_1^n\frac{dt}{1+(t/\alpha)}=\alpha\ \log\left(\frac{\alpha+n}{\alpha+1}\right)\le\mathbb{E}\left(T_n\right)\le\int_0^{n-1}\frac{dt}{1+(t/\alpha)}=\alpha\ \log\left(1+(n-1)/\alpha\right).$$

The formula

$$\mathbb{P}\left(X_{n+1}=i\mid X_1,\ldots,X_n\right)=\frac{\alpha\mu(i)+V_n(i)}{\alpha+n}\quad\text{with}\quad V_n(i)=\sum_{1\le p\le n}1_{X_p}(i)$$

is a direct consequence of (30.22).

This ends the proof of the exercise. ∎

Solution to exercise 16:

For each $s \in S$ and $x = (x_1, \ldots, x_{n+1})$ we let $t_k(s,x) \in \{1, \ldots, n+1\}$, with $k = 1, \ldots, v_{n+1}(s)$ be the times at which $x_{t_k(s,x)} = s$. In this notation, we have

$$\begin{aligned}\mathbb{P}\left(X_1 = x_1, \ldots, X_{n+1} = x_{n+1}\right) &= \prod_{s \in S} \prod_{0 \leq k < v_{n+1}(s)} \frac{\alpha\mu(s) + k}{\alpha + (t_k(s,x) - 1)} \\ &= \left[\prod_{0 \leq t \leq n} \frac{1}{\alpha + t}\right] \prod_{s \in S} \prod_{0 \leq k < v_{n+1}(s)} (\alpha\mu(s) + k).\end{aligned}$$

In the last assertion we have used the fact that $\mathcal{T}(s,x) := \{t_k(s,x)\ ,\ k = 1, \ldots, v_{n+1}(s)\}$, with $s \in S$ is a partition of the set $\{1, \ldots, n+1\}$

$$\cup_{s \in S} \mathcal{T}(s,x) = \{1, \ldots, n+1\}.$$

The formula (2.6) coincides with (4.9) when $a_s = \alpha\mu(s)$. Following the arguments described on page 79, we conclude that $(X_i)_{i \geq 1}$ can be interpreted as a sequence of independent random variables on the set $S := \{1, \ldots, d\}$ with probability distribution given by (2.7). By the law of large numbers, given U check that $\frac{1}{n}\sum_{1 \leq p \leq n} 1_{X_p}(i)$ converges almost surely to U_i, as $n \uparrow \infty$. In addition, we have

$$\mathbb{E}\left(\frac{1}{n}\sum_{1 \leq p \leq n} 1_{X_p}(i) \mid U\right) = U_i \quad \text{and} \quad \mathrm{Var}\left(\frac{1}{n}\sum_{1 \leq p \leq n} 1_{X_p}(i) \mid U\right) = \frac{1}{n}\ (U_i(1 - U_i)).$$

This ends the proof of the exercise. ∎

Solution to exercise 17:

The first assertion is immediate. In addition, we have that

$$\begin{aligned}S_n(f) &= \frac{n}{n+1}\left(\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) + \frac{1}{n}\ f(X_n)\right) \\ &= \frac{n}{n+1}\ S_{n-1}(f) + \frac{1}{n+1}\ f(X_n).\end{aligned}$$

By construction, we have

$$\mathbb{E}(f(X_{n+1}) \mid X_0, \ldots, X_n) = \epsilon\ S_n(f) + (1 - \epsilon)\ \mu(f).$$

In other words, this yields

$$\mathbb{E}([f(X_{n+1}) - \mu(f)] \mid X_0, \ldots, X_n) = \epsilon\ S_n([f - \mu(f)])\ .$$

Thus, for any function f such that $\mu(f) = 0$, we have

$$\mathbb{E}(f(X_{n+1}) \mid X_0, \ldots, X_n) = \epsilon\ S_n(f) \quad \Rightarrow \quad \eta_{n+1}(f) = \epsilon\ \overline{S}_n(f)\ .$$

Recalling that

$$\mathbb{E}(f(X_{n+1})) = \mathbb{E}(\mathbb{E}(f(X_{n+1}) \mid X_0, \ldots, X_n))$$

we prove that

$$\begin{aligned}
\overline{S}_n(f) &= \frac{n}{n+1}\ \overline{S}_{n-1}(f) + \frac{1}{n+1}\ \mathbb{E}(f(X_n)) \\
&= \frac{n+\epsilon}{n+1} \times \overline{S}_{n-1}(f) \\
&= \frac{n+\epsilon}{n+1} \times \frac{(n-1)+\epsilon}{(n-1)+1} \times \overline{S}_{n-2}(f) = \ldots = \left[\prod_{k=1}^{n} \frac{k+\epsilon}{k+1}\right] \times \mathbb{E}(f(X_0)).
\end{aligned}$$

We observe that

$$k \le t \le k+1 \Rightarrow \log\left(1 - \frac{(1-\epsilon)}{k}\right) \le \log\left(1 - \frac{(1-\epsilon)}{t}\right) \le \log\left(1 - \frac{(1-\epsilon)}{k+1}\right).$$

This implies that

$$\sum_{1\le k\le n} \int_k^{k+1} \log\left(1 - \frac{(1-\epsilon)}{t}\right)\, dt \le \log \alpha_\epsilon(n)$$

and

$$\log \alpha_\epsilon(n) \le \sum_{1\le k\le n} \int_{k+1}^{k+2} \log\left(1 - \frac{(1-\epsilon)}{t}\right)\, dt.$$

This ends the proof of (2.8). Using the estimates

$$\forall x \in [0,1[\qquad -\frac{x}{1-x} \le \log(1-x) \le -x$$

we check that

$$\int_2^{n+2} \log\left(1 - \frac{(1-\epsilon)}{t}\right)\, dt \le -(1-\epsilon)\ \log(1+n/2)$$

and

$$\int_1^{n+1} \log\left(1 - \frac{(1-\epsilon)}{t}\right)\, dt \ge -(1-\epsilon)\ \log(1+n/\epsilon).$$

The end of the proof of the exercise is immediate.

■

Solution to exercise 18:
By construction, we have

$$\begin{aligned}
M(f)(i) = \epsilon\ K(f)(i) + (1-\epsilon)\ \nu(f) \quad &\Rightarrow \quad [M(f)(i) - M(f)(j)] = \epsilon\ [K(f)(i) - K(f)(j)] \\
&\Rightarrow \quad \mathrm{osc}(M(f)) \le \epsilon\ \mathrm{osc}(f).
\end{aligned}$$

Assuming that $\mathrm{osc}(M^n(f)) \le \epsilon^n\ \mathrm{osc}(f)$ is true at rank n, we have

$$\mathrm{osc}(M^{n+1}(f)) = \mathrm{osc}(M^n(M(f))) \le \epsilon^n\ \mathrm{osc}(M(f)) \le \epsilon^{n+1}\ \mathrm{osc}(f).$$

Recall that

$$f = 1_k \Rightarrow M^n(f)(i) = M^n(i,k) = \mathbb{P}(X_n = k | X_0 = i).$$

The end of the proof of the exercise is now clear. ■

Solution to exercise 19: The first assertion is immediate since $\frac{d\overline{W}_t}{dt} = W_t \frac{dW_t}{dt}$. To check the second one, we observe that

$$X_n = a_n X_{n-1} + b_n = \left[\prod_{p=1}^{n} a_p\right] X_0 + \sum_{1\leq p\leq n} \left[\prod_{n\geq q>p} a_q\right] b_p$$

with the sequence of random variables

$$a_n = (1-\epsilon_n) + \epsilon_n \ 4^{-1} = 4^{-\epsilon_n} \quad \text{and} \quad b_n = (1-\epsilon_n)\, h$$

Using the fact that

$$\begin{array}{l} \text{Law}\left((a_1,\ldots,a_{p+1},\ldots,a_n),(b_1,\ldots,b_p,\ldots,b_n)\right) \\ = \\ \text{Law}\left((a_n,\ldots,a_{n-p},\ldots,a_1),(b_n,\ldots,b_{n-p+1},\ldots,b_1)\right). \end{array}$$

we check that

$$\sum_{1\leq p\leq n} \left[\prod_{n\geq q>p} a_q\right] b_p \overset{\text{law}}{=} \sum_{1\leq p\leq n} [a_1\ldots a_{n-p}]\, b_{(n-p)+1} = \sum_{0\leq p<n} [a_1\ldots a_p]\, b_{p+1}$$

The end of the proof of the exercise is now clear. ∎

Solution to exercise 20:
We have

$$\begin{array}{l} \mathbb{P}(X_T = x_{\max}|X_0 = x) \\ = \mathbb{E}\left(\mathbb{P}(X_T = x_{\max} \mid X_1)|X_0 = x\right) \\ = p \ \underbrace{\mathbb{P}(X_T = x_{\max} \mid X_1 = x+1)}_{:=P(x+1)} + (1-p) \ \underbrace{\mathbb{P}(X_T = x_{\max} \mid X_1 = x-1)}_{:=P(x-1)}. \end{array}$$

On the other hand

$$\begin{array}{l} P(x) = pP(x) + qP(x) = p \ P(x+1) + q \ P(x-1) \\ \Rightarrow p\left[P(x+1) - P(x)\right] = q\left[P(x) - P(x-1)\right] \\ \Rightarrow \left[P(x+1) - P(x)\right] = \frac{p}{q}\left[P(x) - P(x-1)\right]. \end{array}$$

Recalling that $P(0) = 0$, this yields

$$[P(2) - P(1)] = \frac{p}{q} \ P(1) \ \Rightarrow [P(3) - P(2)] = \frac{p}{q}\left[P(2) - P(1)\right] = \left(\frac{p}{q}\right)^2 P(1).$$

By a simple induction w.r.t. x we find that

$$\begin{array}{l} [P(x) - P(x-1)] = \left(\frac{p}{q}\right)^{x-1} P(1) \\ \Rightarrow [P(x+1) - P(x)] = \frac{p}{q}\left[P(x) - P(x-1)\right] = \left(\frac{p}{q}\right)^{x} P(1). \end{array}$$

On the other hand, we have

$$P(x+1) = [P(x+1) - P(0)] = \sum_{0 \leq y \leq x} [P(y+1) - P(y)] = P(1) \sum_{0 \leq y \leq x} \left(\frac{p}{q}\right)^y.$$

We end the proof using the fact that

$$x = x_{\max} - 1 \Rightarrow P(x+1) = P(x_{\max}) = 1 = P(1) \sum_{0 \leq y < x_{\max}} \left(\frac{p}{q}\right)^y$$

so that

$$P(1) = 1 / \sum_{0 \leq y < x_{\max}} \left(\frac{p}{q}\right)^y.$$

This implies that

$$P(x+1) = \frac{\sum_{0 \leq y \leq x} \left(\frac{p}{q}\right)^y}{\sum_{0 \leq y < x_{\max}} \left(\frac{p}{q}\right)^y} = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^{x+1}}{1 - \left(\frac{q}{p}\right)^{x_{\max}}} & \text{if} \quad p \neq q \\ \dfrac{(x+1)}{x_{\max}} & \text{if} \quad p = q. \end{cases}$$

This ends the proof of the exercise. ∎