Chapter 2

Section 2.2

2-1 Show that (2.2-4) is equal to (2.2-5).

$$D(f, f_X) = F_{x_A}\left\{A_A \operatorname{rect}(x_A / L)\right\} = A_A \int_{-L/2}^{L/2} \exp(+j2\pi f_X x_A)\, dx_A$$

$$\begin{aligned}\int_{-L/2}^{L/2} \exp(+j2\pi f_X x_A)\, dx_A &= \frac{1}{j2\pi f_X} \exp(+j2\pi f_X x_A)\Big|_{-L/2}^{L/2} \\ &= \frac{1}{j2\pi f_X}\left[\exp(+j\pi f_X L) - \exp(-j\pi f_X L)\right]\end{aligned}$$

Since $\sin\alpha = \dfrac{1}{j2}[\exp(+j\alpha) - \exp(-j\alpha)]$,

$$\int_{-L/2}^{L/2} \exp(+j2\pi f_X x_A)\, dx_A = \frac{\sin(\pi f_X L)}{\pi f_X} = L\frac{\sin(\pi f_X L)}{\pi f_X L} = L \operatorname{sinc}(f_X L)$$

$$\therefore\ D(f, f_X) = F_{x_A}\left\{A_A \operatorname{rect}(x_A / L)\right\} = A_A L \operatorname{sinc}(f_X L)$$

2-2 Verify (2.2-9).

$$D(f, f_X) = A_A L \,\mathrm{sinc}(f_X L) = A_A L \frac{\sin(\pi f_X L)}{\pi f_X L}$$

$$D(f, 0) = A_A L \frac{\sin(0)}{0} = \frac{0}{0}$$

Since $0/0$ is an indeterminate form, must use L'Hospital's rule to evaluate $D(f, 0)$.

Numerator: $\frac{d}{df_X} A_A L \sin(\pi f_X L) = \pi A_A L^2 \cos(\pi f_X L)$

Denominator: $\frac{d}{df_X} \pi f_X L = \pi L$

$$\therefore\ D(f, 0) = \left. \frac{\pi A_A L^2 \cos(\pi f_X L)}{\pi L} \right|_{f_X = 0} = A_A L \quad \text{and} \quad \left| D(f, 0) \right| = A_A L$$

2-3 Derive the *normalized*, far-field beam patterns of the following aperture functions, expressing your answers in terms of the spherical angles θ and ψ :

(a) $A(f, y_A) = A_A \operatorname{rect}(y_A / L)$

(b) $A(f, z_A) = A_A \operatorname{rect}(z_A / L)$

Recall: if $A(f, x_A) = A_A \operatorname{rect}(x_A / L)$, then $D_N(f, f_X) = \operatorname{sinc}(f_X L)$. Therefore,

(a) if $A(f, y_A) = A_A \operatorname{rect}(y_A / L)$, then $D_N(f, f_Y) = \operatorname{sinc}(f_Y L)$

Since $f_Y = \dfrac{v}{\lambda} = \dfrac{\sin\theta \sin\psi}{\lambda}$, $D_N(f, \theta, \psi) = \operatorname{sinc}\left(\dfrac{L}{\lambda}\sin\theta\sin\psi\right)$

(b) if $A(f, z_A) = A_A \operatorname{rect}(z_A / L)$, then $D_N(f, f_Z) = \operatorname{sinc}(f_Z L)$

Since $f_Z = \dfrac{w}{\lambda} = \dfrac{\cos\theta}{\lambda}$, $D_N(f, \theta, \psi) = \operatorname{sinc}\left(\dfrac{L}{\lambda}\cos\theta\right)$

2-4 The aperture function of a linear aperture lying along the X axis is given by

$$A(f, x_A) = a(f, x_A) = \begin{cases} -A_A, & -L/2 \le x_A < 0 \\ A_A, & 0 \le x_A \le L/2. \end{cases}$$

Find the unnormalized, far-field beam pattern of the aperture. This amplitude response is an example of an *odd* function of x_A.

Note: The amplitude response along the negative X axis is 180° out-of-phase with the amplitude response along the positive X axis.

$$D(f, f_X) = F_{x_A}\{A(f, x_A)\} = \int_{-L/2}^{L/2} A(f, x_A)\exp(+j2\pi f_X x_A)\,dx_A$$

$$D(f, f_X) = -A_A \int_{-L/2}^{0} \exp(+j2\pi f_X x_A)\,dx_A + A_A \int_{0}^{L/2} \exp(+j2\pi f_X x_A)\,dx_A$$

$$D(f, f_X) = -A_A \frac{1}{j2\pi f_X} \exp(+j2\pi f_X x_A)\Big|_{-L/2}^{0} + A_A \frac{1}{j2\pi f_X} \exp(+j2\pi f_X x_A)\Big|_{0}^{L/2}$$

$$D(f, f_X) = \frac{-A_A}{j2\pi f_X}\left[1 - \exp(-j\pi f_X L)\right] + \frac{A_A}{j2\pi f_X}\left[\exp(+j\pi f_X L) - 1\right]$$

$$D(f, f_X) = \frac{A_A}{j2\pi f_X}\left[-1 + \exp(-j\pi f_X L) + \exp(+j\pi f_X L) - 1\right]$$

$$D(f, f_X) = \frac{A_A}{j2\pi f_X}\left[2\cos(\pi f_X L) - 2\right] = j\frac{A_A}{\pi f_X}\left[1 - \cos(\pi f_X L)\right] = j\frac{A_A}{\pi f_X}\left[1 - \cos(2\pi f_X L/2)\right]$$

$$1 - \cos(2\alpha) = 2\sin^2\alpha\,, \quad \alpha = \pi f_X L/2$$

$$D(f, f_X) = j\frac{2A_A}{\pi f_X}\sin^2(\pi f_X L/2) = j\frac{2A_A}{\pi f_X}\frac{\pi f_X}{\pi f_X}\left[\frac{L/2}{L/2}\right]^2 \sin^2(\pi f_X L/2)$$

$$D(f, f_X) = j\frac{2A_A L^2}{4}\pi f_X \left[\frac{\sin(\pi f_X L/2)}{\pi f_X L/2}\right]^2$$

$$D(f, f_X) = j\frac{A_A L^2}{2}\pi f_X \operatorname{sinc}^2(f_X L/2) \quad \text{unnormalized, far-field beam pattern}$$

$$f_X = u/\lambda\,, \quad u = \sin\theta\cos\psi$$

$$D(f,u) = j\frac{A_A L^2}{2}\pi\frac{u}{\lambda}\text{sinc}^2\left(\frac{L}{2\lambda}u\right)$$

Note: $D(f,u)$ is an *imaginary*, *odd* function of u

At broadside ($u = 0$), $D(f,0) = 0$

$D(f,-u) = -D(f,u)$ odd function of u

2-5 Derive (2.2-20) by evaluating the spatial-domain, Fourier integral.

$$D(f, f_X) = F_{x_A}\{A(f, x_A)\} = \int_{-L/2}^{L/2} A(f, x_A)\exp(+j2\pi f_X x_A)\,dx_A$$

$$A(f, x_A) = A_A \operatorname{tri}\left(\frac{x_A}{L}\right) \quad \text{where} \quad \operatorname{tri}\left(\frac{x_A}{L}\right) \triangleq \begin{cases} 1-\dfrac{|x_A|}{L/2}, & |x_A| \le L/2 \\ 0, & |x_A| > L/2 \end{cases}$$

Since $\operatorname{tri}(x_A/L)$ is a real, even function of x_A,

$$\begin{aligned} D(f, f_X) &= 2A_A \int_0^{L/2} \operatorname{tri}(x_A/L)\cos(2\pi f_X x_A)\,dx_A \\ &= 2A_A \int_0^{L/2} \cos(2\pi f_X x_A)\,dx_A - \frac{4A_A}{L}\int_0^{L/2} x_A \cos(2\pi f_X x_A)\,dx_A \end{aligned}$$

$$\int_0^{L/2} \cos(2\pi f_X x_A)\,dx_A = \frac{1}{2\pi f_X}\sin(2\pi f_X x_A)\Big|_0^{L/2} = \frac{\sin(\pi f_X L)}{2\pi f_X} = \frac{L}{2}\operatorname{sinc}(f_X L)$$

$$\begin{aligned} \int_0^{L/2} x_A\cos(2\pi f_X x_A)\,dx_A &= \left[\frac{\cos(2\pi f_X x_A)}{(2\pi f_X)^2} + \frac{x_A \sin(2\pi f_X x_A)}{2\pi f_X}\right]_0^{L/2} \\ &= \frac{\cos(\pi f_X L) - 1}{(2\pi f_X)^2} + \frac{L}{2}\frac{\sin(\pi f_X L)}{2\pi f_X} \\ &= \frac{L^2}{4}\operatorname{sinc}(f_X L) - \frac{1-\cos(\pi f_X L)}{(2\pi f_X)^2} \end{aligned}$$

$$\therefore\ D(f, f_X) = \frac{4A_A}{L}\frac{1-\cos(\pi f_X L)}{(2\pi f_X)^2} \quad \text{and since} \quad 1-\cos(\pi f_X L) = 2\sin^2\left(\frac{\pi f_X L}{2}\right),$$

$$\begin{aligned} D(f, f_X) &= \frac{4A_A}{L}\frac{2}{(2\pi f_X)^2}\sin^2\left(\frac{\pi f_X L}{2}\right) = A_A L \frac{2}{(\pi f_X L)^2}\sin^2\left(\frac{\pi f_X L}{2}\right) \\ &= \frac{A_A L}{4}\frac{2}{\left(\dfrac{\pi f_X L}{2}\right)^2}\sin^2\left(\frac{\pi f_X L}{2}\right) \end{aligned}$$

$$D(f, f_X) = F_{x_A}\left\{A_A \operatorname{tri}\left(\frac{x_A}{L}\right)\right\} = \frac{A_A L}{2}\operatorname{sinc}^2\left(\frac{f_X L}{2}\right)$$

2-6 Show that

$$F_{f_X}^{-1}\left\{\delta(f_X \pm f_X')\right\} = \exp(\pm j2\pi f_X' x_A) \ .$$

Note: Use the sifting property of impulse functions.

$$F_{f_X}^{-1}\left\{\delta(f_X \pm f_X')\right\} = \int_{-\infty}^{\infty} \delta(f_X \pm f_X') \exp(-j2\pi f_X x_A)\, df_X = \exp\left[-j2\pi(\mp f_X')x_A\right]$$

$$F_{f_X}^{-1}\left\{\delta(f_X \pm f_X')\right\} = \exp(\pm j2\pi f_X' x_A)$$

2-7 Show that

$$g(x) \underset{x}{*} \delta(x \pm x_0) = g(x \pm x_0),$$

where the asterisk denotes convolution with respect to x, and x_0 is an arbitrary constant.

Note: Use the sifting property of impulse functions.

$$g(x) \underset{x}{*} \delta(x \pm x_0) = \int_{-\infty}^{\infty} g(\tau)\delta(x - \tau \pm x_0)\, d\tau = \int_{-\infty}^{\infty} g(\tau)\delta([x \pm x_0] - \tau)\, d\tau$$
$$g(x) \underset{x}{*} \delta(x \pm x_0) = g(x \pm x_0)$$

2-8 The complex frequency response (complex aperture function) of a linear aperture lying along the Y axis is modeled by the cosine amplitude window. Evaluate the *normalized*, far-field beam pattern of the aperture at $\theta = 40°$ and $\psi = 108°$. Use $L/\lambda = 2$.

Replacing f_X with f_Y in (2.2-35) yields

$$D_N(f, f_Y) = \frac{\cos(\pi f_Y L)}{1-(2 f_Y L)^2} \quad \text{where} \quad f_Y = \frac{v}{\lambda} = \frac{\sin\theta \sin\psi}{\lambda}$$

$$D_N(f, \theta, \psi) = \frac{\cos\left(\pi \frac{L}{\lambda} \sin\theta \sin\psi\right)}{1-\left(2\frac{L}{\lambda}\sin\theta\sin\psi\right)^2}$$

$$D_N(f, 40°, 108°) = \frac{\cos(2\pi \sin 40° \sin 108°)}{1-(4\sin 40° \sin 108°)^2} = \frac{\cos(3.841)}{1-(2.445)^2} = \frac{-0.765}{-4.978} = 0.154$$

2-9 If the aperture function of a linear aperture lying along the X axis is modeled by the *sine* amplitude window, that is, if

$$A(f, x_A) = a(f, x_A) = A_A \sin(\pi x_A / L) \operatorname{rect}(x_A / L) \,,$$

then find the unnormalized, far-field beam pattern of the aperture. The sine amplitude window is an example of an *odd* function of x_A.

Note: at $x_A = -L/2$, $a(f, x_A) = -A_A$; at $x_A = 0$, $a(f, x_A) = 0$; at $x_A = L/2$, $a(f, x_A) = A_A$

The sine amplitude window is an *odd* function of x_A. The amplitude response along the negative X axis is 180° out-of-phase with the amplitude response along the positive X axis.

$$\begin{aligned} D(f, f_X) &= F_{x_A}\{A_A \sin(\pi x_A / L) \operatorname{rect}(x_A / L)\} \\ &= A_A F_{x_A}\{\sin(\pi x_A / L)\} \underset{f_X}{*} F_{x_A}\{\operatorname{rect}(x_A / L)\} \\ &= A_A L F_{x_A}\{\sin(\pi x_A / L)\} \underset{f_X}{*} \operatorname{sinc}(f_X L) \end{aligned}$$

$$\sin\alpha = \frac{1}{j2}[\exp(+j\alpha) - \exp(-j\alpha)]$$

$$\sin(\pi x_A / L) = \frac{1}{j2}[\exp(+j\pi x_A / L) - \exp(-j\pi x_A / L)]$$

$$F_{x_A}\{\exp(\pm j2\pi f_X' x_A)\} = \delta(f_X \pm f_X'), \quad f_X' = \frac{1}{2L}$$

$$F_{x_A}\left\{\sin\left(\frac{\pi x_A}{L}\right)\right\} = \frac{1}{j2}\left[\delta\left(f_X + \frac{1}{2L}\right) - \delta\left(f_X - \frac{1}{2L}\right)\right]$$

$$D(f, f_X) = \frac{A_A L}{j2}\left[\delta\left(f_X + \frac{1}{2L}\right) - \delta\left(f_X - \frac{1}{2L}\right)\right] \underset{f_X}{*} \operatorname{sinc}(f_X L)$$

$$D(f, f_X) = \frac{A_A L}{j2}\left[\delta\left(f_X + \frac{1}{2L}\right) \underset{f_X}{*} \operatorname{sinc}(f_X L) - \delta\left(f_X - \frac{1}{2L}\right) \underset{f_X}{*} \operatorname{sinc}(f_X L)\right]$$

$$D(f, f_X) = \frac{A_A L}{j2}\left\{\operatorname{sinc}\left[\left(f_X + \frac{1}{2L}\right)L\right] - \operatorname{sinc}\left[\left(f_X - \frac{1}{2L}\right)L\right]\right\}$$

$$D(f,f_X)=\frac{A_A L}{j2}\left[\frac{\sin\left(\pi f_X L+\frac{\pi}{2}\right)}{\pi f_X L+\frac{\pi}{2}}-\frac{\sin\left(\pi f_X L-\frac{\pi}{2}\right)}{\pi f_X L-\frac{\pi}{2}}\right]$$

$$\sin\left(\pi f_X L\pm\frac{\pi}{2}\right)=\pm\cos(\pi f_X L)$$

$$D(f,f_X)=\frac{A_A L}{j2}\left[\frac{\cos(\pi f_X L)}{\pi f_X L+\frac{\pi}{2}}-\frac{-\cos(\pi f_X L)}{\pi f_X L-\frac{\pi}{2}}\right]=\frac{A_A L}{j2}\cos(\pi f_X L)\left[\frac{1}{\pi f_X L+\frac{\pi}{2}}+\frac{1}{\pi f_X L-\frac{\pi}{2}}\right]$$

$$D(f,f_X)=\frac{A_A L}{j2}\cos(\pi f_X L)\frac{\pi f_X L-\frac{\pi}{2}+\pi f_X L+\frac{\pi}{2}}{\left(\pi f_X L+\frac{\pi}{2}\right)\left(\pi f_X L-\frac{\pi}{2}\right)}=\frac{A_A L}{j2}\cos(\pi f_X L)\frac{2\pi f_X L}{(\pi f_X L)^2-\frac{\pi^2}{4}}$$

$$D(f,f_X)=-jA_A L\cos(\pi f_X L)\frac{\pi f_X L}{\pi^2\left[(f_X L)^2-\frac{1}{4}\right]}=-jA_A L\cos(\pi f_X L)\frac{4 f_X L}{\pi\left[(2f_X L)^2-1\right]}$$

$$D(f,f_X)=j\frac{4A_A L^2}{\pi}\frac{\cos(\pi f_X L)}{1-(2f_X L)^2}f_X \quad \text{unnormalized, far-field beam pattern}$$

$f_X=u/\lambda$, $u=\sin\theta\cos\psi$

$$D(f,u)=j\frac{4A_A L^2}{\pi}\frac{\cos\left(\pi\frac{L}{\lambda}u\right)}{1-\left(2\frac{L}{\lambda}u\right)^2}\frac{u}{\lambda}$$

Note: $D(f,u)$ is an *imaginary*, *odd* function of u

At broadside ($u=0$), $D(f,0)=0$

$D(f,-u)=-D(f,u)$ odd function of u

2-10 Derive the *normalized*, far-field beam patterns of the Hanning, Hamming, and Blackman amplitude windows. Use the following trigonometric identity: $\sin(\alpha \pm \pi) = -\sin\alpha$.

Hanning amplitude window

$$a(f, x_A) = 0.5A_A \operatorname{rect}(x_A/L) + 0.5A_A \cos(2\pi x_A/L)\operatorname{rect}(x_A/L)$$

$$D(f, f_X) = 0.5D_1(f, f_X) + 0.5D_2(f, f_X)$$

$$D_1(f, f_X) = F_{x_A}\{A_A \operatorname{rect}(x_A/L)\} = A_A L \operatorname{sinc}(f_X L)$$

$$D_2(f, f_X) = F_{x_A}\{A_A \cos(2\pi x_A/L)\operatorname{rect}(x_A/L)\}$$

Substituting $d = L$ into (2.2-29) yields

$$\begin{aligned}
D_2(f, f_X) &= \frac{A_A L}{2}\left\{\operatorname{sinc}\left[\left(f_X + \frac{1}{L}\right)L\right] + \operatorname{sinc}\left[\left(f_X - \frac{1}{L}\right)L\right]\right\} \\
&= \frac{A_A L}{2}\left[\frac{\sin(\pi f_X L + \pi)}{\pi(f_X L + 1)} + \frac{\sin(\pi f_X L - \pi)}{\pi(f_X L - 1)}\right] \\
&= -\frac{A_A L}{2\pi}\sin(\pi f_X L)\left[\frac{1}{(f_X L + 1)} + \frac{1}{(f_X L - 1)}\right] \\
&= -\frac{A_A L}{2\pi}\sin(\pi f_X L)\frac{2 f_X L}{(f_X L)^2 - 1}
\end{aligned}$$

$$D_2(f, f_X) = F_{x_A}\left\{A_A \cos\left(\frac{2\pi x_A}{L}\right)\operatorname{rect}\left(\frac{x_A}{L}\right)\right\} = A_A L \operatorname{sinc}(f_X L)\frac{(f_X L)^2}{1 - (f_X L)^2}$$

$$\begin{aligned}
D(f, f_X) &= \frac{A_A L}{2}\operatorname{sinc}(f_X L) + \frac{A_A L}{2}\operatorname{sinc}(f_X L)\frac{(f_X L)^2}{1 - (f_X L)^2} \\
&= \frac{A_A L}{2}\operatorname{sinc}(f_X L)\left[1 + \frac{(f_X L)^2}{1 - (f_X L)^2}\right]
\end{aligned}$$

$$D(f, f_X) = \frac{A_A L}{2}\frac{\operatorname{sinc}(f_X L)}{1 - (f_X L)^2}$$

Since $D_{\max} = |D(f, 0)| = \dfrac{A_A L}{2}$, $D_N(f, f_X) = \dfrac{\operatorname{sinc}(f_X L)}{1 - (f_X L)^2}$

Hamming amplitude window

$$a(f, x_A) = 0.54 A_A \operatorname{rect}(x_A/L) + 0.46 A_A \cos(2\pi x_A/L)\operatorname{rect}(x_A/L)$$

$$D(f, f_X) = 0.54 D_1(f, f_X) + 0.46 D_2(f, f_X)$$

$$D_1(f, f_X) = F_{x_A}\{A_A \operatorname{rect}(x_A/L)\} = A_A L \operatorname{sinc}(f_X L)$$

$$D_2(f, f_X) = F_{x_A}\left\{A_A \cos\left(\frac{2\pi x_A}{L}\right)\operatorname{rect}\left(\frac{x_A}{L}\right)\right\} = A_A L \operatorname{sinc}(f_X L)\frac{(f_X L)^2}{1-(f_X L)^2} \quad \text{(from "Hanning")}$$

$$\begin{aligned} D(f, f_X) &= 0.54 A_A L \operatorname{sinc}(f_X L) + 0.46 A_A L \operatorname{sinc}(f_X L)\frac{(f_X L)^2}{1-(f_X L)^2} \\ &= A_A L \operatorname{sinc}(f_X L)\left[0.54 + 0.46\frac{(f_X L)^2}{1-(f_X L)^2}\right] \\ &= A_A L \operatorname{sinc}(f_X L)\frac{0.54 - 0.54(f_X L)^2 + 0.46(f_X L)^2}{1-(f_X L)^2} \end{aligned}$$

$$D(f, f_X) = A_A L \frac{0.54 - 0.08(f_X L)^2}{1-(f_X L)^2}\operatorname{sinc}(f_X L)$$

Since $D_{\max} = |D(f,0)| = 0.54 A_A L$, $D_N(f, f_X) = \dfrac{1}{0.54}\dfrac{0.54 - 0.08(f_X L)^2}{1-(f_X L)^2}\operatorname{sinc}(f_X L)$

Blackman amplitude window

$$\begin{aligned} a(f, x_A) = {} & 0.42 A_A \operatorname{rect}(x_A/L) + 0.5 A_A \cos(2\pi x_A/L)\operatorname{rect}(x_A/L) + \\ & 0.08 A_A \cos(4\pi x_A/L)\operatorname{rect}(x_A/L) \end{aligned}$$

$$D(f, f_X) = 0.42 D_1(f, f_X) + 0.5 D_2(f, f_X) + 0.08 D_3(f, f_X)$$

$$D_1(f, f_X) = F_{x_A}\{A_A \operatorname{rect}(x_A/L)\} = A_A L \operatorname{sinc}(f_X L)$$

$$D_2(f, f_X) = F_{x_A}\left\{A_A \cos\left(\frac{2\pi x_A}{L}\right)\operatorname{rect}\left(\frac{x_A}{L}\right)\right\} = A_A L \operatorname{sinc}(f_X L)\frac{(f_X L)^2}{1-(f_X L)^2} \quad \text{(from "Hanning")}$$

$$D_3(f, f_X) = F_{x_A}\{A_A \cos(4\pi x_A/L)\operatorname{rect}(x_A/L)\}$$

Substituting $d = L/2$ into (2.2-29) yields

$$D_3(f, f_X) = \frac{A_A L}{2}\left\{\text{sinc}\left[\left(f_X + \frac{2}{L}\right)L\right] + \text{sinc}\left[\left(f_X - \frac{2}{L}\right)L\right]\right\}$$

$$= \frac{A_A L}{2}\left[\frac{\sin(\pi f_X L + 2\pi)}{\pi(f_X L + 2)} + \frac{\sin(\pi f_X L - 2\pi)}{\pi(f_X L - 2)}\right]$$

$$= \frac{A_A L}{2\pi}\sin(\pi f_X L)\left[\frac{1}{(f_X L + 2)} + \frac{1}{(f_X L - 2)}\right]$$

$$= \frac{A_A L}{2\pi}\sin(\pi f_X L)\frac{2 f_X L}{(f_X L)^2 - 4}$$

$$D_3(f, f_X) = F_{x_A}\left\{A_A \cos\left(\frac{4\pi x_A}{L}\right)\text{rect}\left(\frac{x_A}{L}\right)\right\} = A_A L\,\text{sinc}(f_X L)\frac{(f_X L)^2}{(f_X L)^2 - 4}$$

$$D(f, f_X) = 0.42 A_A L\,\text{sinc}(f_X L) + 0.5 A_A L\,\text{sinc}(f_X L)\frac{(f_X L)^2}{1-(f_X L)^2} +$$

$$0.08 A_A L\,\text{sinc}(f_X L)\frac{(f_X L)^2}{(f_X L)^2 - 4}$$

$$= A_A L\,\text{sinc}(f_X L)\left[0.42 + 0.5\frac{(f_X L)^2}{1-(f_X L)^2} - 0.08\frac{(f_X L)^2}{4-(f_X L)^2}\right]$$

$$D(f, f_X) = A_A L\frac{1.68 - 0.18(f_X L)^2}{\left[1-(f_X L)^2\right]\left[4-(f_X L)^2\right]}\text{sinc}(f_X L)$$

Since $D_{\max} = \left|D(f,0)\right| = 0.42 A_A L$, $D_N(f, f_X) = \frac{1}{0.42}\frac{1.68 - 0.18(f_X L)^2}{\left[1-(f_X L)^2\right]\left[4-(f_X L)^2\right]}\text{sinc}(f_X L)$

Section 2.2 Appendix 2A

2-11 Find the transmitter sensitivity functions for the following complex frequency responses of a continuous line source lying along the X axis where $A_A(f)$ is a real, nonnegative function of frequency:

(a) rectangular amplitude window: $A_T(f, x_A) = A_A(f)\,\text{rect}(x_A/L)$

(b) cosine amplitude window: $A_T(f, x_A) = A_A(f)\cos(\pi x_A/L)\,\text{rect}(x_A/L)$

From Appendix 2A: $\mathcal{S}_T(f) = \int_{-\infty}^{\infty} A_T(f, x_A)\, dx_A$ where $\mathcal{S}_T(f)$ has units of $\left(\text{m}^3/\text{sec}\right)/\text{V}$

(a) $\mathcal{S}_T(f) = A_A(f)\int_{-\infty}^{\infty} \text{rect}(x_A/L)\, dx_A = A_A(f)\int_{-L/2}^{L/2} dx_A$

$$\mathcal{S}_T(f) = A_A(f)\,L$$

Note: $\mathcal{S}_T(f)$ is real in this case because $A_T(f, x_A)$ is real. Also, if $A_A(f) = A_A$, then $\mathcal{S}_T(f) = A_A L$ is the normalization factor $D_{\max}$ for the far-field beam pattern of the rectangular amplitude window given by (2.2-2) [see (2.2-9)]. Also see Appendix 2D for computing $D_{\max}$.

(b) $\mathcal{S}_T(f) = A_A(f)\int_{-\infty}^{\infty} \cos(\pi x_A/L)\,\text{rect}(x_A/L)\, dx_A = A_A(f)\int_{-L/2}^{L/2} \cos(\pi x_A/L)\, dx_A$

$$\mathcal{S}_T(f) = A_A(f)\frac{L}{\pi}\sin(\pi x_A/L)\Big|_{-L/2}^{L/2} = A_A(f)\frac{L}{\pi}\left[\sin(\pi/2) - \sin(-\pi/2)\right]$$

$$\mathcal{S}_T(f) = A_A(f)\frac{2L}{\pi}$$

Note: $\mathcal{S}_T(f)$ is real in this case because $A_T(f, x_A)$ is real. Also, if $A_A(f) = A_A$, then $\mathcal{S}_T(f) = \dfrac{2A_A L}{\pi}$ is the normalization factor $D_{\max}$ for the far-field beam pattern of the cosine amplitude window given by (2.2-30) [see (2.2-34)]. Also see Appendix 2D for computing $D_{\max}$.

2-12 If the transmitter sensitivity level (transmitting voltage response) of a continuous line source at 16 kHz is $140\text{ dB re }1\,\mu\text{Pa/V}$ at 1 m, then what is the value of the magnitude of the transmitter sensitivity function in $\left(\text{m}^3/\text{sec}\right)/\text{V}$? Use $\rho_0 = 1026\text{ kg/m}^3$ for the ambient density of seawater. The length L of the continuous line source is less than 0.632 m or 24.9 in.

See Appendix 2A:

transmitter sensitivity level: $\text{TSL}(16\text{ kHz}) = 140\text{ dB re }1\,\mu\text{Pa/V}$ at 1 m

transmitter sensitivity: $\text{TS}(f) = \text{TS}_{\text{ref}}\,10^{[\text{TSL}(f)/20]}$ where $\text{TS}_{\text{ref}} = 1\,\mu\text{Pa/V}$

$$\text{TS}(16\text{ kHz}) = \text{TS}_{\text{ref}}\,10^{[\text{TSL}(16\text{ kHz})/20]} = (1\times10^{-6})10^{[140/20]}\text{ Pa/V} = 10\text{ Pa/V}$$

magnitude of the transmitter sensitivity function:

$$\left|\mathcal{S}_T(f)\right| = \frac{2r}{f\rho_0}\text{TS}(f)\Big|_{r=1\text{ m}} \quad \frac{\text{m}^3/\text{sec}}{\text{V}}, \qquad L < 0.632\text{ m}$$

$$\left|\mathcal{S}_T(16\text{ kHz})\right| = \frac{2r}{(16\times10^3\text{ Hz})\rho_0}\text{TS}(16\text{ kHz})\Big|_{r=1\text{ m}} = \frac{2\text{ m}}{(16\times10^3\text{ Hz})(1026\text{ kg/m}^3)}10\,\frac{\text{Pa}}{\text{V}}$$

$$\left|\mathcal{S}_T(16\text{ kHz})\right| = 1.218\times10^{-6}\,\frac{\text{m}^4\text{- sec}}{\text{kg}}\frac{\text{Pa}}{\text{V}}$$

since $1\text{ Pa} = 1\,\dfrac{\text{N}}{\text{m}^2}$ and $1\text{ N} = 1\,\dfrac{\text{kg-m}}{\text{sec}^2}$, $1\text{ Pa} = 1\,\dfrac{\text{kg}}{\text{m-sec}^2}$

$$\therefore \left|\mathcal{S}_T(16\text{ kHz})\right| = 1.218\times10^{-6}\,\frac{\text{m}^4\text{- sec}}{\text{kg}}\,\frac{\text{kg}}{\text{m-sec}^2}\,\frac{1}{\text{V}} = 1.218\times10^{-6}\,\frac{\text{m}^3/\text{sec}}{\text{V}}$$

2-13 If the receiver sensitivity level (open circuit receiving response) of a continuous line receiver at 2 kHz is $-195\text{ dB re }1\text{ V}/\mu\text{Pa}$, then what is the value of the magnitude of the receiver sensitivity function in $\text{V}/\left(\text{m}^2/\text{sec}\right)$? Use $\rho_0 = 1026\text{ kg}/\text{m}^3$ for the ambient density of seawater.

See Appendix 2A:

receiver sensitivity level: $\text{RSL}(2\text{ kHz}) = -195\text{ dB re }1\text{ V}/\mu\text{Pa}$

receiver sensitivity: $\text{RS}(f) = \text{RS}_{\text{ref}}\,10^{[\text{RSL}(f)/20]}$ where $\text{RS}_{\text{ref}} = 1\text{ V}/\mu\text{Pa}$

$$\text{RS}(2\text{ kHz}) = \text{RS}_{\text{ref}}\,10^{[\text{RSL}(2\text{ kHz})/20]} = \frac{1}{1\times 10^{-6}}10^{[-195/20]}\,\frac{\text{V}}{\text{Pa}} = 10^{6}\times 10^{-9.75}\,\frac{\text{V}}{\text{Pa}}$$

$$\text{RS}(2\text{ kHz}) = 1\times 10^{-3.75}\text{ V}/\text{Pa} = 0.178\text{ mV}/\text{Pa}$$

magnitude of the receiver sensitivity function: $\left|\mathcal{S}_R(f)\right| = 2\pi f \rho_0 \text{RS}(f)\text{ V}/\left(\text{m}^2/\text{sec}\right)$

$$\left|\mathcal{S}_R(2\text{ kHz})\right| = 2\pi(2\times 10^3\text{ Hz})\,\rho_0\,\text{RS}(2\text{ kHz})$$

$$\left|\mathcal{S}_R(2\text{ kHz})\right| = 2\pi(2\times 10^3\text{ Hz})(1026\text{ kg}/\text{m}^3)(0.178\times 10^{-3}\text{ V}/\text{Pa}) = 2295\,\frac{\text{kg}}{\text{m}^3\text{-sec}}\,\frac{\text{V}}{\text{Pa}}$$

since $1\text{ Pa} = 1\,\dfrac{\text{N}}{\text{m}^2}$ and $1\text{ N} = 1\,\dfrac{\text{kg-m}}{\text{sec}^2}$, $1\text{ Pa} = 1\,\dfrac{\text{kg}}{\text{m-sec}^2}$

$$\therefore \left|\mathcal{S}_R(2\text{ kHz})\right| = 2295\,\frac{\text{kg}}{\text{m}^3\text{-sec}}\,\frac{\text{m-sec}^2}{\text{kg}}\text{V} = 2295\,\frac{\text{V}}{\text{m}^2/\text{sec}}$$

Section 2.3

2-14 Compute the 3-dB beamwidths of the horizontal, far-field beam patterns of the rectangular, triangular, cosine, Hanning, Hamming, and Blackman amplitude windows for $L/\lambda = 0.5$, 1 , 2 , and 4 .

$$\Delta\psi = 2\sin^{-1}\left(\frac{\Delta u}{2}\right), \qquad \frac{\Delta u}{2} \le 1$$

From Table 2.3-1:

Amplitude Window	Δu
rectangular	$0.886\,\lambda/L$
triangular	$1.276\,\lambda/L$
cosine	$1.189\,\lambda/L$
Hanning	$1.441\,\lambda/L$
Hamming	$1.303\,\lambda/L$
Blackman	$1.644\,\lambda/L$

	$\Delta\psi$ (deg)					
L/λ	Rec	Tri	Cos	Han	Ham	Blk
0.5	124.7	NA	NA	NA	NA	NA
1	52.6	79.3	73.0	92.2	81.3	110.6
2	25.6	37.2	34.6	42.2	38.0	48.5
4	12.7	18.4	17.1	20.8	18.7	23.7

Note: NA means that $\dfrac{\Delta u}{2} > 1$ so that $\Delta\psi$ does not exist.

2-15 Consider a *linear towed array* (an example of a linear aperture) lying along the X axis. The amplitude response of the aperture is modeled by the rectangular amplitude window. How long must the aperture be in order for the 3-dB beamwidth of the horizontal, far-field beam pattern to be $10°$ at broadside for $f = 100$ Hz ? Use $c = 1500$ m/sec .

$$\Delta\psi = 2\sin^{-1}\left(\frac{\Delta u}{2}\right), \qquad \frac{\Delta u}{2} \le 1$$

$$\Delta u = 0.886\frac{\lambda}{L} \quad \text{so that} \quad \frac{\Delta u}{2} = 0.443\frac{\lambda}{L}$$

$$\Delta\psi = 2\sin^{-1}\left(0.443\frac{\lambda}{L}\right), \quad \sin\left(\frac{\Delta\psi}{2}\right) = 0.443\frac{\lambda}{L}, \quad L = 0.443\frac{\lambda}{\sin(\Delta\psi/2)}$$

$$\lambda = \frac{c}{f} = \frac{1500 \text{ m/sec}}{100 \text{ Hz}} = 15 \text{ m} \quad \text{and} \quad \Delta\psi = 10°$$

$$L = 0.443\frac{\lambda}{\sin(\Delta\psi/2)} = 0.443\frac{15 \text{ m}}{\sin(10°/2)} = 76.2 \text{ m}$$

Section 2.4

2-16 Consider a linear aperture lying along the X axis. If $f = 1\,\text{kHz}$ and $c = 1500\ \text{m/sec}$, then

(a) what must the phase response of the aperture be in order to steer the mainlobe of the aperture's far-field beam pattern to $\theta' = 49°$ and $\psi' = 71°$.

(b) Repeat (a) for a linear aperture lying along the Y axis.

(c) Repeat (a) for a linear aperture lying along the Z axis.

(a) $\theta(f, x_A) = -2\pi f'_X x_A$ where $f'_X = \dfrac{u'}{\lambda} = \dfrac{\sin\theta' \cos\psi'}{\lambda}$ and $\lambda = \dfrac{c}{f}$

$$\lambda = \frac{c}{f} = \frac{1500\ \text{m/sec}}{1000\ \text{Hz}} = 1.5\ \text{m}$$

$$\theta(1\,\text{kHz}, x_A) = -2\pi \frac{\sin 49° \cos 71°}{1.5\ \text{m}} x_A = -1.0292\,\text{m}^{-1}\, x_A\ \text{rad}$$

(b) $\theta(f, y_A) = -2\pi f'_Y y_A$ where $f'_Y = \dfrac{v'}{\lambda} = \dfrac{\sin\theta' \sin\psi'}{\lambda}$

$$\theta(1\,\text{kHz}, y_A) = -2\pi \frac{\sin 49° \sin 71°}{1.5\ \text{m}} y_A = -2.989\,\text{m}^{-1}\, y_A\ \text{rad}$$

(c) $\theta(f, z_A) = -2\pi f'_Z z_A$ where $f'_Z = \dfrac{w'}{\lambda} = \dfrac{\cos\theta'}{\lambda}$

$$\theta(1\,\text{kHz}, z_A) = -2\pi \frac{\cos 49°}{1.5\ \text{m}} z_A = -2.748\,\text{m}^{-1}\, z_A\ \text{rad}$$

Section 2.5

2-17 Compute the 3-dB beamwidths of the horizontal, far-field beam patterns of the rectangular, triangular, cosine, Hanning, Hamming, and Blackman amplitude windows for the following beam-steer angles: $\psi' = 90°$ (broadside), $75°$, $60°$, $45°$, $30°$, and $0°$ (end-fire). Use $L/\lambda = 4$.

For $\psi' = 90°$ (broadside), use $\Delta\psi = 2\sin^{-1}\left(\frac{\Delta u}{2}\right)$, $\quad \frac{\Delta u}{2} \le 1$

For $\psi' = 75°$, $60°$, $45°$, and $30°$; use

$$\Delta\psi = \cos^{-1}\left(\cos\psi' - \frac{\Delta u}{2}\right) - \cos^{-1}\left(\cos\psi' + \frac{\Delta u}{2}\right), \qquad \left|\cos\psi' \pm \frac{\Delta u}{2}\right| \le 1$$

For $\psi' = 0°$ (end-fire), use $\Delta\psi = 2\cos^{-1}\left(1 - \frac{\Delta u}{2}\right)$, $\quad \left|1 - \frac{\Delta u}{2}\right| \le 1$

From Table 2.3-1:

Amplitude Window	Δu
rectangular	$0.886\,\lambda/L$
triangular	$1.276\,\lambda/L$
cosine	$1.189\,\lambda/L$
Hanning	$1.441\,\lambda/L$
Hamming	$1.303\,\lambda/L$
Blackman	$1.644\,\lambda/L$

	$\Delta\psi$ (deg)					
ψ' (deg)	Rec	Tri	Cos	Han	Ham	Blk
90	12.7	18.4	17.1	20.8	18.7	23.7
75	13.2	19.0	17.7	21.5	19.4	24.6
60	14.7	21.4	19.9	24.2	21.8	27.7
45	18.3	26.9	24.9	30.7	27.5	35.8
30	28.6	NA	NA	NA	NA	NA
0	54.4	65.6	63.3	69.9	66.3	74.8

Note: NA means that $\left|\cos\psi' \pm \frac{\Delta u}{2}\right| > 1$ so that $\Delta\psi$ does not exist.

2-18 The complex aperture function of a *linear towed array* (an example of a linear aperture) lying along the X axis is modeled by the rectangular amplitude window. The length of the aperture is 200 m. Using $f = 60 \text{ Hz}$ and $c = 1500 \text{ m/sec}$, compute the 3-dB beamwidth of the horizontal, far-field beam pattern of this aperture at

(a) broadside

(b) a beam-steer angle of $30°$

(c) Compute *both* half-beamwidth angles at a beam-steer angle of $30°$.

(d) Repeat (a) through (c) for $f = 100 \text{ Hz}$.

For a rectangular amplitude window, $\Delta u = 0.886\,\lambda/L$

$$f = 60 \text{ Hz}, \quad \lambda = \frac{c}{f} = \frac{1500 \text{ m/sec}}{60 \text{ Hz}} = 25 \text{ m}$$

(a) $\Delta\psi = 2\sin^{-1}\left(\dfrac{\Delta u}{2}\right), \qquad \dfrac{\Delta u}{2} \le 1$

$$\Delta\psi = 2\sin^{-1}\left(0.443\frac{\lambda}{L}\right) = 2\sin^{-1}\left(0.443\frac{25 \text{ m}}{200 \text{ m}}\right) = 6.3°$$

(b) $\Delta\psi = \psi_- - \psi_+, \quad \psi_- = \cos^{-1}\left(\cos\psi' - \dfrac{\Delta u}{2}\right), \quad \psi_+ = \cos^{-1}\left(\cos\psi' + \dfrac{\Delta u}{2}\right)$

$$\psi_- = \cos^{-1}\left(\cos\psi' - 0.443\frac{\lambda}{L}\right) = \cos^{-1}\left(\cos 30° - 0.443\frac{25 \text{ m}}{200 \text{ m}}\right) = 35.8°$$

$$\psi_+ = \cos^{-1}\left(\cos\psi' + 0.443\frac{\lambda}{L}\right) = \cos^{-1}\left(\cos 30° + 0.443\frac{25 \text{ m}}{200 \text{ m}}\right) = 22.9°$$

$$\Delta\psi = \psi_- - \psi_+ = 35.8° - 22.9° = 12.9°$$

(c) $\psi' - \psi_+ = 30° - 22.9° = 7.1°$

$$\psi_- - \psi' = 35.8° - 30° = 5.8°$$

(d) $f = 100 \text{ Hz}, \quad \lambda = \dfrac{c}{f} = \dfrac{1500 \text{ m/sec}}{100 \text{ Hz}} = 15 \text{ m}$

(a) $\Delta\psi = 2\sin^{-1}\left(0.443\dfrac{\lambda}{L}\right) = 2\sin^{-1}\left(0.443\dfrac{15\text{ m}}{200\text{ m}}\right) = 3.8°$

(b) $\Delta\psi = \psi_- - \psi_+$

$$\psi_- = \cos^{-1}\left(\cos\psi' - 0.443\frac{\lambda}{L}\right) = \cos^{-1}\left(\cos 30° - 0.443\frac{15\text{ m}}{200\text{ m}}\right) = 33.6°$$

$$\psi_+ = \cos^{-1}\left(\cos\psi' + 0.443\frac{\lambda}{L}\right) = \cos^{-1}\left(\cos 30° + 0.443\frac{15\text{ m}}{200\text{ m}}\right) = 25.9°$$

$$\Delta\psi = \psi_- - \psi_+ = 33.6° - 25.9° = 7.7°$$

(c) $\psi' - \psi_+ = 30° - 25.9° = 4.1°$

$$\psi_- - \psi' = 33.6° - 30° = 3.6°$$

2-19 Consider a linear aperture lying along the X axis.

(a) Show that the 3-dB beamwidth of the vertical, far-field beam pattern in the XZ plane is given by

$$\Delta\theta = \sin^{-1}\left(\sin\theta' + \frac{\Delta u}{2}\right) - \sin^{-1}\left(\sin\theta' - \frac{\Delta u}{2}\right), \qquad \left|\sin\theta' \pm \frac{\Delta u}{2}\right| \le 1 ,$$

where θ' is the beam-steer angle in the XZ plane. Note that $\theta' = 0°$ corresponds to an unsteered vertical beam pattern and, in this case, $\Delta\theta$ reduces to (2.3-19) since

$$\sin^{-1}(-x) = -\sin^{-1}(x) .$$

Hint: follow a procedure analogous to the one used to derive (2.5-12).

(b) Show that the 3-dB beamwidth of the vertical, far-field beam pattern in the XZ plane steered to end-fire is given by

$$\Delta\theta = 2\cos^{-1}\left(1 - \frac{\Delta u}{2}\right), \qquad \left|1 - \frac{\Delta u}{2}\right| \le 1 .$$

Hint: follow a procedure analogous to the one used to derive (2.5-19).

(a) In the XZ plane along the positive X axis where $\psi = 0°$, $u = \sin\theta$.

$$\therefore\ \Delta\theta = \theta_+ - \theta_- = \sin^{-1}(u_+) - \sin^{-1}(u_-)$$

$$u_+ = u' + \frac{\Delta u}{2}\ ,\ \ u_- = u' - \frac{\Delta u}{2}\ ,\ \ u' = \sin\theta'$$

Substituting yields

$$\Delta\theta = \sin^{-1}\left(\sin\theta' + \frac{\Delta u}{2}\right) - \sin^{-1}\left(\sin\theta' - \frac{\Delta u}{2}\right), \qquad \left|\sin\theta' \pm \frac{\Delta u}{2}\right| \le 1$$

(b) In the XZ plane along the positive X axis where $\psi = 0°$, $u = \sin\theta$. Let $\theta' = 90°$ (end-fire). $\therefore u' = \sin\theta' = \sin 90° = 1$. As a result,

$$u_- = u' - \frac{\Delta u}{2} = 1 - \frac{\Delta u}{2}$$

Since the beam pattern is symmetric about $\theta = \theta' = 90°$, $\theta_- = 90° - \frac{\Delta\theta}{2}$. As a result,

$$u_- = \sin\theta_- = \sin\left(90° - \frac{\Delta\theta}{2}\right) = \cos\left(\frac{\Delta\theta}{2}\right)$$

Equating the right-hand sides of both equations for u_- yields

$$\cos\left(\frac{\Delta\theta}{2}\right) = 1 - \frac{\Delta u}{2}$$

$$\Delta\theta = 2\cos^{-1}\left(1 - \frac{\Delta u}{2}\right), \qquad \left|1 - \frac{\Delta u}{2}\right| \le 1$$

Section 2.6

2-20 The spherical coordinates of a sound-source (target) as measured from the center of a *linear towed array* (an example of a linear aperture) lying along the X axis are $r_S = 4$ km , $\theta_S = 100°$, and $\psi_S = 75°$. The length of the aperture is 100 m . If $f = 1$ kHz and $c = 1500$ m/sec ,

(a) is the sound-source in the aperture's Fresnel region or far-field?

(b) What must the phase response along the length of the aperture be if the aperture's far-field beam pattern is to be focused (if required) and steered to coordinates (r_S, θ_S, ψ_S)?

(c) If the range to the sound-source r_S and the vertical angle θ_S remain the same, but the bearing angle changes to $\psi_S = 65°$, is the sound-source in the aperture's Fresnel region or far-field?

(a) First check the Fresnel range criterion $1.356 R_A < r < \pi R_A^2 / \lambda$.

$$R_A = \frac{L}{2} = \frac{100 \text{ m}}{2} = 50 \text{ m} , \quad r_{\min} = 1.356 R_A = 1.356 \times 50 \text{ m} = 67.8 \text{ m}$$

$$\lambda = \frac{c}{f} = \frac{1500 \text{ m/sec}}{1000 \text{ Hz}} = 1.5 \text{ m} , \quad r_{\text{NF/FF}} = \frac{\pi R_A^2}{\lambda} = \frac{\pi (50 \text{ m})^2}{1.5 \text{ m}} = 5.236 \text{ km}$$

Since $67.8 \text{ m} < r_S = 4 \text{ km} < 5.236 \text{ km}$, the Fresnel range criterion is satisfied.

Next, check the Fresnel angle criterion $72° \le \phi \le 108°$, where $\phi = \cos^{-1}(\hat{r}_S \bullet \hat{r}_A)$.

$$\hat{r}_S = u_S \hat{x} + v_S \hat{y} + w_S \hat{z}$$

$\hat{r}_A = \pm \hat{x}$ because the linear aperture is lying along the X axis.

$$\hat{r}_S \bullet \hat{r}_A = \pm u_S = \pm \sin\theta_S \cos\psi_S = \pm \sin 100° \cos 75° = \pm 0.255$$

$\phi = \cos^{-1}(0.255) = 75.2°$. The angle criterion $72° \le \phi \le 108°$ is satisfied.

$\phi = \cos^{-1}(-0.255) = 104.8°$. The angle criterion $72° \le \phi \le 108°$ is satisfied.

Since *both* the range and angle criteria are satisfied, the sound-source is in the Fresnel region of the aperture.

(b) Aperture focusing is required along with beam steering.

$$\therefore \theta(f, x_A) = \theta_1(f)\,x_A + \theta_2(f)\,x_A^2$$

$$\theta_1(f) = -2\pi f_X', \quad f_X' = \frac{u'}{\lambda} = \frac{\sin\theta'\cos\psi'}{\lambda}, \quad \theta_1(f) = -2\pi\frac{\sin\theta'\cos\psi'}{\lambda}$$

$$\theta_2(f) = \frac{k}{2r'}, \quad k = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}, \quad \theta_2(f) = \frac{\pi}{\lambda r'}$$

$$\lambda = 1.5 \text{ m}, \quad r' = r_S = 4 \text{ km}, \quad \theta' = \theta_S = 100°, \quad \psi' = \psi_S = 75°$$

$$\theta_1(1 \text{ kHz}) = -2\pi\frac{\sin 100° \cos 75°}{1.5 \text{ m}} = -1.0677 \text{ m}^{-1}$$

$$\theta_2(1 \text{ kHz}) = \frac{\pi}{1.5 \text{ m} \times 4000 \text{ m}} = 5.236 \times 10^{-4} \text{ m}^{-2}$$

Substituting yields

$$\theta(1 \text{ kHz}, x_A) = -1.0677 \text{ m}^{-1}\, x_A + (5.236 \times 10^{-4} \text{ m}^{-2})\, x_A^2 \text{ rad}$$

(c) Since r_S remains the same, the Fresnel range criterion is satisfied. Check the Fresnel angle criterion $72° \le \phi \le 108°$, where $\phi = \cos^{-1}(\hat{r}_S \bullet \hat{r}_A)$.

$$\hat{r}_S \bullet \hat{r}_A = \pm u_S = \pm\sin\theta_S \cos\psi_S = \pm\sin 100° \cos 65° = \pm 0.416$$

$\phi = \cos^{-1}(0.416) = 65.4°$. The angle criterion $72° \le \phi \le 108°$ is *not* satisfied.

$\phi = \cos^{-1}(-0.416) = 114.6°$. The angle criterion $72° \le \phi \le 108°$ is *not* satisfied.

Since the Fresnel angle criterion is not satisfied, the sound-source is *not* in the Fresnel region of the aperture. However, since $r_S < r_{\text{NF/FF}}$, the sound-source is in the near-field region of the aperture. Aperture focusing is still required, but in the near-field region of the aperture outside the Fresnel region, a quadratic phase response can only approximately focus a far-field beam pattern.