Chapter 2

2.1 What are the dimensions of the correlation function R_x and the spectral density S_x if the random function $X(t)$ is

(A) Displacement (in.)
(B) Acceleration (in./sec^2)
(C) Force (lb.)
(D) Stress (psi)

Solution

$$R_x(\tau) = \frac{1}{T}\int_{-T}^{T} x(t)x(t+\tau)d\tau \qquad S_x(\omega) = \frac{1}{2T}\left|x_T^*(i\omega)\right|^2 = \frac{1}{2T}\left|\int_{-\infty}^{\infty} x_T(t)e^{i\omega t}dt\right|^2$$

Part	Function	$R_x(\tau)$	$S_x(\omega)$
A	Displacement (in.)	in.^{2}	in.^{2} $-$ sec
B	Acceleration (in./sec^2)	in.^{2} / sec^4	in.^{2} / sec^3
C	Force (lb.)	lb.^{2}	lb.^{2} $-$ sec
D	Stress (psi)	lb.^{2} / in.^{4}	lb.^{2} $-$ sec/ in.^{4}

2.2. What are the dimensions of the probability function F_x and the probability density function f_x if the random function $X(t)$ is

(A) Displacement (in.)
(B) Acceleration (in./sec^2)
(C) Force (lb.)
(D) Stress (psi)

Solution

$$F_x = \int_{-\infty}^{\infty} f(x)dx \qquad f_x = \frac{1}{\sqrt{2\pi}c}e^{-(x-\bar{x})^2/2c^2}$$

Part	Function	F_x	f_x

A	Displacement (in.)	Dimensionless	$1 / \text{in}$
B	Acceleration (in./sec^2)	Dimensionless	$\text{sec}^2 / \text{in.}$
C	Force (lb.)	Dimensionless	1/lbs.
D	Stress (psi)	Dimensionless	$\text{in.}^2 / \text{lbs}$

2.3. Define a stationary random process.

Solution

A stationary random process is defined by the correlations function of a random variable being constant regardless of the time at which it is evaluated, so long as the period over which each evaluation is taken is unchanged in duration.

2.4. Consider a stationary ergodic process $X(t)$; express $\mathrm{E}\left[X\right]$, $\mathrm{E}\left[X^2\right]$, and $\mathrm{R}_x\left(\tau\right)$ as the time averages.

Solution

For a stationary ergodic process $X(t)$ we have in terms of time averages

$$\mathrm{E}\left[X(t)\right] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$$\mathrm{E}\left[X^2(t)\right] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt$$

$$\mathrm{R}_x\left(\tau\right) = \mathrm{E}\left[X(t)X(t+\tau)\right] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

2.5. Prove the relations between the spectral density $S\left(\omega\right)$ and the correlation function $\mathrm{R}\left(\tau\right)$:

$$\mathrm{R}\left(\tau\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S\left(\omega\right)e^{i\omega\tau}d\omega$$

$$S\left(\omega\right) = \int_{-\infty}^{\infty} R\left(\tau\right)e^{-i\omega\tau}d\tau$$

Solution

$$\mathrm{R}_x(\tau) = \mathrm{E}\left[X(t)X(t+\tau)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = \frac{1}{2T}\int_{-\infty}^{\infty} e^{-i\omega\tau}d\tau\int_{-\infty}^{\infty} x(t)x(t+\tau)dt = \frac{1}{2T}\int_{-\infty}^{\infty} d\tau\int_{-\infty}^{\infty} x(t)x(t+\tau)e^{-i\omega(t+\tau)}e^{i\omega t}dt$$

$$= \frac{1}{2T}\int_{-\infty}^{\infty} x(t)e^{i\omega t}dt\int_{-\infty}^{\infty} x(t+\tau)e^{-i\omega(t+\tau)}d\tau$$

Since $X^*(i\omega) = \int_{-\infty}^{\infty} x(t)e^{i\omega t}dt$ and $X^*(-i\omega) = \int_{-\infty}^{\infty} x(t+\tau)e^{-i\omega(t+\tau)}d\tau$

$$\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = \frac{1}{2T}X^*(i\omega)X^*(-i\omega)$$

Take the limit of both sides as $T\to\infty$

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = \lim_{T\to\infty}\frac{1}{2T}\left|X^*(i\omega)\right|^2 X^*(-i\omega)$$

$$\int_{-\infty}^{\infty}\lim_{T\to\infty} R(\tau)e^{-i\omega\tau}d\tau = S(\omega)$$

$$\underline{S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau}$$

Thus, $S(\omega) = \mathscr{F}\left\{\mathrm{R}(\tau)\right\}$ and $\mathrm{R}(\tau) = \mathscr{F}^{-1}\left\{S(\omega)\right\}$

$$\underline{\mathrm{R}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}d\omega}$$

2.6. Assuming that the input-output relation is $y(t) = \int_{-\infty}^{t}\phi(t-\lambda)x(\lambda)d\lambda$. Show that

$$\mathrm{R}_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\lambda)\phi(\eta)\mathrm{R}_x(\tau+\lambda-\eta)\,d\lambda d\eta \quad \text{and} \quad \mathrm{S}_y(\omega) = \left|\phi^*(i\omega)\right|^2\mathrm{S}_x(\omega)$$

Solution

Given $y(t)$ we know that the correlation function is

$$\mathrm{R}_y(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y(t)y(t+\tau)dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt\int_{-\infty}^{\infty} x(t-\lambda)\phi(\lambda)d\lambda\int_{-\infty}^{\infty} x(t+\tau-\eta)\phi(\eta)d\eta$$

where

$$y(t)=\int_{-\infty}^{t}\phi(t-\lambda)x(\lambda)d\lambda=\int_{-\infty}^{t}x(t-\lambda)\phi(\lambda)d\lambda=\int_{-\infty}^{\infty}x(t-\lambda)\phi(\lambda)d\lambda$$

$$y(t+\tau)=\int_{-\infty}^{t+\tau}x(t+\tau-\lambda)\phi(\eta)d\eta=\int_{-\infty}^{\infty}x(t+\tau-\eta)\phi(\eta)d\eta$$

$$\mathrm{R}_y(\tau)=\lim_{T\to\infty}\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}d\eta\phi(\lambda)\phi(\eta)\frac{1}{2T}\int_{-\infty}^{\infty}x(t-\lambda)x(t+\tau-\eta)dt$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t-\lambda)x(t+\tau-\lambda)dt=\mathrm{R}_x(\tau+\lambda-\eta)$$

$$t-\lambda=t_1 \quad\text{and}\quad t+\tau-\lambda=t_2 \quad\text{so}\quad t_2-t_1=\tau+\lambda-\eta$$

therefore

$$\underline{\mathrm{R}_y(\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\lambda)\phi(\eta)\mathrm{R}_x(\tau+\lambda-\eta)\,d\lambda d\eta}$$

$$\begin{aligned}
S(\omega)=\int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}d\tau &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\lambda)\phi(\eta)\mathrm{R}_x(\tau+\lambda-\eta)e^{-i\omega\tau}d\eta d\lambda d\tau\\
&=\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}d\eta e^{-i\omega\tau}\phi(\lambda)\phi(\eta)\mathrm{R}_x(\tau+\lambda-\eta)\\
&=\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}d\eta e^{i\omega\lambda}e^{-i\omega\eta}e^{-i\omega(\tau+\lambda-\lambda)}\phi(\lambda)\phi(\eta)\mathrm{R}_x(\tau+\lambda-\eta)e^{-i\omega\tau}\\
&=\int_{-\infty}^{\infty}\phi(\eta)e^{-i\omega\eta}d\eta\int_{-\infty}^{\infty}\phi(\lambda)e^{i\omega\lambda}d\lambda\int_{-\infty}^{\infty}\mathrm{R}_x(\tau+\lambda-\eta)e^{-i\omega(\tau+\lambda-\lambda)}d\tau\\
&=\phi^*(i\omega)\phi^*(-i\omega)S_x(\omega)
\end{aligned}$$

therefore

$$\underline{\mathrm{S}_y(\omega)=\left|\phi^*(i\omega)\right|^2\mathrm{S}_x(\omega)}$$

2.7. Derive the expression for the average number of crossings of the value $x=\alpha$ assuming that the joint distribution function $f(\alpha,\beta)$ of $X(t)$ and $\dot{X}(t)$ is

(A) two-dimensional normal
(B) arbitrary

Solution

(a) The joint distribution function $f(\alpha,\beta)$ of $X(t)$ and $\dot{X}(t)$ is given by

$$f(\alpha,\beta) = \frac{1}{2\pi C_1 C_2} e^{-\left[\frac{\alpha^2}{2C_1^2}+\frac{\beta^2}{2C_2^2}\right]}$$

where $C_1^2 = E\left[X^2\right]$ and $C_2^2 = E\left[\dot{X}^2\right]$

The probability that $\alpha < X < \alpha + d\alpha$ and $\beta < \dot{X} < \beta + d\beta$ is $f(\alpha,\beta)d\alpha d\beta$. The duration of the passage from α to $\alpha + d\alpha$ is $d\alpha / |\beta|$. Thus the passages form α to $\alpha + d\alpha$ at a velocity β per unit time is

$$\frac{f(\alpha,\beta)d\alpha d\beta}{\frac{d\alpha}{|\beta|}} = \frac{|\beta| f(\alpha,\beta)d\alpha d\beta}{d\alpha} = \left\{|\beta| f(\alpha,\beta)d\beta\right\}$$

The passages from α to $\alpha + d\alpha$ at any rate is equal to the sum of the passages at each value of β, or

$$N_\alpha = \int_{-\infty}^{\infty} |\beta| f(\alpha,\beta)d\beta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\beta|}{C_1 C_2} e^{-\left[\frac{\alpha^2}{2C_1^2}+\frac{\beta^2}{2C_2^2}\right]} d\beta$$

2.8. The spectral density of the loading (input) is given in the form shown in Figure 2.29

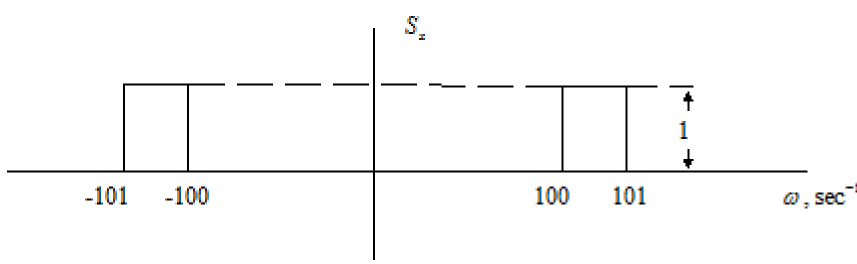


Figure 2.29Loading input

Find the spectral density's of the outputs (displacement, bending moment, etc.) of the systems described in problem 1(a) of Chapter 1.

Solution

From Problem 3(a) of Chapter 1 we know that

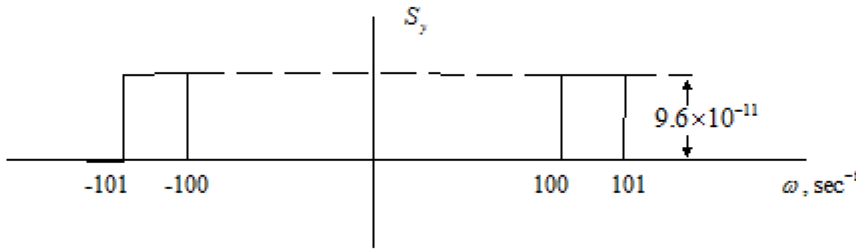
$$U^* = \left[\frac{1}{115,200 - 1.3\omega^2}\right] \quad \text{or} \quad \phi^*(i\omega) = \frac{1}{115,200 - 1.3\omega^2}$$

$$S_y = |\phi^*(i\omega)|^2\, S_x$$

At $\omega = 100$

$$\phi^*(i100) = \frac{1}{115,200 - 1.3(100)^2} = 9.785\times10^{-6} \approx 9.8\times10^{-6}$$

$$|\phi^*(i100)|^2 = \left(9.8\times10^{-6}\right)^2 \approx 9.6\times10^{-11} \text{ so } S_y = \left(9.6\times10^{-11}\right)S_x$$



Also, the mean square value of $u(x,t)$ is

$$E(u^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = \frac{1}{2\pi}\int_{-101}^{-100}(1)\,d\omega + \frac{1}{2\pi}\int_{100}^{1001}(1)\,d\omega = \frac{1}{\pi}$$

2.9. A random function $X(t)$ has its spectral density as in Problem 2.8. Find the mean square value of $X(t)$. Find the average number of crossings of the values $x=0$, $x=1$, and $x=10$.

Solution

From Problem 8 we have

$$E(u^2) = \frac{1}{\pi}$$

For $x = 0$

$$N_0 = \frac{1}{\pi}\left[\frac{\int_0^{\infty}\omega^2 S_x(\omega)\,d\omega}{\int_0^{\infty} S_x(\omega)\,d\omega}\right]^{1/2} = \frac{1}{\pi}\sqrt{\frac{10,000}{1}} = \frac{100}{\pi} \text{ rad/sec}$$

For $x = 1$

$$N_\alpha = N_0 e^{-\frac{\alpha^2}{2C_1^2}} \quad \text{where} \quad C_1^2 = E(u^2) = \frac{1}{\pi}$$

$$N_1 = \frac{100}{\pi} e^{-\pi(1)^2/2} = \frac{100}{\pi} e^{-\pi/2} \text{ rad/sec}$$

For $x = 10$

$$N_1 = \frac{100}{\pi} e^{-\pi(10)^2/2} = \frac{100}{\pi} e^{-50\pi} \text{ rad/sec}$$