Chapter 2

Section 2.1

It is recommended that students work out the solution starting with the expansion and not simply by plugging the given function into the general coefficient formulas. The solution to **1** is given in full as a template. The solutions to **2**–**16** show only the calculation results. The easily-drawn sketches have been omitted.

1. Here $L = 1$, so we seek an expansion of the form

$$f(x) \sim \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi x) + b_n \sin(n\pi x)].$$

Integrating term-by-term over $[-1,1]$ and taking (2.3) and (2.4) into account, we find that

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\,a_0 \int_{-1}^{1} dx + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-1}^{1} \cos(n\pi x)\,dx}_{0} + b_n \underbrace{\int_{-1}^{1} \sin(n\pi x)\,dx}_{0}\right] = a_0$$

$$\Rightarrow \quad a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx = 1.$$

We now multiply the series by $\cos(m\pi x)$ and then integrate over $[-1,1]$, making use of (2.3), (2.4), and (2.6):

$$\int_{-1}^{1} f(x)\cos(m\pi x)\,dx = \frac{1}{2}\,a_0 \underbrace{\int_{-1}^{1} \cos(m\pi x)\,dx}_{0} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-1}^{1} \cos(n\pi x)\cos(m\pi x)\,dx}_{\begin{cases}0, & n \neq m,\\ 1, & n = m\end{cases}} \right.$$

$$\left. + \, b_n \underbrace{\int_{-1}^{1} \sin(n\pi x)\cos(m\pi x)\,dx}_{0}\right] = a_m$$

$$\Rightarrow \quad a_m = \int_{-1}^{1} f(x)\cos(m\pi x)\,dx = \int_{-1}^{0} \cos(m\pi x)\,dx = \frac{1}{m\pi}\sin(m\pi x)\Big|_{-1}^{0} = 0, \quad m = 1, 2, \ldots.$$

Finally, repeating the above operation but after multiplication by $\sin(n\pi x)$ and applying (2.3), (2.5), and (2.6), we have

$$\int_{-1}^{1} f(x)\sin(m\pi x)\,dx = \frac{1}{2}\,a_0 \underbrace{\int_{-1}^{1} \sin(m\pi x)\,dx}_{0} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-1}^{1} \cos(n\pi x)\sin(m\pi x)\,dx}_{0} \right.$$

$$\left. + \, b_n \underbrace{\int_{-1}^{1} \sin(n\pi x)\sin(m\pi x)\,dx}_{\begin{cases}0, & n \neq m,\\ 1, & n = m\end{cases}}\right] = b_m$$

$$\Rightarrow \quad b_m = \int_{-1}^{1} f(x)\sin(m\pi x)\,dx = \int_{-1}^{0} \sin(m\pi x)\,dx = -\frac{1}{m\pi}[\cos(m\pi x)]_{-1}^{0}$$

$$= -\frac{1}{m\pi}[1-\cos(m\pi)] = [(-1)^m - 1]\,\frac{1}{m\pi}, \quad m = 1, 2, \ldots .$$

Changing m to n in the coefficients, we obtain the series

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}[(-1)^n - 1]\frac{1}{n\pi}\sin(n\pi x).$$

The convergence theorem now yields

$$(\text{series}) = \begin{cases} 1, & -1 < x < 0, \\ 0, & 0 < x < 1, \\ 1/2, & x = -1, 0, 1. \end{cases}$$

2. Proceeding as in **1** but with $L = \pi$, we have

$$a_0 = -3, \quad a_m = 0, \quad b_m = [(-1)^m - 1]\frac{3}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim -\frac{3}{2} + \sum_{n=1}^{\infty}[(-1)^n - 1]\frac{3}{n\pi}\sin(nx),$$

$$(\text{series}) = \begin{cases} 0, & -\pi < x < 0, \\ -3, & 0 < x < \pi, \\ -3/2, & x = -\pi, 0, \pi. \end{cases}$$

3. Proceeding as in **1**, we have

$$a_0 = 1, \quad a_m = 0, \quad b_m = [1 - (-1)^m]\frac{5}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}[1 - (-1)^n]\frac{5}{n\pi}\sin(n\pi x),$$

$$(\text{series}) = \begin{cases} -2 & -1 < x < 0, \\ 3, & 0 < x < 1, \\ 1/2, & x = -1, 0, 1. \end{cases}$$

4. Proceeding as in **1** but with $L = \pi/2$, we have

$$a_0 = 3, \quad a_m = 0, \quad b_m = [1 - (-1)^m]\frac{1}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{3}{2} + \sum_{n=1}^{\infty}[1 - (-1)^n]\frac{1}{n\pi}\sin(2nx),$$

$$(\text{series}) = \begin{cases} 1, & -\pi/2 < x < 0, \\ 2, & 0 < x < \pi/2, \\ 3/2, & x = -\pi/2, 0, \pi/2. \end{cases}$$

5. Proceeding as in **1**, we have

$$a_0 = 2, \quad a_m = 0, \quad b_m = (-1)^{m+1}\frac{2}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin(n\pi x),$$

$$(\text{series}) = \begin{cases} x+1, & -1 < x < 1, \\ 1, & x = -1, 1. \end{cases}$$

6. Proceeding as in **1** but with $L = 2$, we have

$$a_0 = 2, \quad a_m = 0, \quad b_m = (-1)^m \frac{8}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim 1 + \sum_{n=1}^{\infty} (-1)^n \frac{8}{n\pi} \sin \frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} 1-2x, & -2 < x < 2, \\ 1, & x = -2, 2. \end{cases}$$

7. Proceeding as in **1** but with $L = 2$, we have

$$a_0 = 0, \quad a_m = \left[1 - (-1)^m\right] \frac{2}{m^2\pi^2}, \quad b_m = \left[3 - (-1)^m\right] \frac{1}{m\pi}, \quad m = 1, 2, \ldots,$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} \left\{ \left[1 - (-1)^n\right] \frac{2}{n^2\pi^2} \cos \frac{n\pi x}{2} + \left[3 - (-1)^n\right] \frac{1}{n\pi} \sin \frac{n\pi x}{2} \right\},$$

$$(\text{series}) = \begin{cases} -1, & -2 < x < 0, \\ 2-x, & 0 < x < 2, \\ 1/2, & x = 0, \\ -1/2, & x = -2, 2. \end{cases}$$

8. Proceeding as in **1** but with $L = 1/2$, we have

$$a_0 = \tfrac{9}{2}, \quad a_m = [1 - (-1)^m] \frac{1}{m^2\pi^2}, \quad b_m = [3 - (-1)^m 4] \frac{1}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \tfrac{9}{4} + \sum_{n=1}^{\infty} \left\{ [1 - (-1)^n] \frac{1}{n^2\pi^2} \cos(2n\pi x) + [3 - (-1)^n 4] \frac{1}{n\pi} \sin(2n\pi x) \right\},$$

$$(\text{series}) = \begin{cases} 1+2x, & -1/2 < x < 0, \\ 4, & 0 < x < 1/2, \\ 5/2, & x = 0, \\ 2, & x = -1/2, 1/2. \end{cases}$$

9. Proceeding as in **1**, we have

$$a_0 = -\tfrac{1}{2}, \quad a_m = [(-1)^m - 1] \frac{1}{m^2\pi^2}, \quad b_m = -[(-1)^m 2 + 1] \frac{1}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim -\tfrac{1}{4} + \sum_{n=1}^{\infty} \left\{ [(-1)^n - 1] \frac{1}{n^2\pi^2} \cos(n\pi x) - [(-1)^n 2 + 1] \frac{1}{n\pi} \sin(n\pi x) \right\},$$

$$(\text{series}) = \begin{cases} x, & -1 < x < 0, \\ 2x-1, & 0 < x < 1, \\ -1/2, & x = 0, \\ 0, & x = -1, 1. \end{cases}$$

10. Proceeding as in **1** but with $L = \pi$, we have

$$a_0 = \frac{\pi}{2}, \quad a_m = [(-1)^m - 1]\frac{1}{m^2\pi}, \quad b_m = [2 - (-1)^m 2 - (-1)^m 3\pi]\frac{1}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ [(-1)^n - 1]\frac{1}{n^2\pi^2} \cos(nx) + [2 - (-1)^n 2 - (-1)^n 3\pi]\frac{1}{n\pi} \sin(nx) \right\},$$

$$(\text{series}) = \begin{cases} x - 1, & -\pi < x < 0, \\ 2x + 1, & 0 < x < \pi, \\ 0, & x = 0, \\ \pi/2, & x = -\pi, \pi. \end{cases}$$

11. Proceeding as in **1**, we have

$$a_0 = \frac{20}{3}, \quad a_m = (-1)^m \frac{4}{m^2\pi^2}, \quad b_m = (-1)^m \frac{4}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{10}{3} + \sum_{n=1}^{\infty} \left[(-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x) + (-1)^n \frac{4}{n\pi} \sin(n\pi x) \right],$$

$$(\text{series}) = \begin{cases} x^2 - 2x + 3, & -1 < x < 1, \\ 4, & x = -1, 1. \end{cases}$$

12. Proceeding as in **1**, we have

$$a_0 = \frac{7}{3}, \quad a_m = [(-1)^m 2 - 1]\frac{2}{m^2\pi^2}, \quad b_m = [1 - (-1)^m 2]\frac{1}{m\pi} + [1 - (-1)^m]\frac{2}{m^3\pi^3}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{7}{6} + \sum_{n=1}^{\infty} \left\{ [(-1)^n 2 - 1]\frac{2}{n^2\pi^2} \cos(n\pi x) + \left[\frac{1 - (-1)^n 2}{n\pi} + 2\,\frac{1 - (-1)^n}{n^3\pi^3} \right] \sin(n\pi x) \right\},$$

$$(\text{series}) = \begin{cases} x^2, & -1 < x < 0, \\ 1 + 2x, & 0 < x < 1, \\ 1/2, & x = 0, \\ 2, & x = -1,\ 1. \end{cases}$$

13. Proceeding as in **1** but with $L = 2$, we have

$$a_0 = \frac{e^4 - 1}{2e^2}, \quad a_m = (-1)^m \frac{2(e^4 - 1)}{e^2(4 + m^2\pi^2)}, \quad b_m = (-1)^{m+1} \frac{m\pi(e^4 - 1)}{e^2(4 + m^2\pi^2)}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{e^4 - 1}{4e^2} + \sum_{n=1}^{\infty} (-1)^n \frac{e^4 - 1}{e^2(4 + n^2\pi^2)} \left(2\cos\frac{n\pi x}{2} - n\pi \sin\frac{n\pi x}{2} \right),$$

$$(\text{series}) = \begin{cases} e^x, & -2 < x < 2, \\ (e^{-2} + e^2)/2, & x = -2, 2. \end{cases}$$

14. Proceeding as in **1** but with $L = \pi/2$, we have

$$a_0 = \frac{2 + \pi}{\pi}, \quad a_m = \frac{2}{(1 - 4m^2)\pi}, \quad b_m = \frac{4m^2 - 1 + (-1)^m}{(1 - 4m^2)m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{2 + \pi}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{2}{(1 - 4n^2)\pi} \cos(2nx) + \frac{4n^2 - 1 + (-1)^n}{(1 - 4n^2)n\pi} \sin(2nx) \right],$$

$$(\text{series}) = \begin{cases} 1, & -\pi/2 \le x < 0, \\ \sin x, & 0 < x \le \pi/2, \\ 1/2, & x = 0. \end{cases}$$

15. Proceeding as in **1** but with $L = 2$, we have

$$a_0 = \tfrac{9}{4}, \quad a_m = \left[(-1)^m - \cos\frac{m\pi}{2}\right]\frac{2}{m^2\pi^2}, \quad b_m = (-1)^{m+1}\frac{3}{m\pi} + \frac{2}{m^2\pi^2}\sin\frac{m\pi}{2}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \tfrac{9}{8} + \sum_{n=1}^{n}\left\{\left[(-1)^n - \cos\frac{n\pi}{2}\right]\frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2} + \left[(-1)^{n+1}\frac{3}{n\pi} + \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}\right\},$$

$$(\text{series}) = \begin{cases} 0, & -2 < x \le -1, \\ 1 + x, & -1 < x < 2, \\ 3/2, & x = -2,\, 2. \end{cases}$$

16. Proceeding as in **1** but with $L = \pi$, we have

$$a_0 = 5, \quad a_m = \frac{2}{m\pi}\sin\frac{m\pi}{2}, \quad b_m = \frac{2}{m\pi}\left[(-1)^m - \cos\frac{m\pi}{2}\right], \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \tfrac{5}{2} + \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos(nx) + \frac{2}{n\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\sin(nx)\right\},$$

$$(\text{series}) = \begin{cases} 3, & -\pi < x < \pi/2, \\ 1, & \pi/2 < x < \pi, \\ 2, & x = -\pi, \pi/2, \pi. \end{cases}$$

Section 2.2

It is recommended that students work out the solution starting with the expansions and not simply by plugging the given function into the general coefficient formulas. The solution to **1** is given in full as a template. The solutions to **2**–**14** show only the calculation results. The easily drawn sketches have been omitted.

1. First, we note that, since $\sin(n\pi x/L)\sin(m\pi x/L)$ is an even function, we have (see Remark 2.9(ii) and (2.5))

$$\int_0^L \sin\frac{n\pi x}{L}\sin m\pi x L\,dx = \frac{1}{2}\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \ne m, \\ L/2, & n = m. \end{cases}$$

Here $L = 2$, so the sine series has the form

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}.$$

Multiplying the series by $\sin(m\pi x/2)$, integrating over $[0, 2]$, and taking into account the result above, we find that

$$\int_0^2 f(x)\sin\frac{m\pi x}{2}\,dx = \sum_{n=1}^{\infty} b_n \underbrace{\int_0^2 \sin\frac{n\pi x}{2}\sin\frac{m\pi x}{2}\,dx}_{\begin{cases} 0, & n \ne m, \\ 1, & n = m \end{cases}} = b_m$$

$$\Rightarrow \quad b_m = \int_0^2 f(x)\sin\frac{m\pi x}{2}\,dx = \int_1^2 \sin\frac{m\pi x}{2}\,dx = -\frac{2}{m\pi}\cos\frac{m\pi x}{2}\Big|_1^2$$

$$= \frac{2}{m\pi}\left[\cos\frac{m\pi}{2} - \cos(m\pi)\right] = \frac{2}{m\pi}\left[\cos\frac{m\pi}{2} - (-1)^m\right], \quad m = 1, 2, \ldots,$$

which, with m replaced by n, yields the series

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[\cos\frac{n\pi}{2} - (-1)^n \right] \sin\frac{n\pi x}{2}.$$

According to the convergence theorem,

$$(\text{series}) = \begin{cases} 0, & 0 \le x < 1,\ x = 2, \\ 1, & 1 < x < 2, \\ 1/2, & x = 1. \end{cases}$$

Since $\cos(m\pi x/L)$ and $\cos(n\pi x/L)\cos(m\pi x/L)$ are even functions, we have

$$\int_0^L \cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L} \cos\frac{m\pi x}{L}\,dx = 0,$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \ne m, \\ L/2, & n = m. \end{cases}$$

The cosine series for f with $L = 2$ has the form

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}.$$

Integrating the series term-by-term over $[0, 2]$, we have

$$\int_0^2 f(x)\,dx = \frac{1}{2}a_0 \int_0^2 dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^2 \cos\frac{n\pi x}{2}\,dx}_{0} = a_0$$

$$\Rightarrow \quad a_0 = \int_0^2 f(x)\,dx = \int_1^2 dx = 1.$$

Doing the same again but after multiplying the series by $\cos(m\pi x/2)$, we find that

$$\int_0^2 f(x)\cos\frac{m\pi x}{2}\,dx = \frac{1}{2}a_0 \underbrace{\int_0^2 \cos\frac{m\pi x}{2}\,dx}_{0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^2 \cos\frac{n\pi x}{2}\cos\frac{m\pi x}{2}\,dx}_{\begin{cases} 0, & n \ne m, \\ 1, & n = m. \end{cases}} = a_m$$

$$\Rightarrow \quad a_m = \int_0^2 f(x)\cos\frac{m\pi x}{2}\,dx = \int_1^2 \cos\frac{m\pi x}{2}\,dx = \frac{2}{m\pi}\sin\frac{m\pi x}{2}\bigg|_1^2 = -\frac{2}{m\pi}\sin\frac{m\pi}{2}, \qquad m = 1, 2, \ldots,$$

which, with m replaced by n, yields

$$f(x) \sim \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}.$$

By the convergence theorem,

$$(\text{series}) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & 1 < x \le 2, \\ 1/2, & x = 1. \end{cases}$$

2. Proceeding as in **1** but with $L = 2\pi$, we have

$$b_m = \frac{4}{m\pi}\left(1 - \cos\frac{m\pi}{2}\right), \quad m = 1, 2, \dots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} \frac{4}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\sin\frac{nx}{2},$$

$$(\text{series}) = \begin{cases} 2, & 0 < x < \pi, \\ 0, & \pi < x \le 2\pi,\ \ x = 0, \\ 1, & x = \pi, \end{cases}$$

and

$$a_0 = 2, \quad a_m = \frac{4}{m\pi}\sin\frac{m\pi}{2}, \quad m = 1, 2, \dots$$

$$\Rightarrow \quad f(x) \sim 1 + \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi}{2}\cos\frac{nx}{2},$$

$$(\text{series}) = \begin{cases} 2, & 0 \le x < \pi, \\ 0, & \pi < x \le 2\pi, \\ 1, & x = \pi. \end{cases}$$

3. Proceeding as in **1**, we have

$$b_m = \left[1 + (-1)^m - 2\cos\frac{m\pi}{2}\right]\frac{2}{m\pi}, \quad m = 1, 2, \dots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} \left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\right]\frac{2}{n\pi}\sin\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} 1, & 0 < x < 1, \\ -1, & 1 < x < 2, \\ 0, & x = 0, 1, 2, \end{cases}$$

and

$$a_0 = 0, \quad a_m = \frac{4}{m\pi}\sin\frac{m\pi}{2}, \quad m = 1, 2, \dots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} 1, & 0 \le x < 1, \\ -1, & 1 < x \le 2, \\ 0, & x = 1. \end{cases}$$

4. Proceeding as in **1** but with $L = \pi$, we have

$$b_m = \frac{2}{m\pi}\left[5\cos\frac{m\pi}{2} - 2 - 3(-1)^m\right], \quad m = 1, 2, \dots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[5\cos\frac{n\pi}{2} - 2 - 3(-1)^n\right]\sin(nx),$$

$$\text{(series)} = \begin{cases} -2, & 0 < x < \pi/2, \\ 3, & \pi/2 < x < \pi, \\ 1/2, & x = \pi/2, \\ 0, & x = 0, \pi, \end{cases}$$

and

$$a_0 = 1, \quad a_m = -\frac{10}{m\pi}\sin\frac{m\pi}{2}, \quad m = 1, 2, \ldots$$
$$\Rightarrow \quad f(x) \sim \tfrac{1}{2} - \sum_{n=1}^{\infty} \frac{10}{n\pi}\sin\frac{n\pi}{2}\cos(nx),$$
$$\text{(series)} = \begin{cases} -2, & 0 \le x < \pi/2, \\ 3, & \pi/2 < x \le \pi, \\ 1/2, & x = \pi/2. \end{cases}$$

5. Proceeding as in **1** but with $L = 1$, we have

$$b_m = [2 - (-1)^m]\frac{2}{m\pi}, \quad m = 1, 2, \ldots$$
$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty}[2 - (-1)^n]\frac{2}{n\pi}\sin(n\pi x),$$
$$\text{(series)} = \begin{cases} 2 - x, & 0 < x < 1, \\ 0, & x = 0, 1, \end{cases}$$

and

$$a_0 = 3, \quad a_m = [1 - (-1)^m]\frac{2}{m^2\pi^2}, \quad m = 1, 2, \ldots$$
$$\Rightarrow \quad f(x) \sim \tfrac{3}{2} + \sum_{n=1}^{\infty}[1 - (-1)^n]\frac{2}{n^2\pi^2}\cos(n\pi x),$$
$$\text{(series)} = 2 - x, \quad 0 \le x \le 1.$$

6. Proceeding as in **1** but with $L = 2\pi$, we have

$$b_m = [1 - (-1)^m - (-1)^m 6\pi]\frac{2}{m\pi}, \quad m = 1, 2, \ldots$$
$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty}[1 - (-1)^n - (-1)^n 6\pi]\frac{2}{n\pi}\sin\frac{nx}{2},$$
$$\text{(series)} = \begin{cases} 3x + 1, & 0 < x < 2\pi, \\ 0, & x = 0, 2\pi, \end{cases}$$

and

$$a_0 = 2 + 6\pi, \quad a_m = [(-1)^m - 1]\frac{12}{m^2\pi}, \quad m = 1, 2, \ldots$$
$$\Rightarrow \quad f(x) \sim 1 + 3\pi + \sum_{n=1}^{\infty}[(-1)^n - 1]\frac{12}{n^2\pi}\cos\frac{nx}{2},$$
$$\text{(series)} = 3x + 1, \quad 0 \le x \le 2\pi.$$

7. Proceeding as in **1**, we have

$$b_m = -\frac{6}{m\pi}\cos\frac{m\pi}{2} + \frac{4}{m^2\pi^2}\sin\frac{m\pi}{2} + (-1)^m\frac{4}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} \left[-\frac{6}{n\pi} \cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2} + (-1)^n \frac{4}{n\pi} \right] \sin\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} x, & 0 \le x < 1, \\ -2, & 1 < x < 2, \\ -1/2, & x = 1, \\ 0, & x = 2, \end{cases}$$

and

$$a_0 = -\tfrac{3}{2}, \quad a_m = \frac{6}{m\pi} \sin\frac{m\pi}{2} + \frac{4}{m^2\pi^2} \cos\frac{m\pi}{2} - \frac{4}{m^2\pi^2}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim -\frac{3}{4} + \sum_{n=1}^{\infty} \left(\frac{6}{n\pi} \sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2} \cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} \right) \cos\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} x, & 0 \le x < 1, \\ -2, & 1 < x \le 2, \\ -1/2, & x = 1. \end{cases}$$

8. Proceeding as in **1**, we have

$$b_m = \left[2\cos\frac{m\pi}{2} - 1 - (-1)^m 5 \right] \frac{2}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} \left[2\cos\frac{n\pi}{2} - 1 - (-1)^n 5 \right] \frac{2}{n\pi} \sin\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} 2x - 1, & 0 < x < 1, \\ 2x + 1, & 1 < x < 2, \\ 2, & x = 1, \\ 0, & x = 0, 2, \end{cases}$$

and

$$a_0 = 4, \quad a_m = -\left[2 - (-1)^m 2 + m\pi \sin\frac{m\pi}{2} \right] \frac{4}{m^2\pi^2}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim 2 + \sum_{n=1}^{\infty} -\left[2 - (-1)^n 2 + n\pi \sin\frac{n\pi}{2} \right] \frac{4}{n^2\pi^2} \cos\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} 2x - 1, & 0 \le x < 1, \\ 2x + 1, & 1 < x \le 2, \\ 2, & x = 1. \end{cases}$$

9. Proceeding as in **1**, we have

$$b_m = -\frac{6}{m\pi} \cos\frac{m\pi}{2} + \frac{4}{m\pi} + \frac{8}{m^2\pi^2} \sin\frac{m\pi}{2} + (-1)^m \frac{2}{m\pi}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} \left[-\frac{6}{n\pi} \cos\frac{n\pi}{2} + \frac{8}{n^2\pi^2} \sin\frac{n\pi}{2} + \frac{4}{n\pi} + (-1)^n \frac{2}{n\pi} \right] \sin\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} 2 + x, & 0 < x < 1, \\ 1 - x, & 1 < x < 2, \\ 0, & x = 0,\ 2, \\ 3/2, & x = 1, \end{cases}$$

and

$$a_0 = 2, \quad a_m = \frac{6}{m\pi}\sin\frac{m\pi}{2} + \frac{8}{m^2\pi^2}\cos\frac{m\pi}{2} - \frac{4}{m^2\pi^2} + (-1)^{m+1}\frac{4}{m^2\pi^2}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim 1 + \sum_{n=1}^{\infty}\left[\frac{6}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} + (-1)^{n+1}\frac{4}{n^2\pi^2}\right]\cos\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} 2+x, & 0 \le x < 1, \\ 1-x, & 1 < x \le 2, \\ 3/2, & x = 1. \end{cases}$$

10. Proceeding as in **1**, we have

$$b_m = \{(-1)^m 8 - 8 - [1 + (-1)^m 5]m^2\pi^2\}\frac{2}{m^3\pi^3}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty}\{(-1)^n 8 - 8 - [1 + (-1)^n 5]n^2\pi^2\}\frac{2}{n^3\pi^3}\sin\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} x^2 + x - 1, & 0 < x < 2, \\ 0, & x = 0, 2, \end{cases}$$

and

$$a_0 = \tfrac{8}{3}, \quad a_m = [(-1)^m 5 - 1]\frac{4}{m^2\pi^2}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \tfrac{4}{3} + \sum_{n=1}^{\infty}[(-1)^n 5 - 1]\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2},$$

$$(\text{series}) = x^2 + x - 1, \quad 0 \le x \le 2.$$

11. Proceeding as in **1**, we have

$$b_m = \left[(-1)^m 8 + m^2\pi^2 - (8 + 3m^2\pi^2)\cos\frac{m\pi}{2} + 2m\pi\sin\frac{m\pi}{2}\right]\frac{2}{m^3\pi^3}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty}\left[(-1)^n 8 + n^2\pi^2 - (8 + 3n^2\pi^2)\cos\frac{n\pi}{2} + 2n\pi\sin\frac{n\pi}{2}\right]\frac{2}{n^3\pi^3}\sin\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} x+1, & 0 < x < 1, \\ x^2 - 2x, & 1 < x \le 2, \\ 1/2, & x = 1, \\ 0, & x = 0, \end{cases}$$

and

$$a_0 = \tfrac{5}{6}, \quad a_m = \left\{[(-1)^m 2 - 1]2m\pi + 2m\pi\cos\frac{m\pi}{2} + (8 + 3m^2\pi^2)\sin\frac{m\pi}{2}\right\}\frac{2}{m^3\pi^3}, \quad m = 1, 2, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{5}{12} + \sum_{n=1}^{\infty}\left\{[(-1)^n 2 - 1]2n\pi + 2n\pi\cos\frac{n\pi}{2} + (8 + 3n^2\pi^2)\sin\frac{n\pi}{2}\right\}\frac{2}{n^3\pi^3}\cos\frac{n\pi x}{2},$$

$$(\text{series}) = \begin{cases} x+1, & 0 \le x < 1, \\ x^2 - 2x, & 1 < x \le 2, \\ 1/2, & x = 1. \end{cases}$$

12. Proceeding as in **1** but with $L = 1$, we have

$$b_m = \{1 - (-1)^m + [2 - (-1)^m - (-1)me]m^2\pi^2\}\frac{2}{m\pi + m^3\pi^3}, \quad m = 1, 2, \ldots$$
$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty}\{1 - (-1)^n + [2 - (-1)^n - (-1)^n e]n^2\pi^2\}\frac{2}{n\pi + n^3\pi^3}\sin(n\pi x),$$
$$(\text{series}) = \begin{cases} 1 + e^x, & 0 < x < 1, \\ 0, & x = 0, 1, \end{cases}$$

and

$$a_0 = 2e, \quad a_m = \frac{2[(-1)^m e - 1]}{1 + m^2\pi^2}, \quad m = 1, 2, \ldots$$
$$\Rightarrow \quad f(x) \sim e + \sum_{n=1}^{\infty}\frac{2[(-1)^n e - 1]}{1 + n^2\pi^2}\cos\frac{n\pi x}{2},$$
$$(\text{series}) = 1 + e^x, \quad 0 \le x \le 1.$$

13. Expanding only the term x and proceeding as in **1** but with $L = \pi$, we have

$$b_m = (-1)^{m+1}\frac{2}{m}, \quad m = 1, 2, \ldots$$
$$\Rightarrow \quad f(x) \sim \sin x + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx) = 3\sin x + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx),$$
$$(\text{series}) = \begin{cases} x + \sin x, & 0 \le x < \pi, \\ 0, & x = \pi, \end{cases}$$

and, considering the case $m = 1$ separately,

$$a_0 = \frac{4}{\pi} + \pi, \quad a_1 = -\frac{4}{\pi}, \quad a_m = \frac{2[2m^2 + (-1)^m - 1]}{(1 - m^2)m^2\pi}, \quad m = 2, 3, \ldots$$
$$\Rightarrow \quad f(x) \sim \frac{2}{\pi} + \frac{\pi}{2} - \frac{4}{\pi}\cos x + \sum_{n=2}^{\infty}\frac{2[2n^2 + (-1)^n - 1]}{(1 - n^2)n^2\pi}\cos(nx),$$
$$(\text{series}) = x + \sin x, \quad 0 \le x \le \pi.$$

The coefficient a_1 in the cosine series is computed separately because the general coefficient a_m assumes the indeterminate form $0/0$ for $m = 1$.

14. Proceeding as in **1** but with $L = \pi$ and considering the case $m = 1$ separately to avoid the form $0/0$, we have

$$b_1 = -\frac{1}{\pi}, \quad b_m = \frac{2}{\pi}\left\{\left[(-1)^m - \cos\frac{m\pi}{2}\right]\frac{1}{m} + \frac{1}{m^2 - 1}\left(m - \sin\frac{m\pi}{2}\right)\right\}, \quad m = 2, 3, \ldots$$
$$\Rightarrow \quad f(x) \sim -\frac{1}{\pi}\sin x + \sum_{n=2}^{\infty}\frac{2}{\pi}\left\{\left[(-1)^n - \cos\frac{n\pi}{2}\right]\frac{1}{n} + \frac{1}{n^2 - 1}\left(n - \sin\frac{n\pi}{2}\right)\right\}\sin(nx),$$
$$(\text{series}) = \begin{cases} \cos x, & 0 < x < \pi/2, \\ -1, & \pi/2 < x < \pi, \\ -1/2, & x = \pi/2, \\ 0, & x = 0, \pi, \end{cases}$$

and

$$a_0 = \frac{2}{\pi} - 1, \quad a_1 = \frac{2}{\pi} + \frac{1}{2}, \quad a_m = \frac{2}{\pi}\left(\frac{1}{m}\sin m\pi 2 + \frac{1}{1 - m^2}\cos\frac{m\pi}{2}\right)\sin\frac{m\pi}{2}, \quad m = 2, 3, \ldots$$

$$\Rightarrow \quad f(x) \sim \frac{1}{\pi} - \frac{1}{2} + \left(\frac{2}{\pi} + \frac{1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2}{\pi}\left(\frac{1}{n}\ \sin n\pi 2 + \frac{1}{1-n^2}\ \cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}\cos(nx),$$

$$(\text{series}) = \begin{cases} \cos x, & 0 \le x < \pi/2, \\ -1, & \pi/2 < x \le \pi, \\ -1/2, & x = \pi/2. \end{cases}$$