Chapter 2

Fourier Trigonometric Series

1. Write $y(t) = 3\cos 2t - 4\sin 2t$ in the form $y(t) = A\cos(2\pi f t + \phi)$.
We can determine the constants by expanding the cosine function,

$$y(t) = A\cos(2\pi f t + \phi) = A\cos\phi\cos 2\pi f t - A\sin\phi\sin 2\pi f t.$$

Comparing this to $y(t) = 3\cos 2t - 4\sin 2t$, we see $2\pi f = 2$ and

$$\begin{aligned} A\cos\phi &= 3, \\ A\sin\phi &= 4. \end{aligned}$$

Adding the squares of these equations,

$$25 = A^2\cos^2\phi + A^2\sin^2\phi = A^2,$$

we obtain $A = 5$. Dividing the first equation into the second, $\tan\phi = 4/3$. So, we find

$$\begin{aligned} y(t) &= 3\cos 2t - 4\sin 2t \\ &= 5\cos\left(2t + \tan^{-1}(\frac{4}{3})\right) \\ &\approx 5\cos(2t + 0.927). \end{aligned}$$

2. Determine the period of the following functions:

a. $f(x) = \cos\frac{x}{3}$.

$T = \frac{2\pi}{\frac{1}{3}} = 6\pi.$

b. $f(x) = \sin 2\pi x$.

$T = \frac{2\pi}{2\pi} = 1.$

c. $f(x) = \sin 2\pi x - 0.1\cos 3\pi x$.

Each term has a different period: $T = \frac{2\pi}{2\pi} = 1$ and $T = \frac{2\pi}{3\pi} = \frac{2}{3}$. Multiples of each give

$$nT = \{1, 2, 3, 4, 5, 6, \ldots\}$$

$$nT = \{\frac{2}{3}, \frac{4}{3}, 2, \frac{8}{3}, \frac{10}{3}, 4, \ldots\}.$$

The smallest common value is the period of $f(x) = \sin 2\pi x - 0.1\cos 3\pi x : T = 2$. This is seen in Figure 2.1.

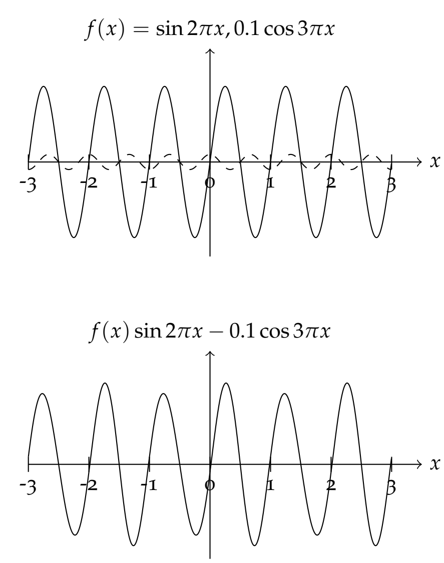


Figure 2.1: Plots of $f(x) = \sin 2\pi x$, $f(x) = 0.1\cos 3\pi x$, and $f(x) = \sin 2\pi x - 0.1\cos 3\pi x$.

d. $f(x) = |\sin 5\pi x|$.

The period of $f(x) = \sin 5\pi x$ is $T = \dfrac{2\pi}{5\pi} = \dfrac{2}{5}$. However, the frequency doubles under the absolute value, so $T = \dfrac{1}{5}$.

e. $f(x) = \cot 2\pi x$.

The periods of the $\tan x$ and $\cot x$ are $T = \pi$. So, for this function we have $T = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$.

f. $f(x) = \cos^2 \dfrac{x}{2}$.

Just like the absolute value, the frequency of the cosine function doubles when the function is squared. So, $T = \dfrac{\pi}{\frac{1}{2}} = 2\pi$.

g. $f(x) = 3\sin \dfrac{\pi x}{2} + 2\cos \dfrac{3\pi x}{4}$.

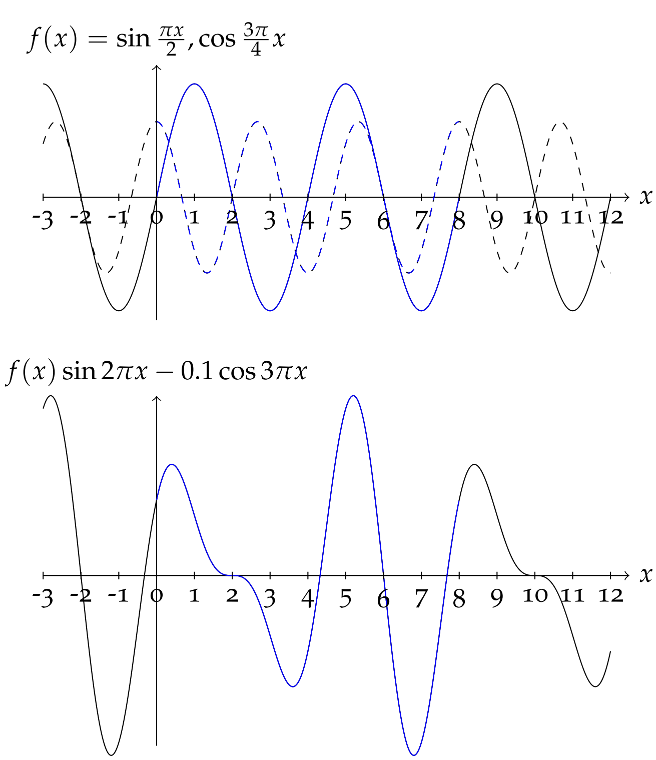
This problem is similar to 2c. Each term has a different period: $T = \dfrac{2\pi}{\frac{\pi}{2}} = 4$ and $T = \dfrac{2\pi}{\frac{3\pi}{4}} = \dfrac{8}{3}$. Multiples of each give

$$nT = \{4, 8, 12, 16, \ldots\}$$

$$nT = \{\frac{8}{3}, \frac{16}{3}, 8, \frac{32}{3}, \frac{40}{3}, 16, \ldots\}.$$

The smallest common value is the period of $f(x) = 3\sin \dfrac{\pi x}{2} + 2\cos \dfrac{3\pi x}{4}$ is $T = 8$. This is seen in Figure 2.2.

Figure 2.2: Plots of $f(x) = 3\sin \frac{\pi x}{2}$, $f(x) = 2\cos \frac{3\pi x}{4}$, and $f(x) = 3\sin \frac{\pi x}{2} + 2\cos \frac{3\pi x}{4}$.



3. Derive the coefficients b_n in Equation (5.24).

This derivation parallels that for the a_n's. We begin with the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right].$$

We multiply this Fourier series by $\sin mx$ for some positive integer m and then integrate:

$$\begin{aligned}
\int_0^{2\pi} f(x) \sin mx \, dx &= \int_0^{2\pi} \frac{a_0}{2} \sin mx \, dx \\
&+ \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right] \sin mx \, dx.
\end{aligned}$$

Integrating term by term, the right side becomes

$$\begin{aligned}
& \int_0^{2\pi} f(x) \sin mx \, dx \\
= & \frac{a_0}{2} \int_0^{2\pi} \sin mx \, dx \\
& + \sum_{n=1}^{\infty} \left[a_n \int_0^{2\pi} \cos nx \sin mx \, dx + b_n \int_0^{2\pi} \sin nx \cos mx \, dx \right].
\end{aligned}$$

We have shown that $\int_0^{2\pi} \sin mx \, dx = 0$, which implies that the first term vanishes. Also, we have that

$$\int_0^{2\pi} \cos nx \sin mx \, dx = 0$$

for integers n and m.

We still need to evaluate $\int_0^{2\pi} \sin nx \sin mx \, dx$ which was not done in the book. We compute this integral by using the product identity for sines. We have for $m \neq n$ that

$$\begin{aligned}
\int_0^{2\pi} \sin nx \sin mx \, dx &= \frac{1}{2} \int_0^{2\pi} \left[\cos(m-n)x - \cos(m+n)x \right] dx \\
&= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{2\pi} \\
&= 0.
\end{aligned}$$

For $n = m$, we have

$$\begin{aligned}
\int_0^{2\pi} \sin^2 mx \, dx &= \frac{1}{2} \int_0^{2\pi} (1 - \cos 2mx) \, dx \\
&= \frac{1}{2} \left[x - \frac{1}{2m} \sin 2mx \right]_0^{2\pi} \\
&= \frac{1}{2}(2\pi) = \pi.
\end{aligned}$$

Now, we can finish the derivation. We have shown the orthogonality of the sines,

$$\int_0^{2\pi} \sin nx \sin mx \, dx = \pi \delta_{nm}.$$

So,

$$\begin{aligned}\int_0^{2\pi} f(x)\sin mx\,dx &= \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin nx \cos mx\,dx \\ &= \sum_{n=1}^{\infty} b_n \pi \delta_{nm} \\ &= b_m \pi.\end{aligned}$$

Solving for b_m, we have

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx.$$

4. Let $f(x)$ be defined for $x \in [-L, L]$. Parseval's identity is given by

$$\frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Assuming the Fourier series of $f(x)$ converges uniformly in $(-L, L)$, prove Parseval's identity by multiplying the Fourier series representation by $f(x)$ and integrating from $x = -L$ to $x = L$. [In Section 5.6.3, we will encounter Plancherel's Formula for Fourier transforms, which is a continuous version of this identity.]

We begin with the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi nx}{L} + b_n \sin\frac{2\pi nx}{L}\right].$$

Multiplying this Fourier series by $f(x)$ and integrating over $x \in [-L, L]$, we obtain

$$\begin{aligned}\int_{-L}^{L} f^2(x)\,dx &= \int_{-L}^{L} \frac{a_0}{2} f(x)\,dx + \int_{-L}^{L}\sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi nx}{L} + b_n \sin\frac{2\pi nx}{L}\right] f(x)\,dx \\ &= \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx \\ &\quad + \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} f(x)\cos\frac{2\pi nx}{L}\,dx + b_n \int_{-L}^{L} f(x)\sin\frac{2\pi nx}{L}\,dx\right]\end{aligned}$$

Recall the Fourier coefficients are given by

$$\begin{aligned}a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{2\pi nx}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{2\pi nx}{L}\,dx.\end{aligned}$$

Replacing the integrals in the integrated Fourier series with Fourier coefficients, we have

$$\int_{-L}^{L} f^2(x)\,dx = \frac{a_0}{2}(a_0 L) + \sum_{n=1}^{\infty}\left[a_n(a_n L) + b_n(b_n L)\right],$$

leading to the sought result,

$$\frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

5. Let $f(x)$ be defined for $x \in [0, L]$. Derive the Parseval identities, similar to Problem 4, for the following series.

a. For the Fourier Cosine series,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},$$

show that

$$\frac{2}{L}\int_0^L [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2.$$

Multiply the Fourier Cosine series by $f(x)$ and integrate from $x = 0$ to $x = L$.

$$\begin{aligned}
\int_0^L [f(x)]^2\,dx &= \int_0^L f(x)\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\right] dx \\
&= \frac{a_0}{2}\int_0^L f(x)\,dx + \sum_{n=1}^{\infty} a_n \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.
\end{aligned}$$

Noting that the integrals are given by

$$\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{L}{2}a_n, \quad n = 0, 1, 2, \ldots,$$

we have

$$\frac{2}{L}\int_0^L [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2.$$

b. For the Fourier Sine series,

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

show that

$$\frac{2}{L}\int_0^L [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2.$$

Multiply the Fourier Sine series by $f(x)$ and integrate from $x = 0$ to $x = L$.

$$\begin{aligned}
\int_0^L [f(x)]^2\,dx &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,dx \\
&= \sum_{n=1}^{\infty} b_n \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.
\end{aligned}$$

Noting that the integrals are given by

$$\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{L}{2}a_n, \quad n = 1, 2, \ldots,$$

we have

$$\frac{2}{L}\int_0^L [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2.$$

6. Consider the square wave function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi. \end{cases}$$

a. Find the Fourier series representation of this function and plot the first 50 terms.

Since $f(x)$ is an odd function on a symmetric interval, $a_n = 0$, $n = 0, 1, \ldots$. We need only compute the b_n's.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx \\ &= -\frac{2}{n\pi}(\cos n\pi - 1). \end{aligned}$$

The Fourier series is then

$$\begin{aligned} f(x) &\sim \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}(\cos n\pi - 1)\right)\sin nx \\ &= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1}. \end{aligned}$$

The first 50 terms $(n = 50)$ are shown in Figure 2.3.

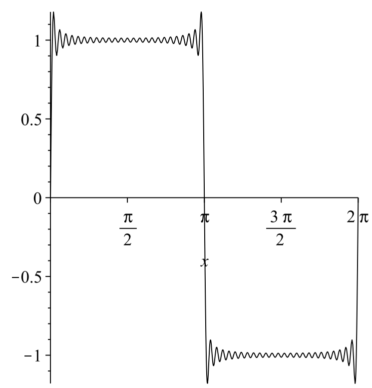


Figure 2.3: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.6.

b. Apply Parseval's identity in Problem 5 to the result in part a.

Parseval's identity states

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \sum_{n=1}^{\infty} b_n^2.$$

From part a we have

$$b_n = -\frac{2}{n\pi}(\cos n\pi - 1).$$

Noting that $f^2(x) = 1$, we have

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx &= \sum_{n=1}^{\infty} b_n^2 \\ \frac{1}{\pi}\int_{-\pi}^{\pi} dx &= \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}(\cos n\pi - 1)\right)^2 \\ 2 &= \sum_{k=1}^{\infty}\left(\frac{4}{(2k-1)\pi}\right)^2 \\ &= \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \\ &= \frac{16}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right). \end{aligned}$$

Thus, we have

$$2 = \frac{16}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right).$$

c. Use the result of part b to show $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Multiplying the series in part b,

$$2 = \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2},$$

by $\pi^2/16$ yields this result.

7. For the following sets of functions: (i) show that each is orthogonal on the given interval, and (ii) determine the corresponding orthonormal set. [See page 288.]

Orthogonality and normalization can be done using simple substitutions and relating the integrals to the basic orthogonality relations between sines and cosines,

$$\int_0^{2\pi} \cos nx \cos mx \, dx = \int_0^{2\pi} \sin nx \sin mx \, dx = \pi \delta_{nm}.$$

a. $\{\sin 2nx\}, \quad n = 1, 2, 3, \ldots, \quad 0 \le x \le \pi.$

Let $y = 2x$, $dy = 2\,dx$. Then,

$$\int_0^{\pi} \sin 2nx \sin 2mx \, dx = \frac{1}{2} \int_0^{2\pi} \sin ny \sin my \, dy = \frac{\pi}{2} \delta_{nm}.$$

These can be normalized by letting $\phi_n(x) = A \sin 2nx$. Then

$$1 = \int_0^{\pi} \phi_n^2(x) \, dx = \int_0^{\pi} A^2 \sin^2 2nx \, dx = \frac{\pi}{2} A^2.$$

So, we have $A = \sqrt{\frac{2}{\pi}}$ and the orthonormal set is given by $\{\sqrt{\frac{2}{\pi}} \sin 2nx\}$, $n = 1, 2, 3, \ldots, 0 \le x \le \pi$.

b. $\{\cos n\pi x\}, \quad n = 0, 1, 2, \ldots, \quad 0 \le x \le 2.$

Let $y = \pi x$, $dy = \pi\, dx$. Then,

$$\int_0^{2} \cos n\pi x \cos m\pi x \, dx = \frac{1}{\pi} \int_0^{2\pi} \cos ny \cos my \, dy = \delta_{nm}.$$

These functions are already orthonormal.

c. $\{\sin \frac{n\pi x}{L}\}, \quad n = 1, 2, 3, \ldots, \quad x \in [-L, L].$

Let $y = \pi x/L$, $dy = \pi/L\, dx$. Then,

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \frac{L}{\pi} \int_{-\pi}^{\pi} \sin ny \sin my \, dy = L\delta_{nm}.$$

These can be normalized by letting $\phi_n(x) = A \sin \frac{n\pi x}{L}$. Then

$$1 = \int_{-L}^{L} \phi_n^2(x) \, dx = \int_{-L}^{L} A^2 \sin^2 \frac{n\pi x}{L} \, dx = LA^2.$$

So, we have $A = \frac{1}{\sqrt{L}}$ and the orthonormal set is given by $\{\frac{1}{\sqrt{L}} \sin \frac{n\pi x}{L}\}$, $n = 1, 2, 3, \ldots, x \in [-L, L]$.

8. Consider $f(x) = 4\sin^3 2x$.

a. Derive the trigonometric identity giving $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$ using DeMoivre's Formula.

We note that $e^{3i\theta} = (e^{i\theta})^3$. Writing both sides of this equation in terms of trigonometric functions, we have

$$\begin{aligned} e^{3i\theta} &= (e^{i\theta})^3 \\ \cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \end{aligned}$$

Equating the real and imaginary parts we have

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta, \\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta. \end{aligned}$$

The second equation can be rearranged to get the result.

$$\begin{aligned} \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta. \end{aligned}$$

Therefore,

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta.$$

b. Find the Fourier series of $f(x) = 4\sin^3 2x$ on $[0, 2\pi]$ without computing any integrals.

We need only let $\theta = 2x$ in part a. Then,

$$f(x) = 4\sin^3 2x = 3\sin 2x - \sin 6x.$$

9. Find the Fourier series representations of the following:

The a_0's are computed separately from a_n's when determining the Fourier coefficients.

a. $f(x) = x$, $x \in [0, 2\pi]$.

We first compute the Fourier coefficients. Note that we compute a_0 separately from a_n.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} x\,dx = \left.\frac{x^2}{2\pi}\right|_0^{2\pi} = 2\pi. \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{2\pi} = 0. \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx \\ &= \frac{1}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{2\pi} = -\frac{2}{n}. \end{aligned}$$

The Fourier series is given by

$$f(x) \sim \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

A plot of the first terms of the series is given in Figure 2.4.

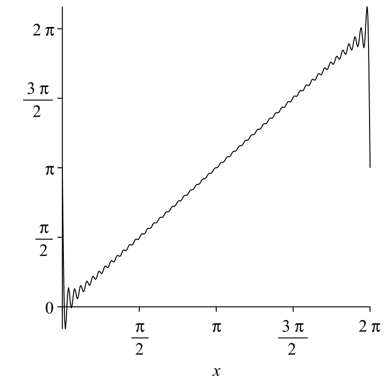


Figure 2.4: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.9a.

b. $f(x) = \dfrac{x^2}{4}$, $|x| < \pi$.

$f(x)$ is an even function on $|x| < \pi$. So, $b_n = 0$ for all n. We only need the a_n's.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{4}\,dx = \frac{2}{4\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{6}. \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{4}\cos nx\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} x^2\cos nx\,dx \\
&= \frac{1}{2\pi}\left[\frac{1}{n}x^2\sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{2}{n^2}\pi\cos n\pi\right) = \frac{\cos n\pi}{n^2} = \frac{(-1)^n}{n^2}.
\end{aligned}$$

Then,the Fourier series is given as

$$f(x) \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx.$$

A plot of the first terms of the series is given in Figure 2.5.

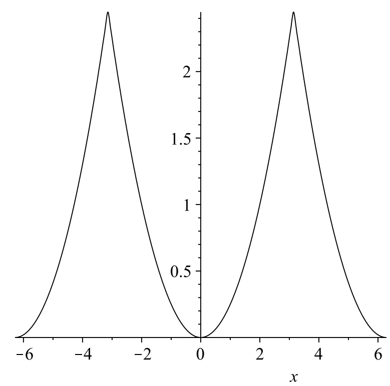


Figure 2.5: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.9b.

c. $f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi, \\ -\frac{\pi}{2}, & \pi < x < 2\pi. \end{cases}$

We first compute the Fourier coefficients.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\right)dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\left(-\frac{\pi}{2}\right)dx \\
&= \frac{1}{2}x\Big|_0^{\pi} - \frac{1}{2}x\Big|_{\pi}^{2\pi} \\
&= \frac{\pi}{2} - (\pi - \frac{\pi}{2}) = 0. \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\right)\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\left(-\frac{\pi}{2}\right)\cos nx\,dx \\
&= \frac{1}{2n}\sin nx\Big|_0^{\pi} - \frac{1}{2n}\sin nx\Big|_{\pi}^{2\pi} = 0. \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\right)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\left(-\frac{\pi}{2}\right)\sin nx\,dx \\
&= -\frac{1}{2n}\cos nx\Big|_0^{\pi} + \frac{1}{2n}\cos nx\Big|_{\pi}^{2\pi} \\
&= \frac{1 - \cos n\pi}{n}.
\end{aligned}$$

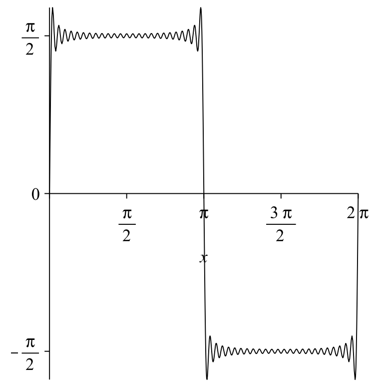


Figure 2.6: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.9c.

The resulting Fourier series is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n} \sin nx = 2\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

A plot of the first terms of the series is given in Figure 2.8.

d. $f(x) = \begin{cases} x, & 0 < x < \pi, \\ \pi, & \pi < x < 2\pi. \end{cases}$

We first compute the Fourier coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\,dx \\ &= \frac{1}{\pi}\frac{x^2}{2}\bigg|_0^{\pi} + x\bigg|_{\pi}^{2\pi} \\ &= \frac{\pi}{2} + (2\pi - \pi) = \frac{3}{2}\pi. \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cos nx\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi} + \frac{1}{n}\sin nx\bigg|_{\pi}^{2\pi} \\ &= \frac{\cos n\pi - 1}{\pi n^2}. \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\sin nx\,dx \\ &= \frac{1}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} - \frac{1}{n}\cos nx\bigg|_{\pi}^{2\pi} \\ &= -\frac{1}{n}. \end{aligned}$$

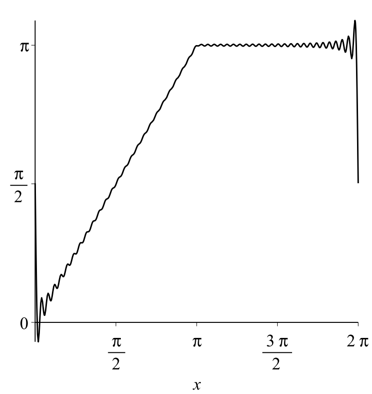


Figure 2.7: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.9d.

The resulting Fourier series is

$$f(x) \sim \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{\cos n\pi - 1}{\pi n^2}\cos nx - \frac{1}{n}\sin nx\right].$$

A plot of the first terms of the series is given in Figure **??**.

e. $f(x) = \begin{cases} \pi - x, & 0 < x < \pi, \\ 0, & \pi < x < 2\pi. \end{cases}$

We first compute the Fourier coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} \pi - x\,dx \\ &= -\frac{1}{\pi}\frac{(\pi-x)^2}{2}\bigg|_0^{\pi} \end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{2}.\\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx\\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx\\
&= \frac{1}{\pi}\left[\frac{1}{n}(\pi - x)\sin nx - \frac{1}{n^2}\cos nx\right]_0^{\pi}\\
&= \frac{1-\cos n\pi}{\pi n^2}.\\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx\\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin nx\,dx\\
&= \frac{1}{\pi}\left[-\frac{1}{n}(\pi - x)\cos nx - \frac{1}{n^2}\sin nx\right]_0^{\pi}\\
&= \frac{1}{n}.
\end{aligned}$$

The resulting Fourier series is

$$f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1-\cos n\pi}{\pi n^2}\cos nx + \frac{1}{n}\sin nx\right].$$

A plot of the first terms of the series is given in Figure **??**.

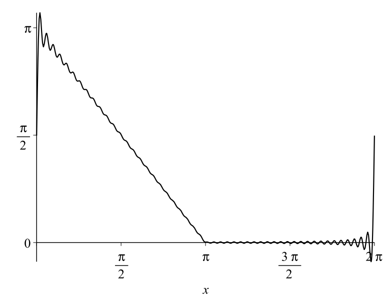


Figure 2.8: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.9e.

10. Find the Fourier series representations of each function $f(x)$ of period 2π. For each series, plot the Nth partial sum,

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N}\left[a_n\cos nx + b_n \sin nx\right],$$

for $N = 5, 10, 50$ and describe the convergence (Is it fast? What is it converging to?, etc.) [Some simple Maple , MATLAB, and Python code for computing partial sums is shown in the notes.]

a. $f(x) = x$, $|x| < \pi$.
Since $f(x) = x$ is an odd function on $|x| < \pi$, the a_n's vanish for all n. So, we just compute the b_n's.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx\\
&= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx\\
&= \frac{2}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi}\\
&= -\frac{2\cos n\pi}{n} = 2\frac{(-1)^{n+1}}{n}.
\end{aligned}$$

The resulting Fourier series is

$$f(x) \sim 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

As seen in Figure 2.9 the convergence is slow as the terms decay like $\frac{1}{n}$. The discontinuities in the periodic extension also are an indication.

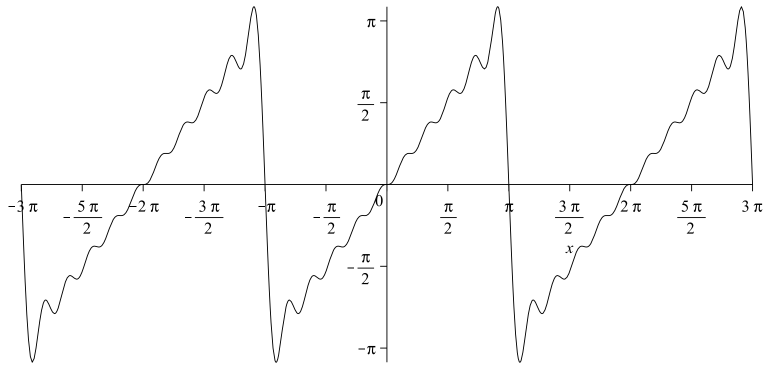


Figure 2.9: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.10a.

b. $f(x) = |x|$, $|x| < \pi$.

Since $f(x) = |x|$ is an even function on $|x| < \pi$, the b_n's vanish for all n. We compute a_0 and a_n.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \\
&= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi} \\
&= \frac{2}{\pi n^2}(\cos n\pi - 1).
\end{aligned}$$

The resulting Fourier series is

$$\begin{aligned}
f(x) &\sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(\cos n\pi - 1)}{\pi n^2}\cos nx \\
&= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}.
\end{aligned}$$

The convergence is fast as the terms decay like $\frac{1}{n^2}$. There are not discontinuities in the periodic extension. See Figure 2.10.

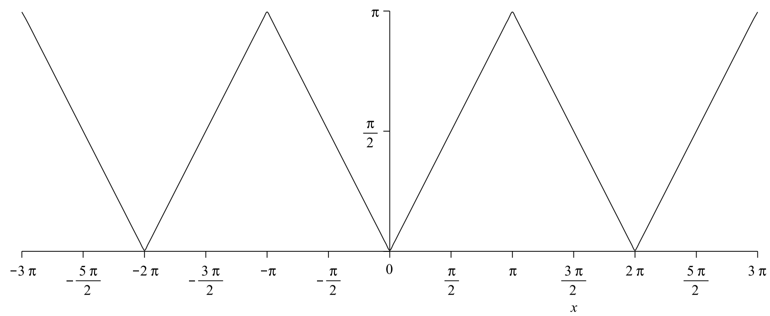


Figure 2.10: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.10b.

c. $f(x) = \cos x$, $|x| < \pi$.

While one can compute the Fourier coefficients by carrying out integrations, it should be noticed that this is a truncated Fourier series and no integration is needed. The result is $f(x) = \cos x$.

d. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

We need to compute all of the Fourier coefficients.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} dx = 1. \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx \\
&= \frac{1}{n\pi}\sin nx\Big|_0^{\pi} = 0. \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx \\
&= -\frac{1}{n\pi}\cos nx\Big|_0^{\pi} \\
&= \frac{1-\cos n\pi}{n\pi}.
\end{aligned}
$$

The resulting Fourier series is

$$
\begin{aligned}
f(x) &\sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n\pi}\sin nx \\
&= \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.
\end{aligned}
$$

The convergence is slow as the terms decay like $\frac{1}{n}$. There are discontinuities in the periodic extension as seen in Figure 2.11.

11. Find the Fourier series representation of $f(x) = x$ on the given interval. Plot the Nth partial sums and describe what you see.

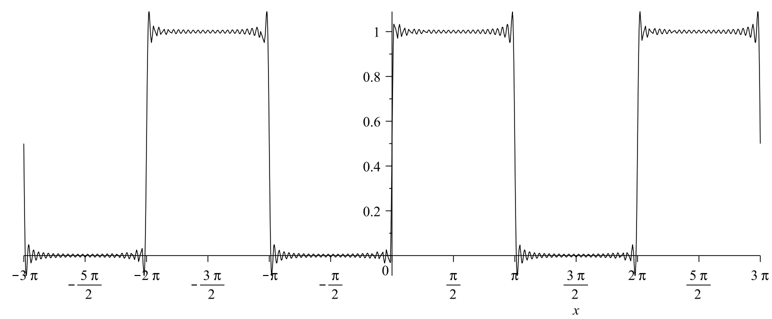


Figure 2.11: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.10c.

a. $0 < x < 2$.

We compute the Fourier coefficients:

$$\begin{aligned}
a_0 &= \frac{2}{2}\int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = 2. \\
a_n &= \frac{2}{2}\int_0^2 x\cos\frac{2n\pi x}{2}\,dx \\
&= \int_0^2 x\cos n\pi x\,dx \\
&= \left[\frac{1}{n\pi}x\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x\right]_0^2 = 0. \\
b_n &= \frac{2}{2}\int_0^2 x\sin\frac{2n\pi x}{2}\,dx \\
&= \int_0^2 x\sin n\pi x\,dx \\
&= \left[-\frac{1}{n\pi}x\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^2 \\
&= -\frac{2}{n\pi}.
\end{aligned}$$

The Fourier series representation is given by

$$f(x) \sim 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}.$$

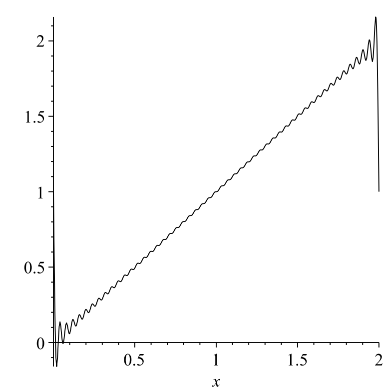


Figure 2.12: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.11a.

A plot of the Fourier series is shown in Figure 2.12.

b. $-2 < x < 2$.

Since $f(x) = x$ is an odd function on $-2 < x < 2$, we only need the b_n's.

$$\begin{aligned}
b_n &= \frac{2}{4}\int_{-2}^2 x\sin\frac{2n\pi x}{4}\,dx \\
&= \int_0^2 x\sin\frac{n\pi x}{2}\,dx \\
&= \left[-\frac{2}{n\pi}x\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2 \\
&= -\frac{4}{n\pi}\cos n\pi.
\end{aligned}$$

The Fourier series is

$$f(x) \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}.$$

A plot of this Fourier series is shown in Figure 2.13.

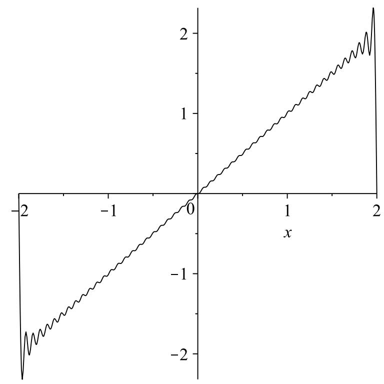


Figure 2.13: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.11b.

c. $1 < x < 2$.

The Fourier coefficients are found as

$$\begin{aligned} a_0 &= 2\int_1^2 x\,dx = x^2\Big|_1^2 = 3. \\ a_n &= 2\int_1^2 x\cos 2n\pi x\,dx \\ &= \left[\frac{1}{2n\pi} x\sin 2n\pi x + \frac{1}{4n^2\pi^2}\cos 2n\pi x\right]_1^2 = 0. \\ b_n &= 2\int_1^2 x\sin 2n\pi x\,dx \\ &= \left[-\frac{1}{2n\pi} x\cos 2n\pi x + \frac{1}{4n^2\pi^2}\sin 2n\pi x\right]_1^2 \\ &= -\frac{1}{n\pi}. \end{aligned}$$

This gives the Fourier series

$$f(x) \sim \frac{3}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{2}.$$

A plot of this Fourier series is shown in Figure 2.14.

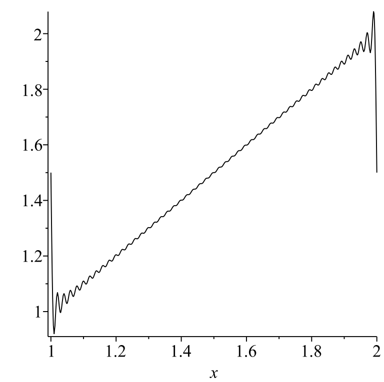


Figure 2.14: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.11c.

12. The result in Problem 9b above gives a Fourier series representation of $\frac{x^2}{4}$. By picking the right value for x and a little arrangement of the series, show that [See Example 2.6.]

a.
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

Using the results in Problem 5.12b, one has that

$$\frac{x^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Letting $x = \pi$, we have

$$\begin{aligned} \frac{\pi^2}{4} &= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi \\ &= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

b.
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Hint: Consider how the series in part a. can be used to do this.

Let
$$S = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Note that
$$\begin{aligned}
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\
&= S + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots \\
&= S + \frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \\
&= S + \frac{1}{4}\left(\frac{\pi^2}{6}\right) \\
&= S + \frac{\pi^2}{24}.
\end{aligned}$$

Therefore, we have
$$\begin{aligned}
S &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \\
&= \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.
\end{aligned}$$

c. Use the Fourier series representation result in Problem 9e. to obtain the series in part b.

The result of Problem 9e is
$$f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1 - \cos n\pi}{\pi n^2} \cos nx + \frac{1}{n} \sin nx\right].$$

Setting $x = \pi$, we have
$$\begin{aligned}
0 &\sim \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{\pi n^2} \cos n\pi \\
&= \frac{\pi}{4} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.
\end{aligned}$$

Therefore,
$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

13. Sketch (by hand) the graphs of each of the following functions over four periods. Then sketch the extensions each of the functions as both an even and odd periodic function. Determine the corresponding Fourier sine and cosine series and verify the convergence to the desired function using Maple.

For these problems we make use of the Fourier cosine series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx,$$

and the Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

a. $f(x) = x^2$, $0 < x < 1$.

The given function is shown in Figure 2.15. In Figure 2.16 this function is reflected about the y-axis and the new function is then periodically extended to give the even periodic extension. In Figure 2.17 the function is reflected about the origin and the new function is then periodically extended to give the odd periodic extension.

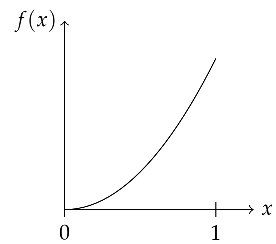


Figure 2.15: Function given in Problem 2.13a.

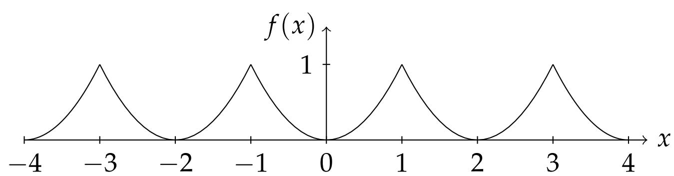


Figure 2.16: Sketch of the even periodic extension of the function given in Problem 2.13a.

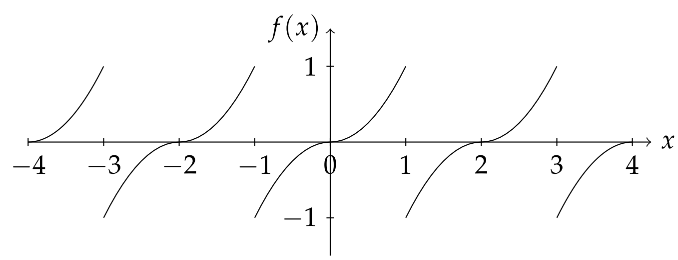


Figure 2.17: Sketch of the odd periodic extension of the function given in Problem 2.13a.

The Fourier cosine series coefficients are given by

$$\begin{aligned} a_0 &= 2\int_0^1 x^2\, dx = \frac{2}{3}. \\ a_n &= 2\int_0^1 x^2 \cos n\pi x\, dx \\ &= 2\left[\frac{1}{n\pi} x^2 \sin n\pi x + \frac{2}{n^2\pi^2} x \cos n\pi x + \frac{2}{(n\pi)^3} \sin n\pi x\right]_0^1 \\ &= \frac{4(-1)^n}{n^2\pi^2}. \end{aligned}$$

The resulting Fourier cosine series is given by

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

A plot of this series representation is shown in Figure 2.18.

The Fourier sine series coefficients are given by

$$\begin{aligned} b_n &= 2\int_0^1 x^2 \sin n\pi x\, dx \\ &= 2\left[-\frac{1}{n\pi}x^2\cos n\pi x + \frac{2}{n^2\pi^2}x\sin n\pi x + \frac{2}{(n\pi)^3}\cos n\pi x\right]_0^1 \\ &= -\frac{2}{n\pi}\cos n\pi + \frac{4}{(n\pi)^3}(\cos n\pi - 1). \end{aligned}$$

The resulting Fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty}(-\frac{2}{n\pi}\cos n\pi + \frac{4}{(n\pi)^3}(\cos n\pi - 1))\sin n\pi x.$$

A plot of this series representation is shown in Figure 2.19.

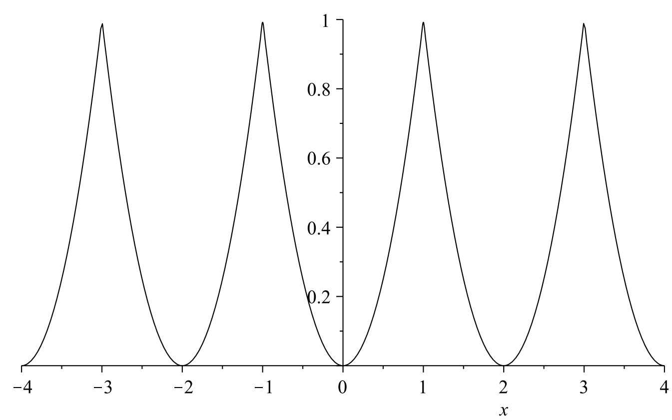


Figure 2.18: A plot of first terms of the Fourier cosine series of $f(x)$ in Problem 2.13a.

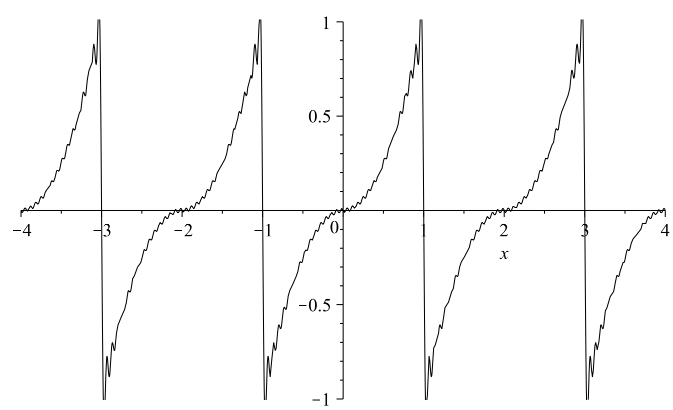


Figure 2.19: A plot of first terms of the Fourier sine series of $f(x)$ in Problem 2.13a.

b. $f(x) = x(2-x)$, $0 < x < 2$.

The given function is shown in Figure 2.20. In Figure 2.21 this function is reflected about the y-axis and the new function is then periodically extended to give the even periodic extension. In Figure 2.22 the function is reflected about the origin and the new function is then periodically extended to give the odd periodic extension.

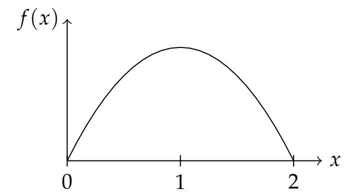


Figure 2.20: Function given in Problem 2.13b.

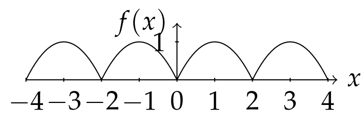


Figure 2.21: Sketch of the even periodic extension of the function given in Problem 2.13b.

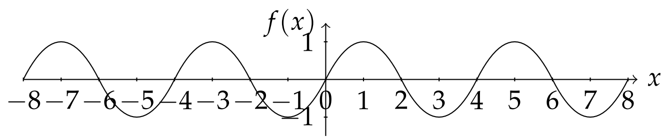


Figure 2.22: Sketch of the odd periodic extension of the function given in Problem 2.13b.

The Fourier cosine series coefficients are given by

$$
\begin{aligned}
a_0 &= \int_0^2 x(2-x)\,dx = \left(x^2 - \frac{x^3}{3}\right)_0^2 = \frac{4}{3}. \\
a_n &= \int_0^2 x(2-x)\cos\frac{n\pi x}{2}\,dx \\
&= \left[\frac{2}{n\pi}x(2-x)\sin\frac{n\pi x}{2} - 2(1-x)\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2} + 2\left(\frac{2}{n\pi}\right)^3\sin\frac{n\pi x}{2}\right]\_0^2 \\
&= \frac{8}{(n\pi)^2}(\cos n\pi - 1).
\end{aligned}
$$

The resulting Fourier cosine series is given by

$$
f(x) = \frac{2}{3} + \sum_{n=1}^{\infty}\frac{8}{(n\pi)^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2} = \frac{2}{3} - \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{(2k-1)\pi x}{2}}{(2k-1)^2}.
$$

A plot of this series representation is shown in Figure 2.23.

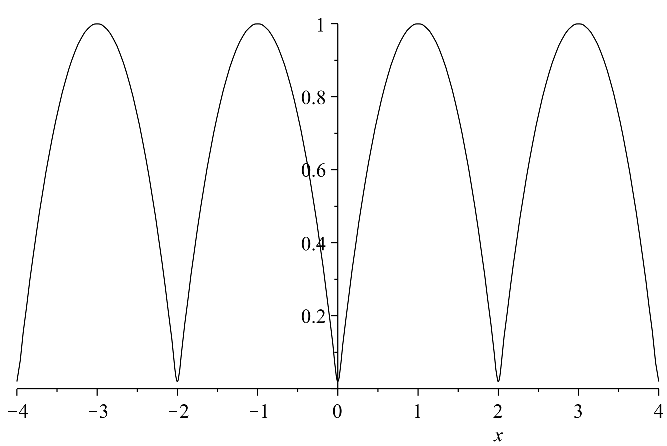


Figure 2.23: A plot of first terms of the Fourier cosine series of $f(x)$ in Problem 2.13b.

The Fourier sine series coefficients are given by

$$\begin{aligned} b_n &= \int_0^2 x(2-x)\sin\frac{n\pi x}{2}\,dx \\ &= \left[-\frac{2}{n\pi}x(2-x)\cos\frac{n\pi x}{2} + 2(1-x)\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi x}{2} - 2\left(\frac{2}{n\pi}\right)^3\cos\frac{n\pi x}{2}\right] \\ &= -\frac{16}{(n\pi)^3}(\cos n\pi - 1). \end{aligned}$$

The resulting Fourier sine series is given by

$$f(x) = -\sum_{n=1}^{\infty}\frac{16}{(n\pi)^3}(\cos n\pi - 1)\sin\frac{n\pi x}{2} = \frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{\sin\frac{(2k-1)\pi x}{2}}{(2k-1)^3}.$$

A plot of this series representation is shown in Figure 2.24.

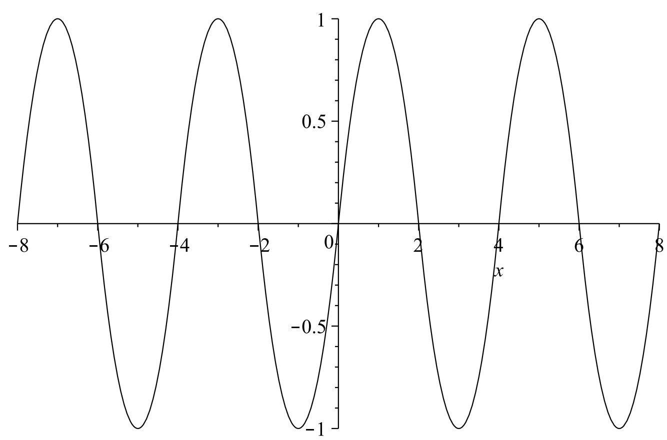


Figure 2.24: A plot of first terms of the Fourier sine series of $f(x)$ in Problem 2.13b.

c. $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$

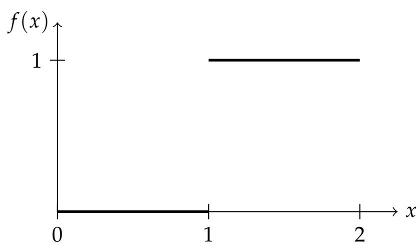


Figure 2.25: Function given in Problem 2.13c.

The given function is shown in Figure 2.25. In Figure 2.26 this function is reflected about the y-axis and the new function is then periodically extended to give the even periodic extension. In Figure 2.27 the function is reflected about the origin and the new function is then periodically extended to give the odd periodic extension.

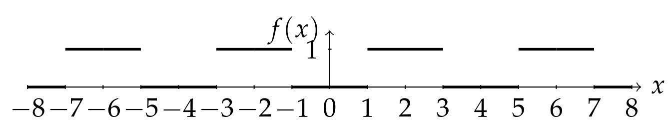


Figure 2.26: Sketch of the even periodic extension of the function given in Problem 2.13c.

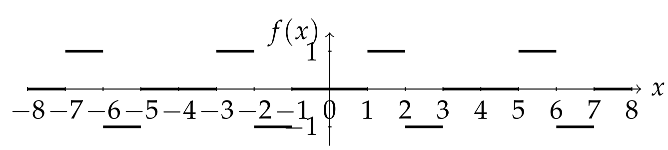


Figure 2.27: Sketch of the odd periodic extension of the function given in Problem 2.13c.

The Fourier cosine series coefficients are given by

$$\begin{aligned} a_0 &= \int_0^2 f(x)\,dx = \int_1^2 dx = 1. \\ a_n &= \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx \\ &= \int_1^2 \cos\frac{n\pi x}{2}\,dx \\ &= \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 = -\frac{2}{n\pi}\sin\frac{n\pi}{2}. \end{aligned}$$

The resulting Fourier cosine series is given by

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}.$$

A plot of this series representation is shown in Figure 2.28.

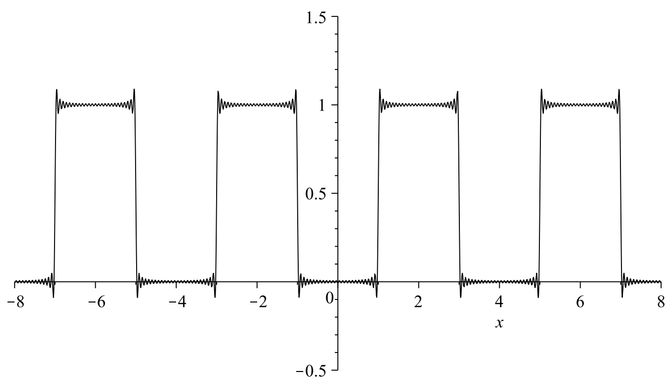


Figure 2.28: A plot of first terms of the Fourier cosine series of $f(x)$ in Problem 2.13c.

The Fourier sine series coefficients are given by

$$\begin{aligned} b_n &= \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx \\ &= \int_1^2 \sin\frac{n\pi x}{2}\,dx \\ &= -\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_1^2 = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right). \end{aligned}$$

The resulting Fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty} \left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin\frac{n\pi x}{2}.$$

A plot of this series representation is shown in Figure 2.29.

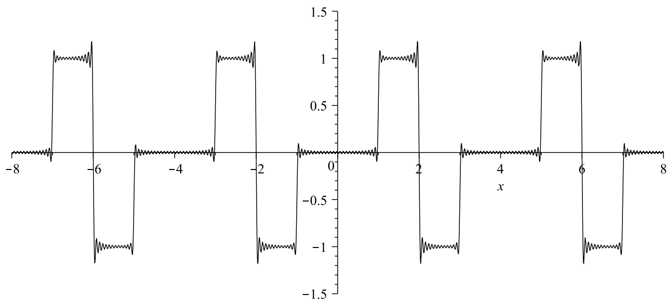


Figure 2.29: A plot of first terms of the Fourier sine series of $f(x)$ in Problem 2.13c.

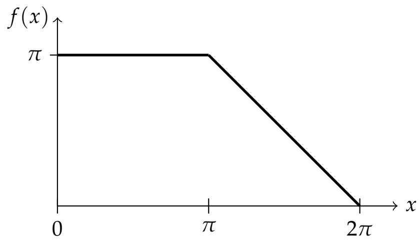


Figure 2.30: Function given in Problem 2.13d.

d. $f(x) = \begin{cases} \pi, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$

The given function is shown in Figure 2.30. In Figure 2.31 this function is reflected about the y-axis and the new function is then periodically extended to give the even periodic extension. In Figure 2.32 the function is reflected about the origin and the new function is then periodically extended to give the odd periodic extension.

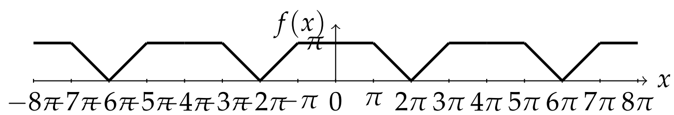


Figure 2.31: Sketch of the even periodic extension of the function given in Problem 2.13d.

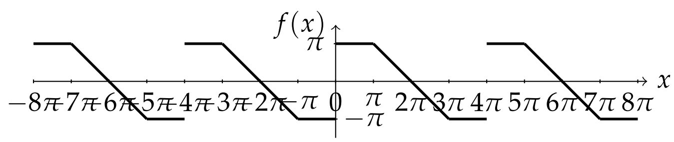


Figure 2.32: Sketch of the odd periodic extension of the function given in Problem 2.13d.

The Fourier cosine series coefficients are given by

$$\begin{aligned}
a_0 &= \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} \pi\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\,dx \\
&= \frac{3\pi}{2}. \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos\frac{nx}{2}\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} \pi\cos\frac{nx}{2}\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\cos\frac{nx}{2}\,dx \\
&= \frac{2}{n}\sin\frac{nx}{2}\Big|_0^{\pi} + \frac{1}{\pi}\left[\frac{2}{n}(2\pi - x)\sin\frac{nx}{2} - \frac{4}{n^2}\cos\frac{nx}{2}\right]_{\pi}^{2\pi} \\
&= -\frac{4}{n^2}(\cos n\pi - \cos\frac{n\pi}{2}).
\end{aligned}$$

The resulting Fourier cosine series is given by

$$f(x) = \frac{3\pi}{4} - 4\sum_{n=1}^{\infty} \frac{\cos n\pi - \cos\frac{n\pi}{2}}{n^2}\cos\frac{nx}{2}.$$

A plot of this series representation is shown in Figure 2.33.

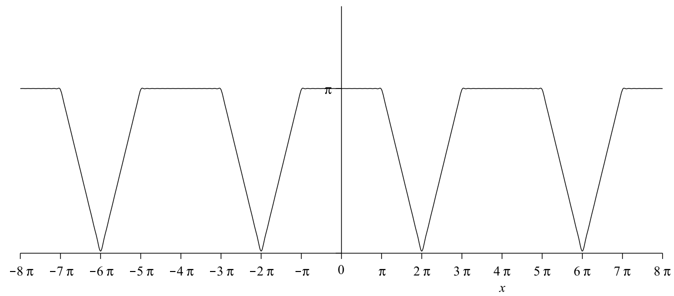


Figure 2.33: A plot of first terms of the Fourier cosine series of $f(x)$ in Problem 2.13d.

The Fourier sine series coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\frac{nx}{2}\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} \pi\sin\frac{nx}{2}\,dx + \frac{1}{\pi}\int_\pi^{2\pi}(2\pi - x)\sin\frac{nx}{2}\,dx \\ &= -\frac{2}{n}\cos\frac{nx}{2}\Big|_0^{\pi} + \frac{1}{\pi}\left[-\frac{2}{n}(2\pi - x)\cos\frac{nx}{2} + \frac{4}{n^2}\sin\frac{nx}{2}\right]_\pi^{2\pi} \\ &= \frac{2}{n} + \frac{4}{\pi n^2}\sin\frac{n\pi}{2}. \end{aligned}$$

The resulting Fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{n} + \frac{4}{\pi n^2}\sin\frac{n\pi}{2}\right)\sin\frac{nx}{2}.$$

A plot of this series representation is shown in Figure 2.34.

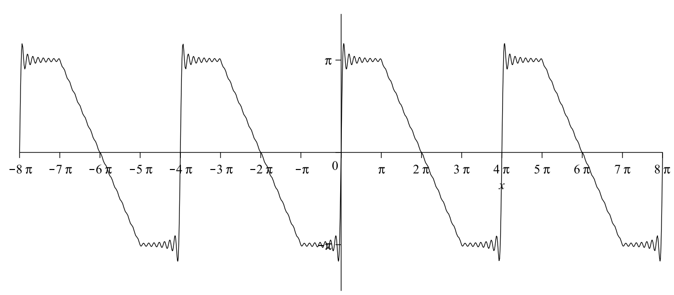


Figure 2.34: A plot of first terms of the Fourier sine series of $f(x)$ in Problem 2.13d.

14. Consider the function $f(x) = x$, $-\pi < x < \pi$.

a. Show that $x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$.

Since $f(x) = x$ is an odd function on $|x| < \pi$, the a_n's vanish for all n.

The Fourier sine series coefficients are

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \end{aligned}$$

$$\begin{aligned} &= \frac{2}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^\pi \\ &= -\frac{2\cos n\pi}{n} = 2\frac{(-1)^{n+1}}{n}. \end{aligned}$$

This gives the Fourier sine series representation

$$f(x) \sim 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

b. Integrate the series in part a and show that $x^2 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}$.

For $f(x) = x$, we consider the integral

$$\int_0^x f(\xi)\,d\xi = \frac{x^2}{2}.$$

Integrating the series as well, we have

$$\begin{aligned} \int_0^x f(\xi)\,d\xi &\sim 2\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^x \frac{\sin n\xi}{n}\,d\xi. \\ &= 2\sum_{n=1}^{\infty}(-1)^n\frac{\cos n\xi}{n^2}\bigg|_0^x \\ &= 2\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2} - 2\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} \end{aligned}$$

Therefore,

$$x^2 = C + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2},$$

where

$$C = -2\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}.$$

We still need to evaluate C. We can use the results in Problem 15 to do this.

$$\begin{aligned} C &= -4\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} \\ &= 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \cdots\right) \\ &= 4\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) - 4\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) \\ &= 4\left(\frac{\pi^2}{8}\right) - 4\left(\frac{\pi^2}{24}\right) = \left(\frac{\pi^2}{3}\right). \end{aligned}$$

Therefore, we have that

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}.$$

c. Find the Fourier cosine series of $f(x) = x^2$ on $[0, \pi]$ and compare to the result in part b.

We first determine the Fourier cosine series coefficients.

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2\pi^2}{3}. \\
a_n &= \frac{2}{\pi}\int_0^\pi x^2 \cos nx\,dx \\
&= \frac{2}{\pi}\left[\frac{1}{n}x^2\sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_0^\pi \\
&= \frac{2}{\pi}\left(\frac{2}{n^2}\pi\cos n\pi\right) = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n.
\end{aligned}$$

This gives the Fourier cosine series representation

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^\infty (-1)^n \frac{\cos nx}{n^2}.$$

d. Apply Parseval's identity in Problem 5 to the series in part a. for $f(x) = x$ on $-\pi < x < \pi$. This gives another means to finding the value $\zeta(4)$, where the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.$$

From the Parseval identity, we have

$$\begin{aligned}
\frac{2}{L}\int_0^L f^2(x)\,dx &= \frac{a_0^2}{2} + \sum_{n=1}^\infty a_n^2 \\
\frac{2}{\pi}\int_0^\pi x^4\,dx &= \frac{2\pi^4}{9} + 16\sum_{n=1}^\infty \frac{1}{n^4} \\
\frac{2}{\pi}\frac{\pi^5}{5} &= \frac{2\pi^4}{9} + \sum_{n=1}^\infty \frac{1}{n^4}.
\end{aligned}$$

This gives

$$16\sum_{n=1}^\infty \frac{1}{n^4} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9} = \frac{8\pi^4}{45},$$

resulting in

$$\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$$

15. Consider the function $f(x) = x$, $0 < x < 2$.

a. Find the Fourier sine series representation of this function and plot the first 50 terms.

The Fourier sine series coefficients are

$$\begin{aligned}
b_n &= \int_0^2 x\sin\frac{n\pi x}{2}\,dx \\
&= \left[-\frac{2}{\pi n}x\cos\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi x}{2}\right]_0^2 \\
&= -\frac{4\cos n\pi}{n\pi} = \frac{4}{n\pi}(-1)^{n+1}.
\end{aligned}$$

This gives the Fourier sine series representation

$$f(x) \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}.$$

The first few terms of this series are shown in Figure 2.35.

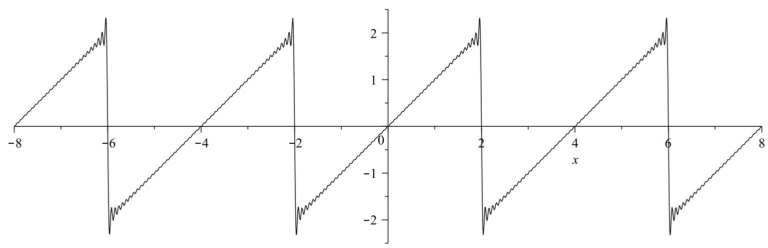


Figure 2.35: A plot of first terms of the Fourier sine series of $f(x)$ in Problem 2.15a.

b. Find the Fourier cosine series representation of this function and plot the first 50 terms.

The Fourier cosine series coefficients are

$$\begin{aligned} a_0 &= \int_0^2 x\, dx = 2. \\ a_n &= \int_0^2 x \cos \frac{n\pi x}{2}\, dx \\ &= \left[\frac{2}{\pi n} x \sin \frac{n\pi x}{2} + \left(\frac{2}{n\pi} \right)^2 \cos \frac{n\pi x}{2} \right]_0^2 \\ &= \frac{4}{n^2\pi^2} (\cos n\pi - 1). \end{aligned}$$

This gives the Fourier cosine series representation

$$\begin{aligned} f(x) &\sim 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^2} \cos \frac{n\pi x}{2} \\ &= 1 - \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos \frac{(2k-1)\pi x}{2}. \end{aligned}$$

The first few terms of this series are shown in Figure 2.36.

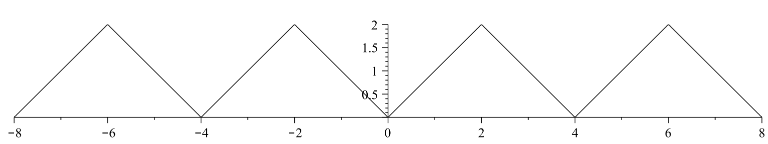


Figure 2.36: A plot of first terms of the Fourier cosine series of $f(x)$ in Problem 2.15b.

c. Apply Parseval's identity in Problem 5 to the result in part b.

Parseval's identity can be extended to Fourier cosine series by slightly modifying the derivation in Problem 5.8.

We begin with the Parseval identity

$$\frac{2}{L}\int_0^L f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2.$$

Applying this result to $f(x) = x$, $x \in [0,2]$. From part b, we have $a_0 = 2$ and

$$a_n = \frac{4}{n^2\pi^2}(\cos n\pi - 1), \quad n = 1,2,\ldots.$$

Then,

$$\begin{aligned}
\frac{2}{L}\int_0^L f^2(x)\,dx &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \\
\int_0^2 x^2\,dx &= \frac{2^2}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}(\cos n\pi - 1)\right)^2 \\
\frac{8}{3} &= 2 + \frac{64}{\pi}^4 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}.
\end{aligned}$$

This gives the sum

$$\begin{aligned}
\frac{\pi^4}{96} &= \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} \\
&= 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots.
\end{aligned}$$

d. Use the result of part c. to find the sum $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

The result in part c is not quite the sum we seek as the terms involve only the odd terms. We can rearrange the series to make use of the result in part c and solve for $S = \sum_{n=1}^{\infty}\frac{1}{n^4}$:

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}\frac{1}{n^4} \\
&= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{3^5} + \frac{1}{4^6} + \cdots \\
&= \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right) \\
&= \frac{\pi^4}{96} + \frac{1}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right) \\
&= \frac{\pi^4}{96} + \frac{1}{16}S \\
\frac{15}{16}S &= \frac{\pi^4}{96} \\
S &= \frac{\pi^4}{90}.
\end{aligned}$$

Therefore, we have shown that $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

16. Differentiate the Fourier sine series term-by-term in the last problem. Show that the result is not the derivative of $f(x) = x$..

The Fourier sine series is given by

$$f(x) \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}.$$

A simple term by term differentiation gives

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{d}{dx} \left(\sin \frac{n\pi x}{2} \right) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi x}{2}.$$

However, this is a divergent series and cannot sum to $f'(x) = 1$.

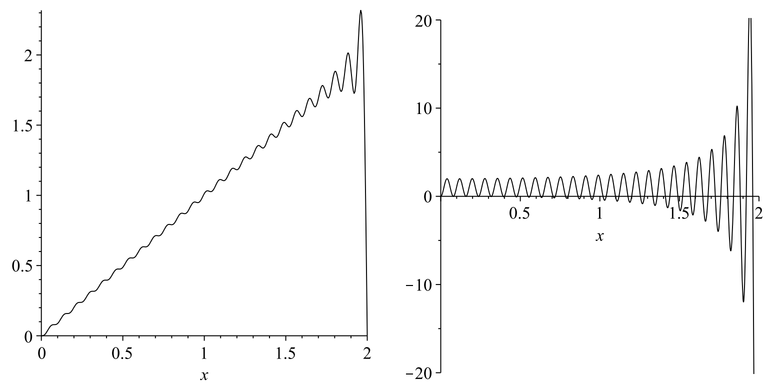


Figure 2.37: Plot of first 50 terms of (left) the Fourier sine series of $f(x) = x$ and (right) the derivative of these terms from Problem 2.16.

17. Consider the function $f(x) = x \sin x$.

a. Find the Fourier series representation of this function if $f(x)$ is defined on $[0, 2\pi]$ and plot the first 50 terms.

We compute the Fourier coefficients:

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx \\
&= \frac{1}{\pi} \int_0^{2\pi} x \sin x\, dx \\
&= \frac{1}{\pi} [-x \cos x + \sin x] \Big|_0^{2\pi} \\
&= -2. \\
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx, \quad n > 1, \\
&= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx\, dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\, dx \\
&= \frac{1}{2\pi} \left[\frac{-x \cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} \right. \\
&\qquad \left. - \frac{-x \cos(n-1)x}{n-1} - \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{2\pi}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{n^2-1}. \\
a_1 &= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx \\
&= \frac{1}{2\pi}\left[-\frac{1}{2}x\cos x+\frac{1}{4}\sin 2x\right]_0^{2\pi} \\
&= -\frac{1}{2}. \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx, \quad n>1, \\
&= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x-\cos(n+1)x]\,dx \\
&= \frac{1}{2\pi}\left[\frac{x\sin(n+1)x}{n+1}+\frac{\cos(n+1)x}{(n+1)^2}\right. \\
&\quad \left.-\frac{x\sin(n-1)x}{n-1}-\frac{\cos(n-1)x}{(n-1)^2}\right]_0^{2\pi} \\
&= 0. \\
b_1 &= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx, \\
&= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx, \\
&= \frac{1}{2\pi}\left[\frac{x^2}{2}-\frac{1}{2}x\sin 2x-\frac{1}{4}\cos 2x\right]_0^{2\pi} \\
&= \pi.
\end{aligned}
$$

The Fourier series representation is given by

$$
f(x) \sim -1-\frac{1}{2}\cos x+\pi\sin x+2\sum_{n=2}^{\infty}\frac{\cos nx}{n^2-1}.
$$

The Fourier series representation is shown in Figure 2.38.

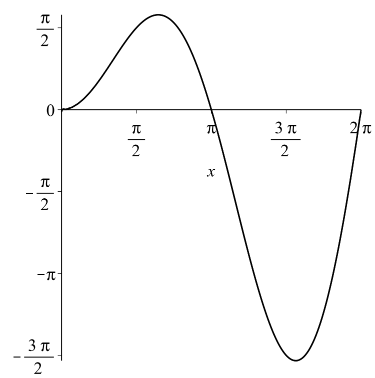


Figure 2.38: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.17a.

b. Find the Fourier series representation of this function if $f(x)$ is defined on $[-\pi,\pi]$ and plot the first 50 terms.

We compute the Fourier coefficients:

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx \\
&= \frac{2}{\pi}[-x\cos x+\sin x]\Big|_0^{\pi} \\
&= 2. \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n>1, \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x-\sin(n-1)x]\,dx
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{\pi}\left[\frac{-x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right. \\
&\qquad \left. - \frac{-x\cos(n-1)x}{n-1} - \frac{\sin(n-1)x}{(n-1)^2}\right]_0^\pi \\
&= \frac{2(-1)^{n+1}}{n^2-1} \\
a_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin 2x\,dx \\
&= \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx \\
&= \frac{1}{\pi}\left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^\pi \\
&= -\frac{1}{2} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\sin nx\,dx \\
&= 0.
\end{aligned}$$

The Fourier series representation is given by

$$f(x) \sim 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2-1}.$$

The Fourier series representation is shown in Figure 2.39.

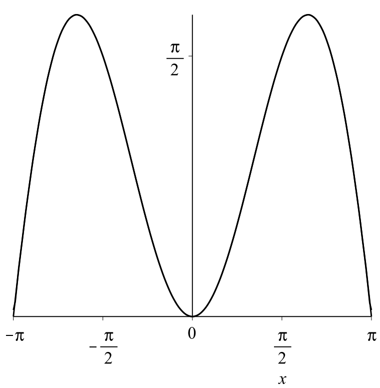


Figure 2.39: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.17b.

18. Consider the function $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1-x, & 1 < x < 2. \end{cases}$

a. Find the Fourier series representation of this function and plot the first 50 terms.

We compute the Fourier coefficients:

$$\begin{aligned}
a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\
&= \int_0^1 x\,dx + \int_0^1 x\,dx + \int_1^2 1-x\,dx \\
&= \frac{x^2}{2}\Big|_0^1 - \frac{(1-x)^2}{2}\Big|_1^2 \\
&= \frac{1}{2} - \frac{1}{2} = 0. \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx \\
&= \int_0^1 x\cos n\pi x\,dx + \int_1^2 (1-x)\cos n\pi x\,dx \\
&= \left[\frac{1}{n\pi}x\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x\right]_0^1 \\
&\quad + \left[\frac{1}{n\pi}(1-x)\sin n\pi x - \frac{1}{n^2\pi^2}\cos n\pi x\right]_1^2
\end{aligned}$$

$$
\begin{aligned}
&= 2\frac{\cos n\pi - 1}{n^2\pi^2}. \\
b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx \\
&= \int_0^1 x\sin n\pi x\,dx + \int_1^2 (1-x)\sin n\pi x\,dx \\
&= \left[-\frac{1}{n\pi}x\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^1 \\
&\quad + \left[-\frac{1}{n\pi}(1-x)\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x\right]_1^2 \\
&= \frac{1-\cos n\pi}{n\pi}.
\end{aligned}
$$

The Fourier series representation is given by

$$
\begin{aligned}
f(x) &\sim \sum_{n=1}^{\infty}\left[2\frac{\cos n\pi - 1}{n^2\pi^2}\cos n\pi x + \frac{1-\cos n\pi}{n\pi}\sin n\pi x\right] \\
&= \sum_{k=1}^{\infty}\left[-\frac{4}{(2k-1)^2\pi^2}\cos(2k-1)\pi x + \frac{2}{(2k-1)\pi}\sin(2k-1)\pi x\right].
\end{aligned}
$$

The Fourier series representation is shown in Figure 2.40.

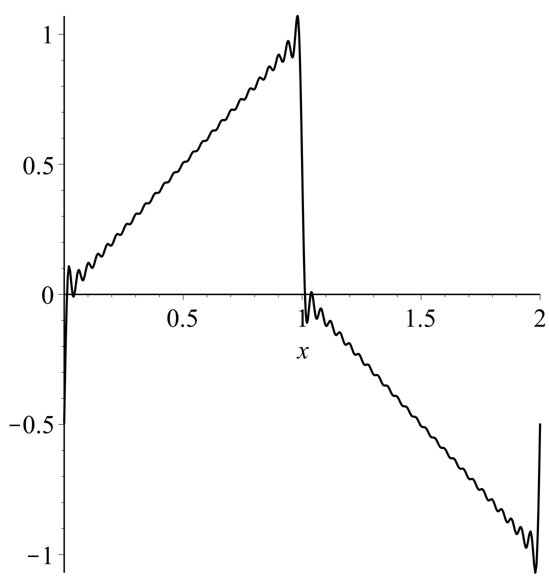


Figure 2.40: A plot of first terms of the Fourier series of $f(x)$ in Problem 2.18.

b. Use the result of part a. to show that

$$
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.
$$

[Hint: Be careful using the discontinuity at $x = 1$ by noting that the Fourier series converges to $f(1) = \frac{1}{2}(f(1^+) + f(1^-))$.]

Setting $x = 1$ in the previous result and noting that

$$
f(1) = \frac{1}{2}(f(1^+) + f(1^-)) = \frac{1}{2},
$$

we have

$$\begin{aligned} \frac{1}{2} &\sim \sum_{k=1}^{\infty} -\frac{4}{(2k-1)^2\pi^2}\cos(2k-1)\pi \\ \frac{\pi^2}{8} &= \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}. \end{aligned}$$

19. The temperature, $u(x,t)$, of a one-dimensional rod of length L satisfies the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

a. Show that the general solution,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2},$$

satisfies the one-dimensional heat equation and the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$.

Computing u_t and u_{xx}, we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= -k\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2}. \\ \frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} \frac{n\pi}{L} b_n \cos\frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2}. \\ \frac{\partial^2 u}{\partial x^2} &= -\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2}, \end{aligned}$$

Comparing these derivatives, we see that $u_t = ku_{xx}$. Note that the vanishing of the function at the interval endpoints allows the differentiation of the sine series.

Furthermore, we have

$$u(0,t) = \sum_{n=1}^{\infty} b_n \sin 0 e^{-n^2\pi^2 kt/L^2} = 0,$$

$$u(L,t) = \sum_{n=1}^{\infty} b_n \sin n\pi e^{-n^2\pi^2 kt/L^2} = 0.$$

b. For $k = 1$ and $L = \pi$, find the solution satisfying the initial condition $u(x,0) = \sin x$. Plot six solutions on the same set of axes for $t \in [0,1]$.

For $k = 1$ and $L = \pi$, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx e^{-n^2 t}.$$

Using the initial condition, $u(x,0) = \sin x$, we have

$$\sin x = \sum_{n=1}^{\infty} b_n \sin nx.$$

The Fourier coefficients are easily found without integration as $b_1 = 1$ and $b_n = 0$, $n > 1$. Then, the solution to the initial-boundary value problem is

$$u(x,t) = \sin x e^{-t}.$$

This solution at six times is shown in Figure 2.41.

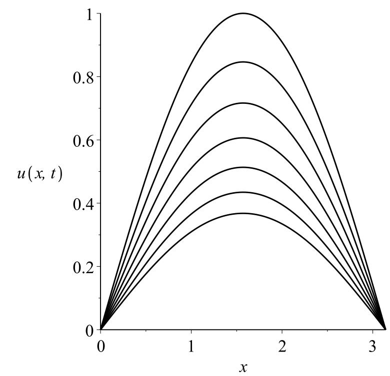


Figure 2.41: A plot of solutions for the heat equation in Problem 2.19b for $t = 0, 1/6, \ldots, 1$.

c. For $k = 1$ and $L = 1$, find the solution satisfying the initial condition $u(x,0) = x(1-x)$. Plot six solutions on the same set of axes for $t \in [0,1]$.

For $k = 1$ and $L = 1$, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}.$$

Using the initial condition, $u(x,0) = x(1-x)$, we have

$$x(1-x) = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

We need to determine the Fourier sine coefficients:

$$\begin{aligned} b_n &= 2\int_0^1 x(1-x)\sin n\pi x\,dx \\ &= 2\left[-\frac{1}{n\pi}x(1-x)\cos n\pi x + \frac{1}{n^2\pi^2}(1-2x)\sin n\pi x\right. \\ &\quad \left. -\frac{2}{(n\pi)^3}\cos n\pi x\right]_0^1 \\ &= \frac{4}{(n\pi)^3}(1-\cos n\pi). \end{aligned}$$

So,

$$u(x,0) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}(1-\cos n\pi)\sin n\pi x,$$

or

$$u(x,0) = \sum_{k=1}^{\infty} \frac{8}{((2k-1)\pi)^3}\sin(2k-1)\pi x.$$

The solution to the initial-boundary value problem is

$$u(x,t) = \sum_{k=1}^{\infty} \frac{8}{((2k-1)\pi)^3}\sin(2k-1)\pi x e^{-(2k-1)^2\pi^2 t}.$$

This solution at six times is shown in Figure 2.42.

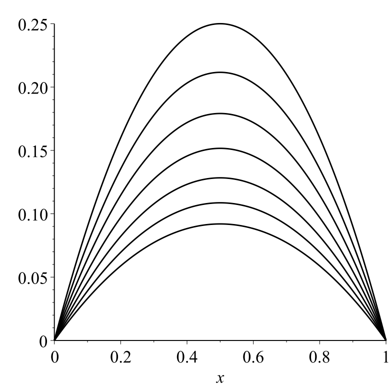


Figure 2.42: A plot of solutions for the heat equation in Problem 2.19c for $t = 0, 1/6, \ldots, 1$.

20. The height, $u(x,t)$, of a one-dimensional vibrating string of length L satisfies the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

a. Show that the general solution,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L} \\ &\quad + B_n \sin\frac{n\pi ct}{L} \sin\frac{n\pi x}{L}, \end{aligned}$$

satisfies the one-dimensional wave equation and the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$.

In order to verify that this is a solution, we compute a few derivatives:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\sum_{n=1}^{\infty} A_n \frac{n\pi c}{L} \sin\frac{n\pi ct}{L} \sin\frac{n\pi x}{L} \\ &\quad + B_n \frac{n\pi c}{L} \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L}. \\ \frac{\partial^2 u}{\partial t^2} &= -\sum_{n=1}^{\infty} A_n \left(\frac{n\pi c}{L}\right)^2 \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L} \\ &\quad - B_n \left(\frac{n\pi c}{L}\right)^2 \sin\frac{n\pi ct}{L} \sin\frac{n\pi x}{L}. \\ \frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \cos\frac{n\pi ct}{L} \cos\frac{n\pi x}{L} \\ &\quad + B_n \frac{n\pi}{L} \sin\frac{n\pi ct}{L} \cos\frac{n\pi x}{L}. \\ \frac{\partial^2 u}{\partial x^2} &= -\sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L} \\ &\quad - B_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi ct}{L} \sin\frac{n\pi x}{L}. \end{aligned}$$

Comparing derivatives, we have

$$u_{tt} = c^2 u_{xx}.$$

Note that the vanishing of the function at the interval endpoints allows the differentiation of the sine series.

We also can verify the boundary conditions:

$$\begin{aligned} u(0,t) &= \sum_{n=1}^{\infty} A_n \cos\frac{n\pi ct}{L} \sin 0 \\ &\quad + B_n \sin\frac{n\pi ct}{L} \sin 0 = 0. \end{aligned}$$

$$\begin{aligned} u(L,t) &= \sum_{n=1}^{\infty} A_n \cos\frac{n\pi ct}{L} \sin n\pi \\ &\quad + B_n \sin\frac{n\pi ct}{L} \sin n\pi = 0. \end{aligned}$$

b. For $c = 1$ and $L = 1$, find the solution satisfying the initial conditions $u(x,0) = x(1-x)$ and $u_t(x,0) = 0$. Plot five solutions for $t \in [0,1]$.

For this problem, the general solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos n\pi t \sin n\pi x + B_n \sin n\pi t \sin n\pi x.$$

The initial conditions give

$$\begin{aligned} x(1-x) &= \sum_{n=1}^{\infty} A_n \sin n\pi x, \\ 0 &= \sum_{n=1}^{\infty} B_n n\pi \sin n\pi x. \end{aligned}$$

The second equation gives $B_n = 0$, $n = 1, 2, \ldots$.

From the previous problem we have

$$A_n = \frac{4}{(n\pi)^3}(1 - \cos n\pi), \quad n = 1, 2, \ldots.$$

Therefore, the solution to the initial-boundary value problem is

$$u(x,t) = \sum_{k=1}^{\infty} \frac{8}{((2k-1)\pi)^3} \sin(2k-1)\pi x \cos(2k-1)\pi t.$$

This solution at five times is shown in Figure 2.43.

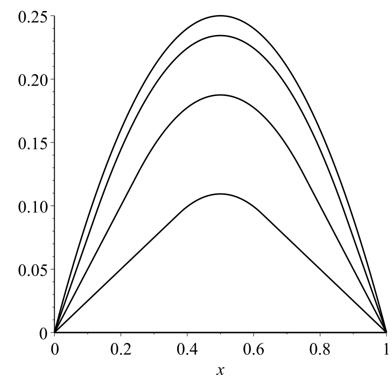


Figure 2.43: A plot of solutions for the wave equation in Problem 2.20b for $t = 0, 1/8, \ldots, 1/2$.

c. For $c = 1$ and $L = 1$, find the solution satisfying the initial condition

$$u(x,0) = \begin{cases} 4x, & 0 \le x \le \frac{1}{4}, \\ \frac{4}{3}(1-x), & \frac{1}{4} \le x \le 1, \end{cases}$$

and $u_t(x,0) = 0$. Plot five solutions for $t \in [0, 0.5]$.

For this problem, the general solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos n\pi t \sin n\pi x + B_n \sin n\pi t \sin n\pi x.$$

The initial conditions give

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} A_n \sin n\pi x, \\ 0 &= \sum_{n=1}^{\infty} B_n n\pi \sin n\pi x. \end{aligned}$$

The second equation gives $B_n = 0$, $n = 1, 2, \ldots$.

The remaining Fourier sine coefficients can be computed using

$$A_n = 2\int_0^1 u(x,0) \sin n\pi x \, dx.$$

Thus,

$$A_n = 8\int_0^{\frac{1}{4}} x \sin n\pi x \, dx + \frac{8}{3}\int_{\frac{1}{4}}^1 (1-x) \sin n\pi x \, dx$$

$$
\begin{aligned}
&= 8\left[-\frac{1}{n\pi}x\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^{\frac14} \\
&\quad +\frac{8}{3}\left[-\frac{1}{n\pi}(1-x)\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x\right]_{\frac14}^{1} \\
&= 8\left[-\frac{1}{4n\pi}\cos\frac{n\pi}{4} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}\right] \\
&\quad -\frac{8}{3}\left[-\frac{3}{4n\pi}\cos\frac{n\pi}{4} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{4}\right] \\
&= \frac{32}{3n^2\pi^2}\sin\frac{n\pi}{4}.
\end{aligned}
$$

So, the solution to the initial-boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{32}{3n^2\pi^2}\sin\frac{n\pi}{4}\sin n\pi x\cos n\pi t.$$

This solution at five times is shown in Figure 2.44.

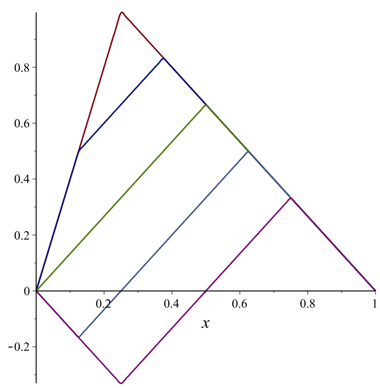


Figure 2.44: A plot of solutions for the wave equation in Problem 2.20c for $t = 0, 1/8, \ldots, 1/2$.

21. Show that

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H},$$

where

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2},$$

satisfies the two-dimensional wave equation

$$u_{tt} = c^2(u_{xx}+u_{yy}), \quad t>0, 0<x<L, 0<y<H,$$

and the boundary conditions,

$$
\begin{aligned}
u(0,y,t) = 0, &\quad u(L,y,t) = 0, \quad t>0, \quad 0<y<H, \\
u(x,0,t) = 0, &\quad u(x,H,t) = 0, \quad t>0, \quad 0<x<L,
\end{aligned}
$$

Computing the needed second parital derivatives of $u(x,y,t)$, we have

$$
\begin{aligned}
u_{tt} &= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\omega_{nm}^2(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \\
u_{xx} &= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n\pi}{L}\right)^2(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \\
u_{yy} &= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{M\pi}{H}\right)^2(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}
\end{aligned}
$$

Inserting these derivatives into the the two-dimensional wave equation, we find

$$\omega_{nm}^2 = c^2\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2.$$

The solution easily satisfies the boundary conditions. For example,

$$
\begin{aligned}
u(0,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin 0\sin\frac{m\pi y}{H} = 0, \\
u(L,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin n\pi\sin\frac{m\pi y}{H} = 0.
\end{aligned}
$$

The boundary conditions, $u(x,0,t)=0, \quad u(x,H,t)=0$, follow in the same way.

22. Find the double Fourier sine series representation of the following:

A function $f(x,y)$ defined on the rectangular region $[0,L]\times[0,H]$ has a double Fourier sine series representation,

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H},$$

where

$$b_{nm}=\frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dxdy \quad n,m=1,2,\ldots.$$

This representation will be used to obtain the series in this problem.

a. $f(x,y)=\sin\pi x\sin 2\pi y$ on $[0,1]\times[0,1]$.

The series expansion for this problem is given by

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin n\pi x\sin m\pi y.$$

It is easy to see that the is only nonzero term, for $n=1$ and $m=2$. Thus, $b_{12}=1$ and $b_{nm}=0$, for $n\neq 1$ and $m\neq 2$.

b. $f(x,y)=x(2-x)\sin y$ on $[0,2]\times[0,\pi]$.

The series expansion for this problem is given by

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\frac{n\pi x}{2}\sin my,$$

Here $b_{nm}=0$ for $m\neq 1$. We need only compute the b_{n1} terms.

$$\begin{aligned}
b_{n1} &= \frac{2}{\pi}\int_0^2\int_0^{\pi} f(x,y)\sin\frac{n\pi x}{2}\sin y\,dxdy\\
&= \frac{2}{\pi}\int_0^2\int_0^{\pi} x(2-x)\sin\frac{n\pi x}{2}\sin^2 y\,dxdy\\
&= \int_0^2 x(2-x)\sin\frac{n\pi x}{2}\,dx\\
&= \left[-\frac{2}{n\pi}x(2-x)\cos\frac{n\pi x}{2}+\frac{4}{n^2\pi^2}(2-2x)\sin\frac{n\pi x}{2}-\frac{16}{n^3\pi^3}\cos\frac{n\pi x}{2}\right]_0^2\\
&= \frac{16}{n^3\pi^3}(1-cosn\pi).
\end{aligned}$$

This gives the series expansion

$$\begin{aligned}
f(x,y) &= \sum_{n=1}^{\infty}\frac{16}{n^3\pi^3}(1-cosn\pi)\sin\frac{n\pi x}{2}\sin y,\\
&= \sum_{k=1}^{\infty}\frac{32}{(2k-1)^3\pi^3}\sin\frac{(2k-1)\pi x}{2}\sin y.
\end{aligned}$$

c. $f(x,y) = x^2y^3$ on $[0,1] \times [0,1]$.

The series expansion for this problem is given by

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin n\pi x \sin m\pi y,$$

The Fourier coefficients can be computed directly:

$$\begin{aligned} b_{nm} &= 4\int_0^1\int_0^1 x^2y^3 \sin n\pi x \sin m\pi y \, dxdy \quad n,m = 1,2,\ldots. \\ &= 4\left(\int_0^1 x^2 \sin n\pi x \, dx\right)\left(\int_0^1 y^3 \sin m\pi y \, dy\right). \end{aligned}$$

Each integral can be computed separately:

$$\begin{aligned} \int_0^1 x^2 \sin n\pi x \, dx &= \left[-\frac{1}{n\pi}x^2 \cos n\pi x + \frac{2}{n^2\pi^2} x \sin n\pi x \right. \\ &\quad \left. + \frac{2}{n^3\pi^3} \cos n\pi x\right]_0^1 \\ &= \frac{2}{n^3\pi^3}(\cos n\pi - 1) - \frac{\cos n\pi}{n\pi}. \end{aligned}$$

$$\begin{aligned} \int_0^1 y^3 \sin m\pi y \, dy &= \left[-\frac{1}{m\pi}y^3 \cos m\pi y + \frac{3}{m^2\pi^2} y^2 \sin m\pi y \right. \\ &\quad \left. + \frac{6}{m^3\pi^3} y \cos m\pi y - \frac{6}{m^4\pi^4} \sin m\pi y\right]_0^1 \\ &= \frac{6\cos m\pi}{m^3\pi^3} - \frac{\cos m\pi}{m\pi}. \end{aligned}$$

This gives

$$b_{nm} = 4\left(\frac{2}{n^3\pi^3}(\cos n\pi - 1) - \frac{\cos n\pi}{n\pi}\right)\left(\frac{6\cos m\pi}{m^3\pi^3} - \frac{\cos m\pi}{m\pi}\right).$$

The resulting Fourier series is

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin n\pi x \sin m\pi y.$$

23. Derive the Fourier coefficients in the double Fourier trigonometric series in Equation (2.124).

The double Fourier trigonometric series in Equation (2.124) is given by

$$\begin{aligned} f(x,y) \sim \; & \frac{a_{00}}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_{n0}\cos\frac{2n\pi x}{L} + d_{n0}\sin\frac{2n\pi x}{L}\right] \\ & + \frac{1}{2}\sum_{m=1}^{\infty}\left[a_{0m}\cos\frac{2m\pi y}{H} + c_{0m}\sin\frac{2m\pi y}{H}\right] \\ & + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\cos\frac{2n\pi x}{L}\cos\frac{2m\pi y}{H}, \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin \frac{2n\pi x}{L} \sin \frac{2m\pi y}{H}, \\
&+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \cos \frac{2n\pi x}{L} \sin \frac{2m\pi y}{H}, \\
&+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm} \sin \frac{2n\pi x}{L} \cos \frac{2m\pi y}{H}.
\end{aligned}
$$

In order to prove this, one needs to consider the Fourier basis on the rectangular region $[0, L] \times [0, H]$,

$$
\begin{aligned}
\phi_{00} &= 1, \\
\phi_{n0} &= \left\{ \cos \frac{2n\pi x}{L}, \sin \frac{2n\pi x}{L}, \right\}, \quad n = 1, 2, \ldots, \\
\phi_{0n} &= \left\{ \cos \frac{2n\pi y}{H}, \sin \frac{2n\pi y}{H}, \right\}, \quad n = 1, 2, \ldots, \\
\phi_{nm} &= \left\{ \cos \frac{2n\pi x}{L} \cos \frac{2m\pi y}{H}, \cos \frac{2n\pi x}{L} \sin \frac{2m\pi y}{H}, \right. \\
&\quad \left. \sin \frac{2n\pi x}{L} \cos \frac{2m\pi y}{H}, \sin \frac{2n\pi x}{L} \sin \frac{2m\pi y}{H}, \right\}, \quad n, m = 1, 2, \ldots.
\end{aligned}
$$

Sample computations are below, using the orthogonality of the trigonometric functions.

$$
\begin{aligned}
\int_0^L \int_0^H f(x, y)\, dxdy &= \int_0^L \int_0^H \frac{a_{00}}{4}\, dxdy = \frac{a_{00} LH}{4}. \\
\int_0^L \int_0^H f(x, y) \cos \frac{2n\pi x}{L}\, dxdy &= \frac{a_{n0}}{2} \int_0^L \int_0^H \cos^2 \frac{2n\pi x}{L}\, dxdy = \frac{a_{n0} LH}{4}. \\
\int_0^L \int_0^H f(x, y) \cos \frac{2n\pi x}{L} \cos \frac{2m\pi y}{H}\, dxdy &= a_{nm} \int_0^L \int_0^H \cos^2 \frac{2n\pi x}{L} \cos^2 \frac{2m\pi y}{H}\, dxdy = \frac{a_{nm} LH}{4}.
\end{aligned}
$$

This gives the Fourier coefficients as

$$
\begin{aligned}
a_{nm} &= \frac{4}{LH} \int_0^L \int_0^H f(x, y) \cos \frac{2n\pi x}{L} \cos \frac{2m\pi y}{H}\, dxdy, \quad n, m = 0, 1, 2, \ldots, \\
b_{nm} &= \frac{4}{LH} \int_0^L \int_0^H f(x, y) \sin \frac{2n\pi x}{L} \sin \frac{2m\pi y}{H}\, dxdy, \quad n, m = 1, 2, \ldots, \\
c_{nm} &= \frac{4}{LH} \int_0^L \int_0^H f(x, y) \cos \frac{2n\pi x}{L} \sin \frac{2m\pi y}{H}\, dxdy, \quad n = 0, 1, 2, \ldots, m = 1, 2, \ldots, \\
d_{nm} &= \frac{4}{LH} \int_0^L \int_0^H f(x, y) \sin \frac{2n\pi x}{L} \cos \frac{2m\pi y}{H}\, dxdy, \quad n = 1, 2, \ldots, m = 0, 1, 2, \ldots.
\end{aligned}
$$