Chapter 2

Combinatorial Methods

2.2 COUNTING PRINCIPLES

1. The total number of six-digit numbers is $9\times10\times10\times10\times10\times10 = 9\times10^5$ since the first digit cannot be 0. The number of six-digit numbers without the digit five is $8 \times 9 \times 9 \times 9 \times 9 \times 9 = 8 \times 9^5$. Hence there are $9 \times 10^5 - 8 \times 9^5 = 427,608$ six-digit numbers that contain the digit five.

2. **(a)** $5^5 = 3125$. **(b)** $5^3 = 125$.

3. There are $26 \times 26 \times 26 = 17,576$ distinct sets of initials. Hence in any town with more than 17,576 inhabitants, there are at least two persons with the same initials. The answer to the question is therefore yes.

4. $4^{15} = 1,073,741,824$.

5. $\dfrac{2}{2^{23}} = \dfrac{1}{2^{22}} \approx 0.00000024$.

6. **(a)** $52^5 = 380,204,032$. **(b)** $52 \times 51 \times 50 \times 49 \times 48 = 311,875,200$.

7. $6/36 = 1/6$.

8. **(a)** $\dfrac{4 \times 3 \times 2 \times 2}{12 \times 8 \times 8 \times 4} = \dfrac{1}{64}$. **(b)** $1 - \dfrac{8 \times 5 \times 6 \times 2}{12 \times 8 \times 8 \times 4} = \dfrac{27}{32}$.

9. $\dfrac{1}{4^{15}} \approx 0.00000000093$.

10. $26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$.

11. There are $26^3 \times 10^2 = 1,757,600$ such codes; so the answer is positive.

12. 2^{nm}.

13. $(2+1)(3+1)(2+1) = 36$. (See the solution to Exercise 24.)

14. There are $(2^6 - 1)2^3 = 504$ possible sandwiches. So the claim is true.

15. **(a)** $5^4 = 625.$ **(b)** $5^4 - 5 \times 4 \times 3 \times 2 = 505.$

16. $2^{12} = 4096.$

17. $1 - \dfrac{48 \times 48 \times 48 \times 48}{52 \times 52 \times 52 \times 52} = 0.274.$

18. $10 \times 9 \times 8 \times 7 = 5040.$ **(a)** $9 \times 9 \times 8 \times 7 = 4536;$ **(b)** $5040 - 1 \times 1 \times 8 \times 7 = 4984.$

19. $1 - \dfrac{(N-1)^n}{N^n}.$

20. By Example 2.6, the probability is 0.507 that among Jenny and the next 22 people she meets randomly there are two with the same birthday. However, it is quite possible that one of these two persons is not Jenny. Let n be the minimum number of people Jenny must meet so that the chances are better than even that someone shares her birthday. To find n, let A denote the event that among the next n people Jenny meets randomly someone's birthday is the same as Jenny's. We have

$$P(A) = 1 - P(A^c) = 1 - \frac{364^n}{365^n}.$$

To have $P(A) > 1/2$, we must find the smallest n for which

$$1 - \frac{364^n}{365^n} > \frac{1}{2},$$

or

$$\frac{364^n}{365^n} < \frac{1}{2}.$$

This gives

$$n > \frac{\log \frac{1}{2}}{\log \frac{364}{365}} = 252.652.$$

Therefore, for the desired probability to be greater than 0.5, n must be 253. To some this might seem counterintuitive.

21. Draw a tree diagram for the situation in which the salesperson goes from I to B first. In this situation, you will find that in 7 out of 23 cases, she will end up staying at island I. By symmetry, if she goes from I to H, D, or F first, in each of these situations in 7 out of 23 cases she will end up staying at island I. So there are $4 \times 23 = 92$ cases altogether and in $4 \times 7 = 28$ of them the salesperson will end up staying at island I. Since $28/92 = 0.3043$, the answer is 30.43%. Note that the probability that the salesperson will end up staying at island I is *not* 0.3043 because not all of the cases are equiprobable.

22. He is at 0 first, next he goes to 1 or -1. If at 1, then he goes to 0 or 2. If at -1, then he goes to 0 or -2, and so on. Draw a tree diagram. You will find that after walking 4 blocks, he is at one of the points $4, 2, 0, -2$, or -4. There are 16 possible cases altogether. Of these 6 end up at 0, none at 1, and none at -1. Therefore, the answer to (a) is 6/16 and the answer to (b) is 0.

23. We can think of a number less than 1,000,000 as a six-digit number by allowing it to start with 0 or 0's. With this convention, it should be clear that there are 9^6 such numbers without the digit five. Hence the desired probability is $1 - (9^6/10^6) = 0.469$.

24. Divisors of N are of the form $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $e_i = 0, 1, 2, \ldots, n_i$, $1 \leq i \leq k$. Therefore, the answer is $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)$.

25. There are 6^4 possibilities altogether. In 5^4 of these possibilities there is no 3. In 5^3 of these possibilities only the first die lands 3. In 5^3 of these possibilities only the second die lands 3, and so on. Therefore, the answer is

$$\frac{5^4 + 4 \times 5^3}{6^4} = 0.868.$$

26. Any subset of the set {salami, turkey, bologna, corned beef, ham, Swiss cheese, American cheese} except the empty set can form a reasonable sandwich. There are $2^7 - 1$ possibilities. To every sandwich a subset of the set {lettuce, tomato, mayonnaise} can also be added. Since there are 3 possibilities for bread, the final answer is $(2^7 - 1) \times 2^3 \times 3 = 3048$ and the advertisement is true.

27. $\dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{11^8} = 0.031.$

28. For $i = 1, 2, 3$, let A_i be the event that no one departs at stop i. The desired quantity is $P(A_1^c A_2^c A_3^c) = 1 - P(A_1 \cup A_2 \cup A_3)$. Now

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\ &\qquad - P(A_1 A_2) - P(A_1 A_3) - P(A_2 A_3) + P(A_1 A_2 A_3) \\ &= \frac{2^6}{3^6} + \frac{2^6}{3^6} + \frac{2^6}{3^6} - \frac{1}{3^6} - \frac{1}{3^6} - \frac{1}{3^6} + 0 = \frac{7}{27}. \end{aligned}$$

Therefore, the desired probability is $1 - (7/27) = 20/27$.

29. For $0 \leq i \leq 9$, the sum of the first two digits is i in $(i + 1)$ ways. Therefore, there are $(i + 1)^2$ numbers in the given set with the sum of the first two digits equal to the sum of the last two digits and equal to i. For $i = 10$, there are 9^2 numbers in the given set with the sum of the first two digits equal to the sum of the last two digits and equal to 10. For $i = 11$, the corresponding numbers are 8^2 and so on. Therefore, there are altogether

$$1^2 + 2^2 + \cdots + 10^2 + 9^2 + 8^2 + \cdots + 1^2 = 670$$

numbers with the desired probability and hence the answer is $670/10^4 = 0.067$.

30. Let A be the event that the number selected contains at least one 0. Let B be the event that it contains at least one 1 and C be the event that it contains at least one 2. The desired quantity is $P(ABC) = 1 - P(A^c \cup B^c \cup C^c)$, where

$$\begin{aligned} P(A^c \cup B^c \cup C^c) = P(A^c) &+ P(B^c) + P(C^c) \\ &- P(A^cB^c) - P(A^cC^c) - P(B^cC^c) + P(A^cB^cC^c) \\ &= \frac{9^r}{9 \times 10^{r-1}} + \frac{8 \times 9^{r-1}}{9 \times 10^{r-1}} + \frac{8 \times 9^{r-1}}{9 \times 10^{r-1}} - \frac{8^r}{9 \times 10^{r-1}} - \frac{8^r}{9 \times 10^{r-1}} \\ &- \frac{7 \times 8^{r-1}}{9 \times 10^{r-1}} + \frac{7^r}{9 \times 10^{r-1}}. \end{aligned}$$

2.3 PERMUTATIONS

1. The answer is $\dfrac{1}{4!} = \dfrac{1}{24} \approx 0.0417$.

2. $3! = 6$.

3. $\dfrac{8!}{3!\,5!} = 56$.

4. The probability that John will arrive right after Jim is $7!/8!$ (consider Jim and John as one arrival). Therefore, the answer is $1 - (7!/8!) = 0.875$.

Another Solution: If Jim is the last person, John will not arrive after Jim. Therefore, the remaining seven can arrive in $7!$ ways. If Jim is not the last person, the total number of possibilities in which John will not arrive right after Jim is $7 \times 6 \times 6!$. So the answer is

$$\frac{7! + 7 \times 6 \times 6!}{8!} = 0.875.$$

5. **(a)** $3^{12} = 531,441$. **(b)** $\dfrac{12!}{6!\,6!} = 924$. **(c)** $\dfrac{12!}{3!\,4!\,5!} = 27,720$.

6. ${}_6P_2 = 30$.

7. $\dfrac{20!}{4!\,3!\,5!\,8!} = 3,491,888,400$.

8. $\dfrac{(5 \times 4 \times 7) \times (4 \times 3 \times 6) \times (3 \times 2 \times 5)}{3!} = 50,400$.

9. There are $8!$ schedule possibilities. By symmetry, in $8!/2$ of them Dr. Richman's lecture precedes Dr. Chollet's and in $8!/2$ ways Dr. Richman's lecture precedes Dr. Chollet's. So the answer is $8!/2 = 20,160$.

10. $\dfrac{11!}{3!\,2!\,3!\,3!} = 92,400.$

11. $1 - (6!/6^6) = 0.985.$

12. **(a)** $\dfrac{11!}{4!\,4!\,2!} = 34,650.$

(b) Treating all P's as one entity, the answer is $\dfrac{10!}{4!\,4!} = 6300.$

(c) Treating all I's as one entity, the answer is $\dfrac{8!}{4!\,2!} = 840.$

(d) Treating all P's as one entity, and all I's as another entity, the answer is $\dfrac{7!}{4!} = 210.$

(e) By (a) and (c), The answer is $840/34650 = 0.024.$

13. $\left(\dfrac{8!}{2!\,3!\,3!}\right)\Big/6^8 = 0.000333.$

14. $\left(\dfrac{9!}{3!\,3!\,3!}\right)\Big/52^9 = 6.043 \times 10^{-13}.$

15. $\dfrac{m!}{(n+m)!}.$

16. Each girl and each boy has the same chance of occupying the 13th chair. So the answer is $12/20 = 0.6$. This can also be seen from $\dfrac{12 \times 19!}{20!} = \dfrac{12}{20} = 0.6.$

17. $\dfrac{12!}{12^{12}} = 0.000054.$

18. Look at the five math books as one entity. The answer is $\dfrac{5! \times 18!}{22!} = 0.00068.$

19. $1 - \dfrac{{}_9P_7}{9^7} = 0.962.$

20. $\dfrac{2 \times 5! \times 5!}{10!} = 0.0079.$

21. $n!/n^n.$

22. $1-(6!/6^6)=0.985$.

23. Suppose that A and B are not on speaking terms. ${}_{134}P_4$ committees can be formed in which neither A serves nor B; $4\times{}_{134}P_3$ committees can be formed in which A serves and B does not. The same numbers of committees can be formed in which B serves and A does not. Therefore, the answer is ${}_{134}P_4+2(4\times{}_{134}P_3)=326,998,056$.

24. **(a)** m^n. **(b)** ${}_mP_n$. **(c)** $n!$.

25. $\Big(3\cdot\dfrac{8!}{2!\,3!\,2!\,1!}\Big)\Big/6^8=0.003$.

26. **(a)** $\dfrac{20!}{39\times 37\times 35\times\cdots\times 5\times 3\times 1}=7.61\times 10^{-6}$.

(b) $\dfrac{1}{39\times 37\times 35\times\cdots\times 5\times 3\times 1}=3.13\times 10^{-24}$.

27. Thirty people can sit in $30!$ ways at a round table. But for each way, if they rotate 30 times (everybody move one chair to the left at a time) no new situations will be created. Thus in $30!/30=29!$ ways 15 married couples can sit at a round table. Think of each married couple as one entity and note that in $15!/15=14!$ ways 15 such entities can sit at a round table. We have that the 15 couples can sit at a round table in $(2!)^{15}\cdot 14!$ different ways because if the couples of each entity change positions between themselves, a new situation will be created. So the desired probability is

$$\frac{14!(2!)^{15}}{29!}=3.23\times 10^{-16}.$$

The answer to the second part is

$$\frac{24!(2!)^5}{29!}=2.25\times 10^{-6}.$$

28. In $13!$ ways the balls can be drawn one after another. The number of those in which the first white appears in the second or in the fourth or in the sixth or in the eighth draw is calculated as follows. (These are Jack's turns.)

$$8\times 5\times 11!+8\times 7\times 6\times 5\times 9!+8\times 7\times 6\times 5\times 4\times 5\times 7!$$
$$+\,8\times 7\times 6\times 5\times 4\times 3\times 2\times 5\times 5!=2,399,846,400.$$

Therefore, the answer is $2,399,846,400/13!=0.385$.

2.4 COMBINATIONS

1. $\binom{20}{6} = 38,760.$

2. $\sum_{i=51}^{100} \binom{100}{i} = 583,379,627,841,332,604,080,945,354,060 \approx 5.8 \times 10^{29}.$

3. $\binom{20}{6}\binom{25}{6} = 6,864,396,000.$

4. $\dfrac{\binom{12}{3}\binom{40}{2}}{\binom{52}{5}} = 0.066.$

5. $\binom{N-1}{n-1}\Big/\binom{N}{n} = \dfrac{n}{N}.$

6. $\binom{5}{3}\binom{2}{2} = 10.$

7. $\binom{8}{3}\binom{5}{2}\binom{3}{3} = 560.$

8. $\binom{18}{6} + \binom{18}{4} = 21,624.$

9. $\binom{10}{5}\Big/\binom{12}{7} = 0.318.$

10. The coefficient of 2^3x^9 in the expansion of $(2+x)^{12}$ is $\binom{12}{9}$. Therefore, the coefficient of x^9 is $2^3\binom{12}{9} = 1760.$

11. The coefficient of $(2x)^3(-4y)^4$ in the expansion of $(2x-4y)^7$ is $\binom{7}{4}$. Thus the coefficient of x^3y^2 in this expansion is $2^3(-4)^4\binom{7}{4} = 71,680.$

12. $\binom{9}{3}\Big[\binom{6}{4} + 2\binom{6}{3}\Big] = 4620.$

13. **(a)** $\binom{10}{5}\Big/2^{10} = 0.246$; **(b)** $\sum_{i=5}^{10}\binom{10}{i}\Big/2^{10} = 0.623$.

14. If their minimum is larger than 5, they are all from the set $\{6, 7, 8, \dots, 20\}$. Hence the answer is $\binom{15}{5}\Big/\binom{20}{5} = 0.194$.

15. **(a)** $\dfrac{\binom{6}{2}\binom{28}{4}}{\binom{34}{6}} = 0.228$; **(b)** $\dfrac{\binom{6}{6}+\binom{6}{6}+\binom{10}{6}+\binom{12}{6}}{\binom{34}{6}} = 0.00084$.

16. $\dfrac{\binom{50}{5}\binom{150}{45}}{\binom{200}{50}} = 0.00206$.

17. $\sum_{i=0}^{n} 2^i\binom{n}{i} = \sum_{i=0}^{n}\binom{n}{i}2^i 1^{n-i} = (2+1)^n = 3^n$.

$$\sum_{i=0}^{n} x^i\binom{n}{i} = \sum_{i=0}^{n}\binom{n}{i}x^i 1^{n-i} = (x+1)^n.$$

18. $\left[\binom{6}{2}5^4\right]\Big/6^6 = 0.201$.

19. $2^{12}\Big/\binom{24}{12} = 0.00151$.

20. **Royal Flush:** $\dfrac{4}{\binom{52}{5}} = 0.0000015$.

Straight flush: $\dfrac{36}{\binom{52}{5}} = 0.000014$.

Four of a kind: $\dfrac{13\times 12\binom{4}{1}}{\binom{52}{5}} = 0.00024$.

Full house: $$\frac{13\binom{4}{3}\cdot 12\binom{4}{2}}{\binom{52}{5}} = 0.0014.$$

Flush: $$\frac{4\binom{13}{5} - 40}{\binom{52}{5}} = 0.002.$$

Straight: $$\frac{10(4)^5 - 40}{\binom{52}{5}} = 0.0039.$$

Three of a kind: $$\frac{13\binom{4}{3}\cdot\binom{12}{2}4^2}{\binom{52}{5}} = 0.021.$$

Two pairs: $$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\cdot 11\binom{4}{1}}{\binom{52}{5}} = 0.048.$$

One pair: $$\frac{13\binom{4}{2}\cdot\binom{12}{3}4^3}{\binom{52}{5}} = 0.42.$$

None of the above: $1-$ the sum of all of the above cases $= 0.5034445$.

21. The desired probability is
$$\frac{\binom{12}{6}\binom{12}{6}}{\binom{24}{12}} = 0.3157.$$

22. The answer is the solution of the equation $\binom{x}{3} = 20$. This equation is equivalent to $x(x-1)(x-2) = 120$ and its solution is $x = 6$.

23. There are $9\times 10^3 = 9000$ four-digit numbers. From every 4-combination of the set $\{0, 1, \dots, 9\}$, exactly one four-digit number can be constructed in which its ones place is less than its tens place, its tens place is less than its hundreds place, and its hundreds place is less than its thousands place. Therefore, the number of such four-digit numbers is $\binom{10}{4} = 210$. Hence the desired probability is 0.023333.

24.

$$
\begin{aligned}
(x+y+z)^2 &= \sum_{n_1+n_2+n_3=2} \frac{n!}{n_1!\,n_2!\,n_3!}\,x^{n_1}y^{n_2}z^{n_3} \\
&= \frac{2!}{2!\,0!\,0!}\,x^2y^0z^0 + \frac{2!}{0!\,2!\,0!}\,x^0y^2z^0 + \frac{2!}{0!\,0!\,2!}\,x^0y^0z^2 \\
&\quad + \frac{2!}{1!\,1!\,0!}\,x^1y^1z^0 + \frac{2!}{1!\,0!\,1!}\,x^1y^0z^1 + \frac{2!}{0!\,1!\,1!}\,x^0y^1z^1 \\
&= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.
\end{aligned}
$$

25. The coefficient of $(2x)^2(-y)^3(3z)^2$ in the expansion of $(2x-y+3z)^7$ is $\dfrac{7!}{2!\,3!\,2!}$. Thus the coefficient of $x^2y^3z^2$ in this expansion is $2^2(-1)^3(3)^2\dfrac{7!}{2!\,3!\,2!} = -7560$.

26. The coefficient of $(2x)^3(-y)^7(3)^3$ in the expansion of $(2x-y+3)^{13}$ is $\dfrac{13!}{3!\,7!\,3!}$. Therefore, the coefficient of x^3y^7 in this expansion is $2^3(-1)^7(3)^3\dfrac{13!}{3!\,7!\,3!} = -7,413,120$.

27. In $\dfrac{52!}{13!\,13!\,13!\,13!} = \dfrac{52!}{(13!)^4}$ ways 52 cards can be dealt among four people. Hence the sample space contains $52!/(13!)^4$ points. Now in $4!$ ways the four different suits can be distributed among the players; thus the desired probability is $4!/[52!/(13!)^4] \approx 4.47 \times 10^{-28}$.

28. The theorem is valid for $k=2$; it is the binomial expansion. Suppose that it is true for all integers $\leq k-1$. We show it for k. By the binomial expansion,

$$
\begin{aligned}
(x_1+x_2+\cdots+x_k)^n &= \sum_{n_1=0}^{n} \binom{n}{n_1} x_1^{n_1}(x_2+\cdots+x_k)^{n-n_1} \\
&= \sum_{n_1=0}^{n} \binom{n}{n_1} x_1^{n_1} \sum_{n_2+n_3+\cdots+n_k=n-n_1} \frac{(n-n_1)!}{n_2!\,n_3!\,\cdots\,n_k!}\,x_2^{n_2}x_3^{n_3}\cdots x_k^{n_k} \\
&= \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1}\frac{(n-n_1)!}{n_2!\,n_3!\,\cdots\,n_k!}\,x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k} \\
&= \sum_{n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1!\,n_2!\,\cdots\,n_k!}\,x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}.
\end{aligned}
$$

29. We must have 8 steps. Since the distance from M to L is ten 5-centimeter intervals and the first step is made at M, there are 9 spots left at which the remaining 7 steps can be made. So the answer is $\binom{9}{7} = 36$.

30. **(a)** $\dfrac{\binom{2}{1}\binom{98}{49} + \binom{98}{48}}{\binom{100}{50}} = 0.753$; **(b)** $2^{50} \Big/ \binom{100}{50} = 1.16 \times 10^{-14}$.

31. **(a)** It must be clear that

$$
\begin{aligned}
n_1 &= \binom{n}{2}\\
n_2 &= \binom{n_1}{2} + nn_1\\
n_3 &= \binom{n_2}{2} + n_2(n+n_1)\\
n_4 &= \binom{n_3}{2} + n_3(n+n_1+n_2)\\
&\vdots\\
n_k &= \binom{n_{k-1}}{2} + n_{k-1}(n+n_1+\cdots+n_{k-1}).
\end{aligned}
$$

(b) For $n = 25,000$, successive calculations of n_k's yield,

$$
\begin{aligned}
n_1 &= 312,487,500,\\
n_2 &= 48,832,030,859,381,250,\\
n_3 &= 1,192,283,634,186,401,370,231,933,886,715,625,\\
n_4 &= 710,770,132,174,366,339,321,713,883,042,336,781,236,\\
&\quad\; 550,151,462,446,793,456,831,056,250.
\end{aligned}
$$

For $n = 25,000$, the total number of all possible hybrids in the first four generations, $n_1+n_2+n_3+n_4$, is 710,770,132,174,366,339,321,713,883,042,337,973,520,184,337, 863,865,857,421,889,665,625. This number is approximately 710×10^{63}.

32. For $n = 1$, we have the trivial identity

$$x + y = \binom{1}{0}x^0y^{1-0} + \binom{1}{1}x^1y^{1-1}.$$

Assume that

$$(x+y)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-1-i}.$$

This gives

$$\begin{aligned}
(x+y)^n &= (x+y) \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-1-i} \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} x^{i+1} y^{n-1-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right] x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} + y^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.
\end{aligned}$$

33. The desired probability is computed as follows.

$$\binom{12}{6}\left[\binom{30}{2}\binom{28}{2}\binom{26}{2}\binom{24}{2}\binom{22}{2}\binom{20}{2}\binom{18}{3}\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}\right] \Big/ 12^{30} \approx 0.000346.$$

34. **(a)** $\dfrac{\binom{10}{6}2^6}{\binom{20}{6}} = 0.347;$ **(b)** $\dfrac{\binom{10}{1}\binom{9}{4}2^4}{\binom{20}{6}} = 0.520;$

(c) $\dfrac{\binom{10}{2}\binom{8}{2}2^2}{\binom{20}{6}} = 0.130;$ **(d)** $\dfrac{\binom{10}{3}}{\binom{20}{6}} = 0.0031.$

35. $\dfrac{\binom{26}{13}\binom{26}{13}}{\binom{52}{26}} = 0.218.$

36. Let a 6-element combination of a set of integers be denoted by $\{a_1, a_2, \ldots, a_6\}$, where $a_1 < a_2 < \cdots < a_6$. It can be easily verified that the function $h\colon \mathcal{B} \to \mathcal{A}$ defined by

$$h\big(\{a_1, a_2, \ldots, a_6\}\big) = \{a_1, a_2 + 1, \ldots, a_6 + 5\}$$

is one-to-one and onto. Therefore, there is a one-to-one correspondence between $\mathcal{B}$ and $\mathcal{A}$. This shows that the number of elements in $\mathcal{A}$ is $\binom{44}{6}$. Thus the probability that no consecutive integers are selected among the winning numbers is $\binom{44}{6}\Big/\binom{49}{6} \approx 0.505$. This implies that the probability of at least two consecutive integers among the winning numbers is approximately $1 - 0.505 = 0.495$. Given that there are 47 integers between 1 and 49, this high probability might be counter-intuitive. Even without knowledge of expected value, a keen student might observe that, on the average, there should be $(49-1)/7 = 6.86$ numbers between each a_i and a_{i+1}, $1 \le i \le 5$. Thus he or she might erroneously think that it is unlikely to obtain consecutive integers frequently.

37. **(a)** Let E_i be the event that car i remains unoccupied. The desired probability is

$$P(E_1^c E_2^c \cdots E_n^c) = 1 - P(E_1 \cup E_2 \cup \cdots \cup E_n).$$

Clearly,

$$\begin{aligned}
P(E_i) &= \frac{(n-1)^m}{n^m}, \quad 1 \le i \le n;\\
P(E_i E_j) &= \frac{(n-2)^m}{n^m}, \quad 1 \le i, j \le n,\ i \ne j;\\
P(E_i E_j E_k) &= \frac{(n-3)^m}{n^m}, \quad 1 \le i, j, k \le n,\ i \ne j \ne k;
\end{aligned}$$

and so on. Therefore, by the inclusion-exclusion principle,

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{(n-i)^m}{n^m}.$$

So

$$\begin{aligned}
P(E_1^c E_2^c \cdots E_n^c) &= 1 - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{(n-i)^m}{n^m} = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \frac{(n-i)^m}{n^m}\\
&= \frac{1}{n^m} \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (n-i)^m.
\end{aligned}$$

(b) Let F be the event that cars $1, 2, \ldots, n-r$ are all occupied and the remaining cars are unoccupied. The desired probability is $\binom{n}{r} P(F)$. Now by part (a), the number of ways m

passengers can be distributed among $n-r$ cars, no car remaining unoccupied is

$$\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}(n-r-i)^m.$$

So

$$P(F)=\frac{1}{n^m}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}(n-r-i)^m$$

and hence the desired probability is

$$\frac{1}{n^m}\binom{n}{r}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}(n-r-i)^m.$$

38. Let the n indistinguishable balls be represented by n identical oranges and the n distinguishable cells be represented by n persons. We should count the number of different ways that the n oranges can be divided among the n persons, and the number of different ways in which exactly one person does not get an orange. The answer to the latter part is $n(n-1)$ since in this case one person does not get an orange, one person gets exactly two oranges, and the remaining persons each get exactly one orange. There are n choices for the person who does not get an orange and $n-1$ choices for the person who gets exactly two oranges; $n(n-1)$ choices altogether. To count the number of different ways that the n oranges can be divided among the n persons, add $n-1$ identical apples to the oranges and note that by Theorem 2.4, the total number of permutations of these $n-1$ apples and n oranges is $\dfrac{(2n-1)!}{n!\,(n-1)!}$. (We can arrange $n-1$ identical apples and n identical oranges in a row in $(2n-1)!/\big[n!\,(n-1)!\big]$ ways.) Now each one of these $\dfrac{(2n-1)!}{n!\,(n-1)!}=\dbinom{2n-1}{n}$ permutations corresponds to a way of dividing the n oranges among the n persons and vice versa. Give all of the oranges preceding the first apple to the first person, the oranges between the first and the second apples to the second person, the oranges between the second and the third apples to the third person and so on. Therefore, if, for example, an apple appears in the beginning of the permutation, the first person does not get an orange, and if two apples are at the end of the permutations, the $(n-1)$st and the nth persons get no oranges. Thus the answer is $n(n-1)\Big/\dbinom{2n-1}{n}$.

39. The left side of the identity is the binomial expansion of $(1-1)^n=0$.

40. Using the hint, we have

$$\begin{aligned}
&\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r}\\
&=\binom{n}{0}+\binom{n+2}{1}-\binom{n+1}{0}+\binom{n+3}{2}-\binom{n+2}{1}\\
&\quad+\binom{n+4}{3}-\binom{n+3}{2}+\cdots+\binom{n+r+1}{r}-\binom{n+r}{r-1}\\
&=\binom{n}{0}-\binom{n+1}{0}+\binom{n+r+1}{r}=\binom{n+r+1}{r}.
\end{aligned}$$

41. The identity expresses that to choose r balls from n red and m blue balls, we must choose either r red balls, 0 blue balls or $r-1$ red balls, one blue ball or $r-2$ red balls, two blue balls or $\cdots$ 0 red balls, r blue balls.

42. Note that $\dfrac{1}{i+1}\dbinom{n}{i}=\dfrac{1}{n+1}\dbinom{n+1}{i+1}$. Hence

$$\text{The given sum } =\frac{1}{n+1}\left[\binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{n+1}\right]=\frac{1}{n+1}(2^{n+1}-1).$$

43. $\left[\dbinom{5}{2}3^3\right]\Big/4^5=0.264.$

44. **(a)** $P_N=\dfrac{\dbinom{t}{m}\dbinom{N-t}{n-m}}{\dbinom{N}{n}}.$

(b) From part (a), we have

$$\frac{P_N}{P_{N-1}}=\frac{(N-t)(N-n)}{N(N-t-n+m)}.$$

This implies $P_N>P_{N-1}$ if and only if $(N-t)(N-n)>N(N-t-n+m)$ or, equivalently, if and only if $N\le nt/m$. So P_N is increasing if and only if $N\le nt/m$. This shows that the maximum of P_N is at $[nt/m]$, where by $[nt/m]$ we mean the greatest integer $\le nt/m$.

45. The sample space consists of $(n+1)^4$ elements. Let the elements of the sample be denoted by x_1, x_2, x_3, and x_4. To count the number of samples (x_1, x_2, x_3, x_4) for which $x_1+x_2=x_3+x_4$, let $y_3=n-x_3$ and $y_4=n-x_4$. Then y_3 and y_4 are also random elements from the set $\{0,1,2,\dots,n\}$. The number of cases in which $x_1+x_2=x_3+x_4$ is identical to the number of cases in which $x_1+x_2+y_3+y_4=2n$. By Example 2.23, the number of nonnegative integer solutions to this equation is $\binom{2n+3}{3}$. However, this also counts the solutions in which one of x_1, x_2, y_3, and y_4 is greater than n. Because of the restrictions $0 \le x_1, x_2, y_3, y_4 \le n$, we must subtract, from this number, the total number of the solutions in which one of x_1, x_2, y_3, and y_4 is greater than n. Such solutions are obtained by finding all nonnegative integer solutions of the equation $x_1 + x_2 + y_3 + y_4 = n - 1$, and then adding $n + 1$ to exactly one of x_1, x_2, y_3, and y_4. Their count is 4 times the number of nonnegative integer solutions of $x_1 + x_2 + y_3 + y_4 = n - 1$; that is, $4\binom{n+2}{3}$. Therefore, the desired probability is

$$\frac{\binom{2n+3}{3} - 4\binom{n+2}{3}}{(n+1)^4} = \frac{2n^2 + 4n + 3}{3(n+1)^3}.$$

46. **(a)** The $n - m$ unqualified applicants are "ringers." The experiment is not affected by their inclusion, so that the probability of any one of the qualified applicants being selected is the same as it would be if there were only qualified applicants. That is, $1/m$. This is because in a random arrangement of m qualified applicants, the probability that a given applicant is the first one is $1/m$.

(b) Let A be the event that a given qualified applicant is hired. We will show that $P(A) = 1/m$. Let E_i be the event that the given qualified applicant is the ith applicant interviewed, *and* he or she is the first qualified applicant to be interviewed. Clearly,

$$P(A) = \sum_{i=1}^{n-m+1} P(E_i),$$

where

$$P(E_i) = \frac{{}_{n-m}P_{i-1} \cdot 1 \cdot (n-i)!}{n!}.$$

Therefore,

$$\begin{aligned}
P(A) &= \sum_{i=1}^{n-m+1} \frac{{}_{n-m}P_{i-1} \cdot (n-i)!}{n!} \\
&= \sum_{i=1}^{n-m+1} \frac{\dfrac{(n-m)!}{(n-m-i+1)!}(n-i)!}{n!} \\
&= \sum_{i=1}^{n-m+1} \frac{1}{m!} \cdot \frac{1}{\dfrac{n!}{m!\,(n-m)!}} \cdot \frac{(n-i)!}{(n-m-i+1)!\,(m-1)!}(m-1)!
\end{aligned}$$

$$= \sum_{i=1}^{n-m+1} \frac{1}{m} \cdot \frac{1}{\binom{n}{m}} \binom{n-i}{m-1}$$

$$= \frac{1}{m} \cdot \frac{1}{\binom{n}{m}} \sum_{i=1}^{n-m+1} \binom{n-i}{m-1}. \tag{4}$$

To calculate $\sum_{i=1}^{n-m+1} \binom{n-i}{m-1}$, note that $\binom{n-i}{m-1}$ is the coefficient of x^{m-1} in the expansion of $(1+x)^{n-i}$. Therefore, $\sum_{i=1}^{n-m+1} \binom{n-i}{m-1}$ is the coefficient of x^{m-1} in the expansion of

$$\sum_{i=1}^{n-m+1} (1+x)^{n-i} = \frac{(1+x)^n - (1+x)^{m-1}}{x}.$$

This shows that $\sum_{i=1}^{n-m+1} \binom{n-i}{m-1}$ is the coefficient of x^m in the expansion of $(1+x)^n - (1+x)^{m-1}$, which is $\binom{n}{m}$. So (4) implies that

$$P(A) = \frac{1}{m} \cdot \frac{1}{\binom{n}{m}} \cdot \binom{n}{m} = \frac{1}{m}.$$

47. Clearly, $N = 6^{10}$, $N(A_i) = 5^{10}$, $N(A_iA_j) = 4^{10}$, $i \neq j$, and so on. So S_1 has $\binom{6}{1}$ equal terms, S_2 has $\binom{6}{2}$ equal terms, and so on. Therefore, the solution is

$$6^{10} - \binom{6}{1}5^{10} + \binom{6}{2}4^{10} - \binom{6}{3}3^{10} + \binom{6}{4}2^{10} - \binom{6}{5}1^{10} + \binom{6}{6}0^{10} = 16,435,440.$$

48. $|A_0| = \frac{1}{2}\binom{n}{3}\binom{n-3}{3}$, $\quad |A_1| = \frac{1}{2}\binom{n}{3}\binom{3}{1}\binom{n-3}{2}$, $\quad |A_2| = \frac{1}{2}\binom{n}{3}\binom{3}{2}\binom{n-3}{1}$.

The answer is

$$\frac{|A_0|}{|A_0| + |A_1| + |A_2|} = \frac{(n-4)(n-5)}{n^2+2}.$$

49. The coefficient of x^n in $(1+x)^{2n}$ is $\binom{2n}{n}$. Its coefficient in $(1+x)^n(1+x)^n$ is

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}$$
$$=\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2,$$

since $\binom{n}{i}=\binom{n}{n-1}$, $\;0\le i\le n.$

50. Consider a particular set of k letters. Let M be the number of possibilities in which only these k letters are addressed correctly. The desired probability is the quantity $\binom{n}{k}M\Big/n!$. All we got to do is to find M. To do so, note that the remaining $n-k$ letters are all addressed incorrectly. For these $n-k$ letters, there are $n-k$ addresses. But the addresses are written on the envelopes at random. The probability that none is addressed correctly on one hand is $M/(n-k)!$, and on the other hand, by Example 2.24, is

$$1-\sum_{i=1}^{n-k}\frac{(-1)^{i-1}}{i!}=\sum_{i=2}^{n}\frac{(-1)^{i-1}}{i!}.$$

So M satisfies

$$\frac{M}{(n-k)!}=\sum_{i=2}^{n}\frac{(-1)^{i-1}}{i!},$$

and hence

$$M=(n-k)!\sum_{i=2}^{n}\frac{(-1)^{i-1}}{i!}.$$

The final answer is

$$\frac{\binom{n}{k}M}{n!}=\frac{\binom{n}{k}(n-k)!\displaystyle\sum_{i=2}^{n}\frac{(-1)^{i-1}}{i!}}{n!}=\frac{1}{k!}\sum_{i=2}^{n}\frac{(-1)^{i-1}}{i!}.$$

51. The set of all sequences of H's and T's of length i with no successive H's are obtained either by adding a T to the tails of all such sequences of length $i-1$, or a TH to the tails of all such sequences of length $i-2$. Therefore,

$$x_i=x_{i-1}+x_{i-2},\quad i\ge 2.$$

Clearly, $x_1=2$ and $x_3=3$. For consistency, we define $x_0=1$. From the theory of recurrence relations we know that the solution of $x_i=x_{i-1}+x_{i-2}$ is of the form $x_i=Ar_1^i+Br_2^i$, where r_1 and r_2 are the solutions of $r^2 = r + 1$. Therefore, $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$ and so

$$x_i = A\Big(\frac{1+\sqrt{5}}{2}\Big)^i + B\Big(\frac{1-\sqrt{5}}{2}\Big)^i.$$

Using the initial conditions $x_0 = 1$ and $x_2 = 2$, we obtain $A = \dfrac{5+3\sqrt{5}}{10}$ and $B = \dfrac{5-3\sqrt{5}}{10}$. Hence the answer is

$$\begin{aligned}\frac{x_n}{2^n} &= \frac{1}{2^n}\Big[\Big(\frac{5+3\sqrt{5}}{10}\Big)\Big(\frac{1+\sqrt{5}}{2}\Big)^n + \Big(\frac{5-3\sqrt{5}}{10}\Big)\Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big]\\ &= \frac{1}{10\times 2^{2n}}\Big[\big(5+3\sqrt{5}\big)\big(1+\sqrt{5}\big)^n + \big(5-3\sqrt{5}\big)\big(1-\sqrt{5}\big)^n\Big].\end{aligned}$$

52. For this exercise, a solution is given by Abramson and Moser in the October 1970 issue of the *American Mathematical Monthly*.

2.5 STIRLING's FORMULA

1. **(a)** $\dbinom{2n}{n}\dfrac{1}{2^{2n}} = \dfrac{(2n)!}{n!\,n!}\dfrac{1}{2^{2n}} \sim \dfrac{\sqrt{4\pi n}\,(2n)^{2n}\,e^{-2n}}{(2\pi n)\,n^{2n}\,e^{-2n}\,2^{2n}} \sim \dfrac{1}{\sqrt{\pi n}}.$

(b) $\dfrac{\big[(2n)!\big]^3}{(4n)!\,(n!)^2} \sim \dfrac{\big[\sqrt{4\pi n}\,(2n)^{2n}\,e^{-2n}\big]^3}{\sqrt{8\pi n}\,(4n)^{4n}\,e^{-4n}\,(2\pi n)\,n^{2n}\,e^{-2n}} = \dfrac{\sqrt{2}}{4^n}.$

REVIEW PROBLEMS FOR CHAPTER 2

1. The desired quantity is equal to the number of subsets of all seven varieties of fruit minus 1 (the empty set); so it is $2^7 - 1 = 127$.

2. The number of choices Virginia has is equal to the number of subsets of $\{1, 2, 5, 10, 20\}$ minus 1 (for empty set). So the answer is $2^5 - 1 = 31$.

3. $(6 \times 5 \times 4 \times 3)/6^4 = 0.278.$

4. $10\Big/\dbinom{10}{2} = 0.222.$

5. $\dfrac{9!}{3!\,2!\,2!\,2!} = 7560.$

6. $5!/5 = 4! = 24.$

7. $3! \cdot 4! \cdot 4! \cdot 4! = 82,944.$

8. $1 - \dfrac{\binom{23}{6}}{\binom{30}{6}} = 0.83.$

9. Since the refrigerators are identical, the answer is 1.

10. $6! = 720.$

11. (Draw a tree diagram.) In 18 out of 52 possible cases the tournament ends because John wins 4 games without winning 3 in a row. So the answer is 34.62%.

12. Yes, it is because the probability of what happened is $1/7^2 = 0.02.$

13. $9^8 = 43,046,721.$

14. **(a)** $26 \times 25 \times 24 \times 23 \times 22 \times 21 = 165,765,600;$

(b) $26 \times 25 \times 24 \times 23 \times 22 \times 5 = 39,468,000;$

(c) $\binom{5}{2}26\binom{3}{1}25\binom{2}{1}24\binom{1}{1}23 = 21,528,000.$

15. $\dfrac{\binom{6}{3}+\binom{6}{1}+\binom{6}{1}+\binom{6}{1}\binom{2}{1}\binom{2}{1}}{\binom{10}{3}} = 0.467.$

Another Solution: $\dfrac{\binom{6}{3}+\binom{6}{1}\binom{4}{2}}{\binom{10}{3}} = 0.467.$

16. $\dfrac{8 \times 4 \times {}_6P_4}{{}_8P_6} = 0.571.$

17. $1 - \dfrac{27^8}{28^8} = 0.252.$

18. $\dfrac{(3!/3)(5!)^3}{15!/15} = 0.000396.$

19. $3^{12} = 531,441.$

20. $\dfrac{\binom{4}{1}\binom{48}{12}\binom{3}{1}\binom{36}{12}\binom{2}{1}\binom{24}{12}\binom{1}{1}\binom{12}{12}}{\dfrac{52!}{13!\,13!\,13!\,13!}} = 0.1055.$

21. Let A_1, A_2, A_3, and A_4 be the events that there is no professor, no associate professor, no assistant professor, and no instructor in the committee, respectively. The desired probability is

$$P(A_1^c A_2^c A_3^c A_4^c) = 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4),$$

where $P(A_1 \cup A_2 \cup A_3 \cup A_4)$ is calculated using the inclusion-exclusion principle:

$$\begin{aligned}
P(A_1 \cup A_2 \cup A_3 \cup A_4) &= P(A_1) + P(A_2) + P(A_3) + P(A_4) \\
&\quad - P(A_1A_2) - P(A_1A_3) - P(A_1A_4) - P(A_2A_3) - P(A_2A_4) - P(A_3A_4) \\
&\quad + P(A_1A_2A_3) + P(A_1A_3A_4) + P(A_1A_2A_4) + P(A_2A_3A_4) - P(A_1A_2A_3A_4) \\
&= \left[1\Big/\binom{34}{6}\right]\left[\binom{28}{6} + \binom{28}{6} + \binom{24}{6} + \binom{22}{6} - \binom{22}{6} - \binom{18}{6} - \binom{16}{6} - \binom{18}{6}\right. \\
&\quad \left. - \binom{16}{6} - \binom{12}{6} + \binom{12}{6} + \binom{6}{6} + \binom{10}{6} + \binom{6}{6} - 0\right] = 0.621.
\end{aligned}$$

Therefore, the desired probability equals $1 - 0.621 = 0.379$.

22. $\dfrac{(15!)^2}{30!/(2!)^{15}} = 0.0002112.$

23. $(N - n + 1)\Big/\dbinom{N}{n}.$

24. **(a)** $\dfrac{\binom{4}{2}\binom{48}{24}}{\binom{52}{26}} = 0.390;$ **(b)** $\dfrac{\binom{40}{1}}{\binom{52}{13}} = 6.299 \times 10^{-11};$

(c) $\dfrac{\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}}{\binom{52}{13}\binom{39}{13}} = 0.00000261.$

25. $12!/(3!)^4 = 369,600$.

26. There is a one-to-one correspondence between all cases in which the eighth outcome obtained is not a repetition and all cases in which the first outcome obtained will not be repeated. The answer is

$$\frac{6\times5\times5\times5\times5\times5\times5\times5}{6\times6\times6\times6\times6\times6\times6\times6} = \left(\frac{5}{6}\right)^7 = 0.279.$$

27. There are $9\times10^3 = 9,000$ four-digit numbers. To count the number of desired four-digit numbers, note that if 0 is to be one of the digits, then the thousands place of the number must be 0, but this cannot be the case since the first digit of an n-digit number is nonzero. Keeping this in mind, it must be clear that from every 4-combination of the set $\{1,2,\ldots,9\}$, exactly one four-digit number can be constructed in which its ones place is greater than its tens place, its tens place is greater than it hundreds place, and its hundreds place is greater than its thousands place. Therefore, the number of such four-digit numbers is $\binom{9}{4} = 126$. Hence the desired probability is $= 0.014$.

28. Since the sum of the digits of 100,000 is 1, we ignore 100,000 and assume that all of the numbers have five digits by placing 0's in front of those with less than five digits. The following process establishes a one-to-one correspondence between such numbers, $d_1d_2d_3d_4d_5$, $\sum_{i=1}^{5} d_i = 8$, and placement of 8 identical objects into 5 distinguishable cells: Put d_1 of the objects into the first cell, d_2 of the objects into the second cell, d_3 into the third cell, and so on. Since this can be done in $\binom{8+5-1}{5-1} = \binom{12}{8} = 495$ ways, the number of integers from the set $\{1,2,3,\ldots,100000\}$ in which the sum of the digits is 8 is 495. Hence the desired probability is $495/100,000 = 0.00495$.