Chapter 2

Torsion Spring Oscillator with Dry Friction

2.2 Review of the Principal Formulas

The differential equation of motion of an oscillator acted upon by dry friction:

$$J\ddot{\varphi} = -D(\varphi + \varphi_\mathrm{m}) \qquad \text{for} \qquad \dot{\varphi} > 0, \tag{2.1}$$
$$J\ddot{\varphi} = -D(\varphi - \varphi_\mathrm{m}) \qquad \text{for} \qquad \dot{\varphi} < 0, \tag{2.2}$$

where φ_m is the angle corresponding to the boundaries of the dead zone. If in addition, viscous friction is present, a term proportional to the angular velocity is also present:

$$\ddot{\varphi} = -\omega_0^2(\varphi + \varphi_\mathrm{m}) - 2\gamma\dot{\varphi} \qquad \text{for} \qquad \dot{\varphi} > 0, \tag{2.3}$$
$$\ddot{\varphi} = -\omega_0^2(\varphi - \varphi_\mathrm{m}) - 2\gamma\dot{\varphi} \qquad \text{for} \qquad \dot{\varphi} < 0, \tag{2.4}$$

where ω_0 is the natural frequency of oscillations in the absence of friction:

$$\omega_0^2 = \frac{D}{J}. \tag{2.5}$$

The damping factor γ that characterizes the viscous friction is related to the quality factor Q by the equation:

$$Q = \frac{\omega_0}{2\gamma}. \tag{2.6}$$

The boundary value of the amplitude that delimits the two cases in which the effects either of viscous friction or of dry friction predominate:

$$a = \frac{4\varphi_\mathrm{m}}{\gamma T} = \frac{4}{\pi}\varphi_\mathrm{m}Q \approx \varphi_\mathrm{m}Q. \tag{2.7}$$

2.3 Questions and Problems with Answers and Solutions

2.3.1 Damping Caused by Dry Friction

The strength of dry friction in the system is characterized by the width of the dead zone. This interval is defined in the program when you input the value of the angle φ_m which sets the limits of the dead zone on both sides of the middle position at which the spring is unstrained. Total width of this dead zone is $2\varphi_{\rm m}$. The value of $\varphi_{\rm m}$ must be expressed in degrees.

2.3.1.1 Oscillations without Dry Friction. Begin with the value $\varphi_{\rm m} = 0$ corresponding to the absence of dry friction. Show that in this case the system displays the familiar behavior of a linear oscillator, i.e., simple harmonic oscillations with a constant amplitude in the absence of friction and with an exponentially decaying amplitude in the presence of viscous friction. The strength of viscous friction is characterized by the quality factor Q.

2.3.1.2 Dry Friction after an Initial Displacement. To display the role of dry friction clearly, choose a large value of the angle $\varphi_{\rm m}$ which determines the limits of the dead zone (say, 15 to 20 degrees), and let viscous friction be zero. Such conditions are somewhat unrealistic. They are far unlike the situation characteristic of measuring instruments using a needle, such as moving-coil galvanometers. These instruments are constructed so that the dead zone is as small as possible, and critical viscous damping is deliberately introduced in order to avoid taking a reading from an oscillating needle. When an instrument is critically damped, its moving system just fails to oscillate, and it comes to rest in the shortest possible time. If the dead zone is narrow, the needle stops at a position very close to the dial point which gives the true value of the measured quantity. Here, on the other hand, conditions are chosen to clarify the role of dry friction.

(a) What can you say about the succession of maximal deflections if damping is caused only by dry friction with the ideal z-characteristic? What is the law of their diminishing? How is the difference of consecutive maximal deflections related to the half-width of the dead zone?

> When dry friction is described by an idealized z-characteristic, each half-cycle of oscillations (while the flywheel is rotating in one direction) can be treated as a half-cycle of harmonic (pure sinusoidal) oscillation about a mid-point which is displaced to one of the boundaries of the dead zone. The displacement of the middle point is caused by the action of constant torque of dry friction. During the next half-cycle, while the flywheel is rotating in the opposite direction, the mid-point of sinusoidal oscillation is displaced to the other boundary of the dead zone. This alternation of the mid-points means that the amplitude of oscillations is reduced during each half-cycle by the same value. This value equals the width of the dead zone $2\varphi_m$. The

succession of maximal displacements from the middle point of the dead zone forms an arithmetic progression. The oscillations cease when the next maximal displacement occurs within the dead zone.

(b) Let the angle φ_{m} that defines the boundaries of the stagnation zone be, say, $15°$, the initial angle of deflection φ_0 be $160°$, and the initial angular velocity be zero. Calculate the point of the dial at which the needle eventually comes to rest. How many semi-ellipses form the phase trajectory of this motion, from its initial point to the point at which the motion stops? Verify your predictions by simulating the motion on the computer.

The width of the dead zone $2\varphi_m$ is $30°$; so after one complete cycle the maximal deflection is $160° - 60° = 100°$ to the right side. After one more full cycle the maximal deflection is $40°$. After the next half-cycle $\varphi_{\max} = -10°$. At this point the motion ceases. The phase trajectory consists of five semi-ellipses.

(c) In the graph of the time dependence of the deflection angle, where are the midpoints of the half-cycles of the sinusoidal oscillations located? Note how these individual segments of the sine curves are joined to form a continuous plot of damped oscillations.

All odd half-cycles are described by segments of sine curves the mid-points of which coincides with the right boundary of the dead zone (located at $\varphi_m = 15°$). The mid-point of the segments of sine curves for even half-cycles is at the left boundary ($-15°$) of the dead zone.

(d) In the graph of the angular velocity versus time, note the abrupt bends in the curve at the instants at which the midpoints abruptly replace one another. What is the reason for these bends? Prove that these instants are separated by half the period of harmonic oscillations in the absence of dry friction. (Note that points on the time scale of the graphs correspond to integral multiples of the period.)

The kinks on the graph of the angular velocity occur when the graph crosses the abscissa axis, i.e., at instants when the direction of rotation is reversed. At these moments the sign of the torque due to dry friction changes while the magnitude of the torque remains the same. The angular acceleration of the flywheel (the slope of the velocity graph) changes abruptly.

2.3.1.3* **Dry Friction after an Initial Push.** Choose different initial conditions: let the initial deflection be zero, and the initial angular velocity be, say, $2\omega_0$ (where ω_0 is the natural frequency of oscillations). Use the same value $\varphi_{\mathrm{m}} = 15°$ as above.

(a) Calculate the maximal deflection of the needle.

After an initial push to the right, the flywheel rotates clockwise, and so the torque of dry friction displaces the mid-point of sinusoidal oscillation to the left boundary of the dead zone, located at $\varphi = -\varphi_m = -15^\circ = -0.26$ rad. The motion starts at $\varphi = 0$, that is, at the point, displaced from the mid-point of oscillations by an angle φ_m to the right. The amplitude of this oscillation (i.e., the maximal displacement from the mid-point) is $\sqrt{\Omega^2/\omega_0^2 + \varphi_m^2}$. Thus, the maximal deflection $\varphi_{\max}$ with respect to the zero point of the dial is:

$$\varphi_{\max} = \sqrt{\Omega^2/\omega_0^2 + \varphi_m^2} - \varphi_m \approx \Omega/\omega_0 - \varphi_m = 1.75 \text{ rad} = 100.6^\circ.$$

(b) To what position on the dial does the needle point when oscillations cease? How many turns are present in the complete phase trajectory of this motion? Verify your answer using a simulation experiment on the computer.

After this maximal deflection, further motion has the same character as in the preceding problem 2.3.1.2. After the next complete cycle the maximal deflection equals $100.6^\circ - 60^\circ = 40.6^\circ$. Then one more half-cycle of oscillation brings the needle to its greatest deflection to the left, -10.6°. Here the motion ceases. The entire phase trajectory makes $1\frac{3}{4}$ turns: a quarter of a turn during the first stage of motion after the initial push to the first maximal deflection, and one-and-a-half turns during the following three half-cycles.

2.3.1.4* **Damping by Dry Friction at Various Initial Conditions.** Assuming the same width of the dead zone as above, calculate the maximal angle of deflection and the final position on the dial to which the needle points when oscillations cease, for the more complicated initial conditions:

(a) The initial deflection angle $\varphi(0) = 135^\circ$, and the initial angular velocity $\dot{\varphi}(0) = 1.5\omega_0$ (ω_0 is the natural frequency of the oscillator). Verify your calculated values in a simulation experiment on the computer.

Initially the flywheel rotates clockwise, and the constant torque of dry friction displaces the mid-point of the first half-cycle of sinusoidal oscillation to the left boundary of the dead zone, i.e., to the point $-\varphi_m = -15^\circ = -0.26$ rad. In this case the motion starts at $\varphi = 135^\circ$, that is, at the point, displaced from the mid-point of oscillations by an angle $135^\circ + \varphi_m = 135^\circ + 15^\circ = 150^\circ = 2.62$ rad to the right. The amplitude of this oscillation (i.e., the maximal displacement from the mid-point) is $\sqrt{1.5^2 + 2.62^2} = 3.02 \text{ rad} = 172.9^\circ$. Thus, the maximal deflection $\varphi_{\max}$ with respect to the zero point of the dial is $172.9^\circ - 15^\circ = 157.9^\circ$. After two more complete cycles the needle reaches the point $157.9^\circ - 120^\circ = 37.9^\circ$, and after the next half-cycle it stops dead at the point -7.9° of the dial.

(b) The initial deflection angle $\varphi(0) = -135°$, and the initial angular velocity $\dot{\varphi}(0) = 1.5\omega_0$. Verify your calculated values in a simulation experiment on the computer.

When the motion starts at $\varphi = -135°$, the initial point is displaced from the mid-point of the first sinusoidal oscillation through an angle $-135° + 15° = -120° = -2.09$ rad. The maximal displacement from the mid-point is $\sqrt{1.5^2 + 2.09^2} = 2.58$ rad = $147.6°$. Hence the maximal deflection from the zero point of the dial is $147.6° - 15° = 132.6°$. After two more complete cycles the motion stops at the point $132.6° - 120° = 12.6°$ of the dial.

2.3.1.5* Energy Dissipation at Dry Friction.

(a) The graph of the total mechanical energy versus the angle of deflection consists of rectilinear segments joining the slopes of the parabolic potential well (when you work in the section "Energy transformations" of the relevant computer program). Suggest an explanation.

The mechanical energy of the oscillator is dissipated because friction does work. The magnitude of the torque of dry friction is constant (it is independent of the angular velocity). Therefore, this work is proportional to the angular path $\Delta\varphi$ of the flywheel. The total mechanical energy E of the oscillator depends linearly on the angular path $\Delta\varphi$ through which the flywheel has passed after the initial moment:

$$E(\Delta\varphi) = E_0 - N_{\text{fr}}\Delta\varphi. \tag{2.8}$$

Here E_0 is the initial total energy (when $\Delta\varphi = 0$). Since E depends linearly on $\Delta\varphi$ the graph of total energy E versus the angle of deflection φ consists of rectilinear segments lying between the walls of the parabolic potential well. All the segments have the same inclination, which is determined by the constant magnitude of the frictional torque $N_{\text{fr}} = D\varphi_m$.

(b) Letting the initial angular velocity $\dot{\varphi}(0) = 2\omega_0$, where ω_0 is the natural frequency, and using energy considerations, calculate the entire angular path of the flywheel, excited from the midpoint of the dead zone by an initial push if the half-width of the dead zone $\varphi_{\text{m}} = 10°$.

To estimate the entire angular path $\Delta\varphi$ of the flywheel after its excitation by an initial push from the mid-point of the dead zone, we can express in Eq. (2.8) the initial energy E_0 of the oscillator in terms of its initial angular velocity Ω: $E_0 = J\Omega^2/2$. Assuming the final energy at the end of motion to be zero, we find from Eq. (2.8):

$$\Delta\varphi = \frac{1}{2}J\Omega^2/N_{\text{fr}} = \frac{1}{2}\frac{J\Omega^2}{D\varphi_m} = \frac{1}{2}\left(\frac{\Omega}{\omega_0}\right)^2\frac{1}{\varphi_m}.$$

Substituting $\Omega = 2\omega_0$ and $\varphi_m = 10° = 0.175$ rad, we find that the entire angular path $\Delta\varphi = 11.46$ rad = $656°$.

To solve the problem, we can also use the method described in the solution of Problem 2.3.1.3. After the initial push, the first maximal deflection equals $\Omega/\omega_0 - \varphi_m \approx 105°$, the next one to the left side equals $-85°$, and so on. The final position of the flywheel is $-5°$. The entire angular path is thus $655°$.

2.3.1.6 Oscillations in the Case of a Narrow Dead Zone. Choose a small value for the angle $\varphi_{\rm m}$ (less than $5°$), and set the initial angular displacement to be many times the width of the dead zone, $2\varphi_{\rm m}$.

(a) How many cycles does the flywheel execute before stopping?

Let us take $\varphi_m = 3°$, and the initial angular displacement $\varphi_0 = 120°$. After each complete cycle the amplitude is reduced by $4\varphi_m = 12°$. Therefore the oscillator will execute 10 cycles before stopping.

(b) When the number of cycles is large, the plots clearly demonstrate the linear decay of the amplitude and the equidistant character of the loops in the phase diagram. What can you say about the time dependence of the total energy, averaged over a cycle?

For damping due to dry friction, the amplitude of oscillation decreases linearly with time. Averaged over a period, the value of the total energy is proportional to the square of the amplitude. Therefore the decrease with time of the averaged total energy during a large number of cycles is described by a quadratic function: $\langle E_{\rm tot}\rangle \sim (t_f - t)^2$, where t_f is the time when the oscillations stop.

2.3.2 Influence of Viscous Friction

2.3.2.1* **Transition of the Main Role from Viscous to Dry Friction.** When damping is caused both by dry and viscous friction, it is interesting to observe the change in the character of damping when the main contribution passes from viscous to dry friction.

Let the angle $\varphi_{\rm m}$ that determines the width of the dead zone be about $1°$ and let the quality factor Q which characterizes the strength of viscous friction be about 30. Let the initial angular deflection be $120°$ and the initial angular velocity be zero.

(a) Does dry or does viscous friction determine the initial damping effects?

The damping of oscillations is influenced mainly by viscous friction under conditions in which the amplitude exceeds a value $\tilde{a} = (4/\pi)Q\varphi_m \approx Q\varphi_m$. In the case under consideration $\tilde{a} \approx 30°$. The initial displacement from the equilibrium position is $120°$. Therefore the initial damping is determined mainly by viscous friction.

(b) At what value of the amplitude does the character of damping change? How does this change manifest itself on the plots of time dependence of the angle of deflection and of the angular velocity? On the phase trajectory?

During the initial stage of damping the amplitude diminishes almost exponentially, and the coils of the phase trajectory gradually condense. the character of damping change when the amplitude approaches the value $\tilde{a} \approx q\varphi_m \approx 30°$. further damping is determined mainly by dry friction. the amplitude diminishes almost linearly, the coils of the phase trajectory become nearly equidistant, and oscillations cease after a final number of cycles.

2.3.2.2* **Both Viscous and Dry Friction.** Let the boundaries of the stagnation interval be at $\varphi_{\mathrm{m}} = 10°$ and the quality factor $Q = 5$. Let the initial velocity be $2\omega_0$ and the initial deflection be zero.

(a) Calculate the maximal angular deflection of the needle at these initial conditions. Verify your answer experimentally.

(b) What kind of friction, dry or viscous, initially dominates the damping of oscillations?

(c)** Let the boundaries of the stagnation zone be determined by the angle $\varphi_{\mathrm{m}} = 10°$. Let the quality factor Q be 3, the initial deflection be $65°$, and the initial angular velocity be $-2\omega_0$. Calculate the maximal angular deflection of the needle in the direction opposite the initial deflection. Verify your answer experimentally.

(a,b,c) When the oscillator experiences both dry and viscous friction, its motion is described by the following differential equation:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2(\varphi \pm \varphi_m) = 0, \tag{2.9}$$

where the sign before φ_m depends on the direction of rotation of the flywheel (on the sign of $\dot{\varphi}$). The general solution of Eq. (2.9) can be expressed in the form:

$$\varphi(t) = Ae^{-\gamma t}\cos(\omega_0 t + \delta) \mp \varphi_m, \tag{2.10}$$

where the upper sign corresponds to $\dot{\varphi} > 0$. Differentiating Eq. (2.10) with respect to time yields the following expression for the angular velocity:

$$\dot{\varphi}(t) = Ae^{-\gamma t}[\omega_0 \sin(\omega_0 t + \delta) + \gamma\cos(\omega_0 t + \delta)]. \tag{2.11}$$

The values of the constants A and δ are determined by the initial conditions $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = \Omega$:

$$\varphi_0 = A\cos\delta \mp \varphi_m, \qquad \Omega = -A(\omega_0\sin\delta + \gamma\cos\delta). \tag{2.12}$$

From Eqs. (2.12) we obtain the following expressions that are convenient for finding δ and A:

$$\tan\delta = -\frac{\gamma}{\omega_0} - \frac{\Omega/\omega_0}{\varphi_0 \pm \varphi_m} = -\frac{1}{2Q} - \frac{\Omega/\omega_0}{\varphi_0 \pm \varphi_m}, \quad A = \frac{\varphi_0 \pm \varphi_m}{\cos\delta}. \tag{2.13}$$

At the instant t_1 of maximal deflection the angular velocity becomes zero: $\dot\varphi(t_1) = 0$. Introducing the notation $\delta_1 = \omega_0 t_1 + \delta$ for the phase of sine and cosine functions in Eq. (2.11) at $t = t_1$, we obtain from Eq. (2.11) the following expression for finding δ_1:

$$\tan\delta_1 = -\frac{\gamma}{\omega_0} = -\frac{1}{2Q}. \tag{2.14}$$

We can substitute the calculated values of A, δ_1, and $t_1 = (\delta_1 - \delta)/\omega_0$ in Eq. (2.10) in order to determine the maximal deflection:

$$\varphi_{\max} = \varphi(t_1) = Ae^{-\gamma t_1}\cos\delta_1 \mp \varphi_m = A\exp\left(-\frac{\delta_1 - \delta}{2Q}\right)\cos\delta_1 \mp \varphi_m. \tag{2.15}$$

(a) In this case $\varphi_m = 10° = 0.174$ rad, $Q = 5$, $\varphi(0) = \varphi_0 = 0$, $\dot\varphi(0) = \Omega = 2\omega_0$. For the initial motion $\dot\varphi > 0$, and we must take the upper sign in the above formulas. From Eq. (2.13) we find $\tan\delta = -11.56$, $\delta = -1.48$ rad, $A = 2.025$ rad. We note that the value of δ is close to $-\pi/2$, and the time dependence of the angle of deflection, according to Eq. (2.10), is approximately described by the function $\varphi(t) \approx Ae^{-\gamma t}\sin\omega_0 t + \varphi_m$. So the maximal deflection occurs approximately at $\omega_0 t_1 = \pi/2$, where the sine function reaches its maximum. Thus, the value of $\delta_1 = \omega_0 t + \delta$ in Eq. (2.15) is approximately zero.

To determine the time instant t_1 of maximal deflection more accurately, we should use Eq. (2.14): $\delta_1 = \arctan(-1/10) \approx -0.1$, and $\cos\delta_1 = 0.995$. Substituting these values into Eq. (2.15), we find the maximal deflection:

$$\varphi_{\max} = 2.025 \cdot \exp(-0.138) \cdot 0.995 - 0.174 = 1.581 \text{ rad} = 90.6°.$$

(b) In this case $\varphi(0) = \varphi_0 = 65° = 1.134$ rad, $\dot\varphi(0) = \Omega = -2\omega_0$. The motion initially occurs to the left ($\dot\varphi < 0$), and so we must take the lower sign in the formulas. The amplitude A and the initial phase δ can be calculated with the help of Eqs. (2.13): $\tan\delta = 1.917$, $\delta = 1.09$, $\cos\delta = 0.463$, $A = 2.075$ rad. Then from Eq. (2.14) we find $\tan\delta_1 = -1/6$. It is clear from Eq. (2.10) that in this case we should take the smallest value of δ_1 which is greater than $\pi/2$:

$\delta_1 = \arctan(-1/6) + \pi = 2.976$, $\cos\delta_1 = -0.9864$. Substituting these values into Eq. (2.15), we finally obtain:

$$\varphi_{\max} = 2.075 \cdot \exp\left(\frac{1.09 - 2.976}{2 \cdot 3}\right) \cdot (-0.9864) + 0.174 =$$
$$= -1.32 \text{ rad} = -76^\circ.$$

2.3.2.3 Dry Friction and Critical Viscous Damping.

(a) Choose the quality factor Q to be near the critical value 0.5 and investigate the character of damping experimentally. Where within the limits of the dead zone is the needle most likely to stop if the quality Q is slightly greater than the critical value? Give some physical explanation of your observations.

If the quality factor Q is slightly greater than the critical value 0.5, the needle, being displaced from the equilibrium position beyond the limits of the dead zone and released without a push, is most likely to stop almost at once after crossing the nearest boundary of the dead zone. The same is true if the displaced flywheel is given a push in the direction of its initial displacement.

The situation may be different if the displaced flywheel is given a push towards the equilibrium position. The flywheel may pass through the whole dead zone, if the initial shove is strong enough. Then, after reaching some extreme point, the needle will move back, and is most likely to stop within the dead zone almost at once after crossing its nearest boundary (this boundary lies in the opposite direction with regard to the initial displacement). When the initial shove towards the equilibrium position is not strong enough for the flywheel to cross the entire dead zone, the needle may stop at any point of the dead zone. This point depends on the initial angular velocity.

(b) Where would the needle stop if the quality factor Q is less than 0.5 (that is, if the system is overdamped)? Does the answer depend on the initial conditions?

In general, the behavior of an overdamped system differs from the situation described above only in one aspect: the needle approaches the nearest boundary of the dead zone from the outside during an infinitely long time, but does not cross it. So the needle is most likely to stop exactly at one of the boundaries, except the cases when the displaced flywheel is given a shove towards the equilibrium position strong enough to cross the nearest boundary of the dead zone, but not strong enough to cross the entire dead zone.